MATH 3630
Actuarial Mathematics I
Class Test 2 - 3:35-4:50 PM
Wednesday, 11 November 2015
Time Allowed: 1 hour, 15 minutes
Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: Emil

Student ID: Suggested Solutions

- There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.

Question No. 1:

You are given:

- $A_x = 0.3127$
- $A_{x+20} = 0.5970$
- $A_{x:\overline{20|}} = 0.4225$

Calculate $A^{\,1}_{x:\overline{20|}}$.

Since $A_x = A^{\,1}_{x:\overline{20|}} + {}_{20}E_x A_{x+20}$, then ${}_{20}E_x = \dfrac{A_x - A^{\,1}_{x:\overline{20|}}}{A_{x+20}}$

Thus, $A^{\,1}_{x:\overline{20|}} = A_{x:\overline{20|}} - {}_{20}E_x$

$$= A_{x:\overline{20|}} - \frac{A_x - A^{\,1}_{x:\overline{20|}}}{A_{x+20}}$$

$$\Rightarrow \quad A^{\,1}_{x:\overline{20|}}\left(1 - \frac{1}{A_{x+20}}\right) = A_{x:\overline{20|}} - A_x / A_{x+20}$$

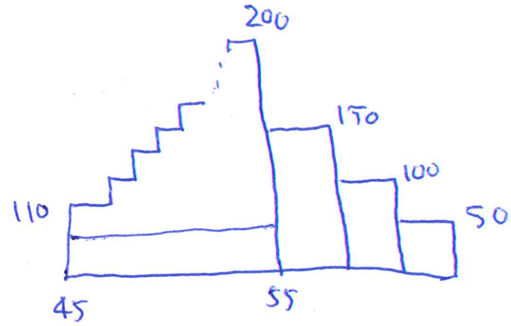
$$\Rightarrow \quad A^{\,1}_{x:\overline{20|}} = \frac{0.4225 - .3127/.5970}{1 - 1/.5970}$$

$$= \underline{\underline{0.1500434}}$$

Question No. 2:

For a special term life insurance policy issued to (45), you are given:

- Death benefits are payable at the end of the year of death.
- The death benefit in the first year is \$110, increasing by \$10 each year subsequently until reaching \$200, after which the benefit decreases by \$50 each year until reaching zero.
- $A^{\;1}_{45:\overline{10}|} = 0.0957$
- $(IA)^{\;1}_{45:\overline{10}|} = 0.5376$
- $(DA)^{\;1}_{55:\overline{3}|} = 1.8850$
- $(DA)^{\;1}_{55:\overline{4}|} = 2.7400$
- ${}_{10}E_{45} = 0.5354$



Payment pattern

Calculate the actuarial present value of the death benefits.

$$APV(\text{benefits}) = 100\, A^{\;1}_{45:\overline{10}|} + 10\,(IA)^{\;1}_{45:\overline{10}|} + 50\,(DA)^{\;1}_{55:\overline{3}|}\cdot {}_{10}E_{45}$$

$$= 100(.0957) + 10(0.5376) + 50(1.8850)(0.5354)$$

$$= \underline{\underline{65.40745}}$$

Question No. 3:

For a cohort of individuals all age x consisting of 70% males (m) and 30% females (f), you are given the following extract from a mortality table:

k	q^{m}_{x+k}	q^{f}_{x+k}
0	0.02	0.01
1	0.05	0.03
2	0.08	0.05

Let Z be the present value random variable for a 2-year term life insurance benefit of 1 payable at the end of the year of death of (x). Assume $i = 5\%$.

Calculate $\mathrm{Var}[Z]$ for a randomly chosen individual from this cohort.

$$A^{m\,1}_{x:\overline{2}|} = v\underbrace{q^m_x}_{.02} + v^2(1-\underbrace{q^m_x}_{.98})\underbrace{q^m_{x+1}}_{.05} = .06349206$$

$$A^{f\,1}_{x:\overline{2}|} = v\underbrace{q^f_x}_{.01} + v^2(1-\underbrace{q^f_x}_{.99})\underbrace{q^f_{x+1}}_{.03} = .03646259$$

$$A^{1}_{x:\overline{2}|} = .70(.06349206) + .30(.03646259) = .05538322$$

$$^2A^{m\,1}_{x:\overline{2}|} = v^2 q^m_x + v^4(1-q^m_x)q^m_{x+1} = .05845301$$

$$^2A^{f\,1}_{x:\overline{2}|} = v^2 q^f_x + v^4(1-q^f_x)q^f_{x+1} = .03350456$$

$$^2A^{1}_{x:\overline{2}|} = .70(.05845301) + .30(.03350456) = .05096848$$

$$\mathrm{Var}[Y] = {}^2A^1_{x:\overline{2}|} - (A^1_{x:\overline{2}|})^2$$

$$= .05096848 - (.05538322)^2$$

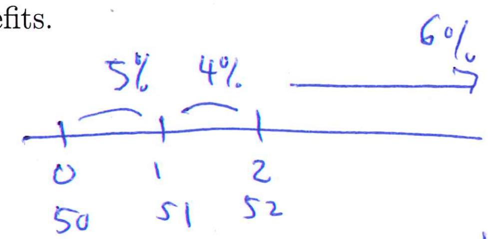
$$= \underline{\underline{.04790117}}$$

Question No. 4:

For a whole life insurance of \$10,000 issued to (50), you are given:

- Death benefits are payable at the end of the year of death.
- Mortality follows the `Illustrative Life Table` with the exception of the first year where you are given that $q_{50} = 0.015$.
- The annual effective interest rate is 5% in the first year, 4% in the second year, and 6% each year thereafter.

Calculate the actuarial present value of the death benefits.



$$A_{50} = \frac{1}{1.05}\underbrace{q_{50}}_{.015} + \frac{1}{1.05}\frac{1}{1.04}\underbrace{(1-q_{50})}_{.985}\underbrace{q_{51}^{ILT}}_{\frac{6.42}{1000}} + \frac{1}{1.05}\frac{1}{1.04}(1-q_{50})(1-q_{51}^{ILT})\underbrace{A_{52}^{ILT}}_{.27050}$$

$$= 0.2625052$$

$$APV(\text{benefits}) = 10{,}000\, A_{50}$$

$$= 10000\,(.2625052)$$

$$= \underline{\underline{2{,}625.052}}$$

Question No. 5:

You are given:

x	q_x	$\ddot{a}_x$
55	0.015	15.84
56	0.020	15.60

Calculate the annual effective interest rate i.

$$\ddot{a}_{55} = 1 + v p_{55} \ddot{a}_{56} \Rightarrow v = \frac{\ddot{a}_{55} - 1}{p_{55}\ddot{a}_{56}} = \frac{1}{1+i}$$

Thus, we have

$$i = \frac{p_{55}\ddot{a}_{56}}{\ddot{a}_{55} - 1} - 1$$

$$= \frac{(1-.015)(15.60)}{14.84} - 1$$

$$= .0354474 \approx \underline{\underline{3.5\%}}$$

Question No. 6:

For a whole life annuity-due issued to (40), you are given:

- For age before 65, mortality pattern follows the constant force with $\mu = 0.005$.
- $\delta = 0.03$
- $A_{65} = 0.425$

Calculate $\ddot{a}_{40}$.

$$A_{40} = A^{1}_{40:\overline{25}|} + {}_{25}E_{40} A_{65}$$

$$A^{1}_{40:\overline{25}|} = \sum_{K=0}^{24} v^{K+1} {}_{K}p_{40}\, q_{40+K}$$

$$= \sum_{K=0}^{24} v^{K+1} e^{-.005K}(1-e^{-.005})$$

$$= (1-e^{-.005}) v \sum_{K=0}^{24} (ve^{-.005})^{K}$$

$$= (1-e^{-.005}) e^{-.03} \frac{1-(ve^{-.005})^{25}}{1-ve^{-.005}}$$

$$= .08206107$$

$${}_{25}E_{40} = v^{25} e^{-.005(25)} = .416862$$

$$A_{40} = .08206107 + .416862(.425) = .2592274$$

$$\ddot{a}_{40} = \frac{1-A_{40}}{d} = \frac{1-.2592274}{1-e^{-.03}} = \underline{\underline{25.06466}}$$

Question No. 7:

For a special life annuity issued to [55], you are given:

- The benefit payments are \$200 at the beginning of age 57 and \$500 at the beginning of age 59. No other benefit payments are made.
- The following select-and-ultimate mortality table with a 3-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
55	882	877	871	864	58
56	875	870	863	856	59
57	868	863	856	849	60
58	861	855	848	840	61
59	854	847	840	832	62
60	846	839	832	823	63

- $i = 0.05$

Calculate actuarial present value of this life annuity.

$$\text{APV(annuity)} = 200\, v^2 \frac{\ell_{[55]+2}}{\ell_{[55]}} + 500\, v^4 \frac{\ell_{59}}{\ell_{[55]}}$$

$$= 200 \frac{1}{1.05^2} \frac{871}{882} + 500 \frac{1}{1.05^4} \frac{856}{882}$$

$$= \underline{\underline{578.3687}}$$

Question No. 8:

Each of 100 independent lives purchases a single premium whole life insurance of \$10 payable at the moment of death. You are given:

- Mortality follows a constant force with $\mu = 0.01$.
- $\delta = 0.05$
- F is the total amount of premium the insurer receives from the 100 lives.
- The 95th percentile of the standard normal distribution is 1.645.

Using a normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

Because of constant force, distribution of future lifetime is the same regardless of age. Let $Y = Y_1 + \dots + Y_{100}$

$$E[Y] = 100 A_x (10) = 1000 \frac{\mu}{\mu+\delta} = 1000\left(\frac{.01}{.06}\right) = 166.6667$$

$$Var[Y] = 100(10)^2\left[{}^2A_x - (A_x)^2\right]$$

$$= 100(10)^2\left[\frac{.01}{.11} - \left(\frac{.01}{.06}\right)^2\right] = 631.3131$$

$$Pr[F \geq Y] = Pr\left[\text{Normal} \leq \underbrace{\frac{F - E[Y]}{\sqrt{Var[Y]}}}_{1.645}\right] = .95$$

$$F = E[Y] + 1.645\sqrt{Var[Y]}$$

$$= 166.6667 + 1.645\sqrt{631.3131}$$

$$= \underline{\underline{207.9988}}$$

Question No. 9:

You are given:

- Mortality follows de Moivre's law with $\omega = 100$.
- $\delta = 5\%$
- Y is the present value random variable for a whole life annuity-due of \$1 per year issued to (45).

Calculate $\Pr[Y > 12]$. $T_{45} \sim \text{Uniform}(0, 55)$

$$Y = \ddot{a}_{\overline{K+1}|} = \frac{1 - v^{K+1}}{d}$$

$$\Pr[Y > 12] = \Pr\left[\frac{1 - v^{K+1}}{d} > 12\right] = \Pr\left[v^{K+1} < 1 - 12d\right]$$

$v = e^{-\delta}$

$$= \Pr\left[(K+1)\underbrace{\cancel{\log} v}_{-\delta} < \log(1 - 12d)\right]$$

$$= \Pr\left[K > \underbrace{\frac{\log(1-12d)}{-\delta} - 1}_{16.60144}\right]$$

$$= \Pr[K \geq 17] = \Pr[T > 17]$$

$$= {}_{17}p_{45} = \int_{17}^{55} \frac{1}{55}\, dt$$

$$= \frac{55 - 17}{55} = \underline{\underline{.6909091}}$$

Question No. 10:

For a whole life insurance on (40) with death benefits payable at the moment of death, you are given:

- The benefit amount at time t is $b_t = 10(1.02)^t$, for $t \geq 0$.
- Mortality follows a constant force with $\mu = 0.015$.
- $\delta = 5\%$

Calculate the actuarial present value for this insurance.

$$\text{APV(insurance)} = \int_0^\infty e^{-\delta t}\, b_t\, \mu e^{-\mu t}\, dt$$

$$= \int_0^\infty e^{-.05t}\, 10(1.02)^t (.015)\, e^{-.015t}\, dt$$

$$= 10(.015) \underbrace{\int_0^\infty \left(1.02\, e^{-.065}\right)^t dt}_{\frac{1}{\log(1.02\, e^{-.065})}\left[\cancel{\left(1.02 e^{-.065}\right)^\infty} - 1\right]}$$

$$= \underline{\underline{3.318777}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK

Illustrative Life Table: Basic Functions and Single Benefit Premiums at $i = 0.06$

x	l_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000(^2A_x)$	$1000\,{}_5E_x$	$1000\,{}_{10}E_x$	$1000\,{}_{20}E_x$	x
0	10,000,000	20.42	16.8010	49.00	25.92	728.54	541.95	299.89	0
5	9,749,503	0.98	17.0379	35.59	8.45	743.89	553.48	305.90	5
10	9,705,588	0.85	16.9119	42.72	9.37	744.04	553.34	305.24	10
15	9,663,731	0.91	16.7384	52.55	11.33	743.71	552.69	303.96	15
20	9,617,802	1.03	16.5133	65.28	14.30	743.16	551.64	301.93	20
21	9,607,896	1.06	16.4611	68.24	15.06	743.01	551.36	301.40	21
22	9,597,695	1.10	16.4061	71.35	15.87	742.86	551.06	300.82	22
23	9,587,169	1.13	16.3484	74.62	16.76	742.68	550.73	300.19	23
24	9,576,288	1.18	16.2878	78.05	17.71	742.49	550.36	299.49	24
25	9,565,017	1.22	16.2242	81.65	18.75	742.29	549.97	298.73	25
26	9,553,319	1.27	16.1574	85.43	19.87	742.06	549.53	297.90	26
27	9,541,153	1.33	16.0873	89.40	21.07	741.81	549.05	297.00	27
28	9,528,475	1.39	16.0139	93.56	22.38	741.54	548.53	296.01	28
29	9,515,235	1.46	15.9368	97.92	23.79	741.24	547.96	294.92	29
30	9,501,381	1.53	15.8561	102.48	25.31	740.91	547.33	293.74	30
31	9,486,854	1.61	15.7716	107.27	26.95	740.55	546.65	292.45	31
32	9,471,591	1.70	15.6831	112.28	28.72	740.16	545.90	291.04	32
33	9,455,522	1.79	15.5906	117.51	30.63	739.72	545.07	289.50	33
34	9,438,571	1.90	15.4938	122.99	32.68	739.25	544.17	287.82	34
35	9,420,657	2.01	15.3926	128.72	34.88	738.73	543.18	286.00	35
36	9,401,688	2.14	15.2870	134.70	37.26	738.16	542.11	284.00	36
37	9,381,566	2.28	15.1767	140.94	39.81	737.54	540.92	281.84	37
38	9,360,184	2.43	15.0616	147.46	42.55	736.86	539.63	279.48	38
39	9,337,427	2.60	14.9416	154.25	45.48	736.11	538.22	276.92	39
40	9,313,166	2.78	14.8166	161.32	48.63	735.29	536.67	274.14	40
41	9,287,264	2.98	14.6864	168.69	52.01	734.40	534.99	271.12	41
42	9,259,571	3.20	14.5510	176.36	55.62	733.42	533.14	267.85	42
43	9,229,925	3.44	14.4102	184.33	59.48	732.34	531.12	264.31	43
44	9,198,149	3.71	14.2639	192.61	63.61	731.17	528.92	260.48	44
45	9,164,051	4.00	14.1121	201.20	68.02	729.88	526.52	256.34	45
46	9,127,426	4.31	13.9546	210.12	72.72	728.47	523.89	251.88	46
47	9,088,049	4.66	13.7914	219.36	77.73	726.93	521.03	247.08	47
48	9,045,679	5.04	13.6224	228.92	83.06	725.24	517.91	241.93	48
49	9,000,057	5.46	13.4475	238.82	88.73	723.39	514.51	236.39	49
50	8,950,901	5.92	13.2668	249.05	94.76	721.37	510.81	230.47	50
51	8,897,913	6.42	13.0803	259.61	101.15	719.17	506.78	224.15	51
52	8,840,770	6.97	12.8879	270.50	107.92	716.76	502.40	217.42	52
53	8,779,128	7.58	12.6896	281.72	115.09	714.12	497.64	210.27	53
54	8,712,621	8.24	12.4856	293.27	122.67	711.24	492.47	202.70	54
55	8,640,861	8.96	12.2758	305.14	130.67	708.10	486.86	194.72	55
56	8,563,435	9.75	12.0604	317.33	139.11	704.67	480.79	186.32	56
57	8,479,908	10.62	11.8395	329.84	147.99	700.93	474.22	177.53	57
58	8,389,826	11.58	11.6133	342.65	157.33	696.85	467.12	168.37	58
59	8,292,713	12.62	11.3818	355.75	167.13	692.41	459.46	158.87	59
60	8,188,074	13.76	11.1454	369.13	177.41	687.56	451.20	149.06	60
61	8,075,403	15.01	10.9041	382.79	188.17	682.29	442.31	139.00	61
62	7,954,179	16.38	10.6584	396.70	199.41	676.56	432.77	128.75	62
63	7,823,879	17.88	10.4084	410.85	211.13	670.33	422.54	118.38	63
64	7,683,979	19.52	10.1544	425.22	223.34	663.56	411.61	107.97	64
65	7,533,964	21.32	9.8969	439.80	236.03	656.23	399.94	97.60	65

Illustrative Life Table: Basic Functions and Single Benefit Premiums at $i = 0.06$

x	l_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000(^2A_x)$	$1000\,{}_5E_x$	$1000\,{}_{10}E_x$	$1000\,{}_{20}E_x$	x
66	7,373,338	23.29	9.6362	454.56	249.20	648.27	387.53	87.37	66
67	7,201,635	25.44	9.3726	469.47	262.83	639.66	374.36	77.38	67
68	7,018,432	27.79	9.1066	484.53	276.92	630.35	360.44	67.74	68
69	6,823,367	30.37	8.8387	499.70	291.46	620.30	345.77	58.54	69
70	6,616,155	33.18	8.5693	514.95	306.42	609.46	330.37	49.88	70
71	6,396,609	36.26	8.2988	530.26	321.78	597.79	314.27	41.86	71
72	6,164,663	39.62	8.0278	545.60	337.54	585.25	297.51	34.53	72
73	5,920,394	43.30	7.7568	560.93	353.64	571.81	280.17	27.96	73
74	5,664,051	47.31	7.4864	576.24	370.08	557.43	262.31	22.19	74
75	5,396,081	51.69	7.2170	591.49	386.81	542.07	244.03	17.22	75
76	5,117,152	56.47	6.9493	606.65	403.80	525.71	225.46	13.04	76
77	4,828,182	61.68	6.6836	621.68	421.02	508.35	206.71	9.61	77
78	4,530,360	67.37	6.4207	636.56	438.42	489.97	187.94	6.88	78
79	4,225,163	73.56	6.1610	651.26	455.95	470.57	169.31	4.77	79
80	3,914,365	80.30	5.9050	665.75	473.59	450.19	151.00	3.19	80
81	3,600,038	87.64	5.6533	680.00	491.27	428.86	133.19	2.05	81
82	3,284,542	95.61	5.4063	693.98	508.96	406.62	116.06	1.27	82
83	2,970,496	104.28	5.1645	707.67	526.60	383.57	99.81	0.75	83
84	2,660,734	113.69	4.9282	721.04	544.15	359.79	84.59	0.42	84
85	2,358,246	123.89	4.6980	734.07	561.57	335.40	70.56	0.22	85
86	2,066,090	134.94	4.4742	746.74	578.80	310.56	57.83	0.11	86
87	1,787,299	146.89	4.2571	759.03	595.79	285.44	46.50	0.05	87
88	1,524,758	159.81	4.0470	770.92	612.51	260.21	36.61	0.02	88
89	1,281,083	173.75	3.8442	782.41	628.92	235.11	28.17	0.01	89
90	1,058,491	188.77	3.6488	793.46	644.96	210.36	21.13	0.00	90
91	858,676	204.93	3.4611	804.09	660.61	186.21	15.41	0.00	91
92	682,707	222.27	3.2812	814.27	675.83	162.90	10.91	0.00	92
93	530,959	240.86	3.1091	824.01	690.59	140.69	7.47	0.00	93
94	403,072	260.73	2.9450	833.30	704.86	119.79	4.93	0.00	94
95	297,981	281.91	2.7888	842.14	718.61	100.43	3.13	0.00	95
96	213,977	304.45	2.6406	850.53	731.83	82.78	1.90	0.00	96
97	148,832	328.34	2.5002	858.48	744.50	66.97	1.10	0.00	97
98	99,965	353.60	2.3676	865.99	756.60	53.09	0.60	0.00	98
99	64,617	380.20	2.2426	873.06	768.13	41.14	0.31	0.00	99
100	40,049	408.12	2.1252	879.70	779.08	31.12	0.15	0.00	100
101	23,705	437.28	2.0152	885.93	789.44	22.91	0.07	0.00	101
102	13,339	467.61	1.9123	891.76	799.21	16.37	0.03	0.00	102
103	7,101	498.99	1.8164	897.19	808.41	11.33	0.01	0.00	103
104	3,558	531.28	1.7273	902.23	817.02	7.56	0.00	0.00	104
105	1,668	564.29	1.6447	906.90	825.06	4.86	0.00	0.00	105
106	727	597.83	1.5685	911.22	832.53	2.99	0.00	0.00	106
107	292	631.64	1.4984	915.19	839.46	1.76	0.00	0.00	107
108	108	665.45	1.4341	918.82	845.84	0.98	0.00	0.00	108
109	36	698.97	1.3755	922.14	851.69	0.52	0.00	0.00	109
110	11	731.87	1.3223	925.15	857.04	0.26	0.00	0.00	110