Life Annuities

Annuities

series of payments

Lecture: Weeks 8-9

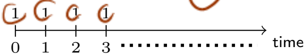
P.V. random variables

What are annuities?

An annuity is a series of payments that could vary according to:

- timing of payment due right now

beginning of year (annuity-due)



end of year (annuity-immediate)

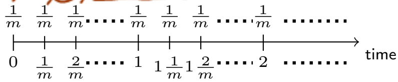
- with fixed maturity

n-year annuity-due

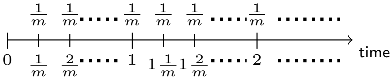
n-year annuity-immediate

- more frequently than once a year

annuity-due payable m-thly



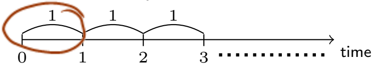
annuity-immediate payable m-thly



- payable continuously

not practical.

continuous annuity



- varying benefits

$\ddot{a}$ a

Review of annuities-certain

annuity-due

annuity-immediate

- payable annually

$1+v+v^2+\dots+v^{n-1}$

$$\ddot{a}_{\overline{n}|} = \sum_{k=0}^{n} v^{k-1} = \frac{1-v^n}{d}$$

due

$v+v^2+\dots+v^n$

$$a_{\overline{n}|} = \sum_{k=1}^{n} v^{k} = \frac{1-v^n}{i}$$

imm

- payable m times a year

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m} \sum_{k=0}^{mn-1} v^{k/m} = \frac{1-v^n}{d^{(m)}}$$

$$a^{(m)}_{\overline{n}|} = \frac{1}{m} \sum_{k=1}^{mn} v^{k} = \frac{1-v^n}{i^{(m)}}$$

continuous annuity

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1-v^n}{\delta}$$

continuous rate

Chapter summary

Y = PV r.v. symbol for life annuities

- Life annuities
  - series of benefits paid contingent upon survival of a given life
  - single life considered
  - actuarial present values (APV) or expected present values (EPV)
  - actuarial symbols and notation
- Types of annuities
  - discrete - due or immediate
    - payable more frequently than once a year
  - continuous
  - varying payments

  - insurance product
  - premium
- "Current payment techniques" APV formulas
- Chapter 5 of Dickson, et al.

Whole life annuity-due

- Pays a benefit of a unit $\$1$ at the beginning of each year that the annuitant (x) survives.
- The present value random variable is

$$Y = \ddot{a}_{\overline{K+1}|}$$

where K, in short for K_x, is the curtate future lifetime of (x).

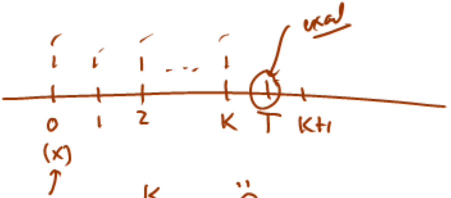
- The actuarial present value of a whole life annuity-due is

$$\begin{aligned} \ddot{a}_x &= \mathsf{E}[Y] = \mathsf{E}\left[\ddot{a}_{\overline{K+1}|}\right] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \mathsf{Pr}[K=k] \\ &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \cdot {}_{k}p_x \, q_{x+k} \end{aligned}$$

(discrete) life annuity due - whole life

K = curtate lifetime

$K+1$ payments



$$Y = \text{PV of these annuity payments} = \underbrace{1 + v + \cdots + v^K} = \ddot{a}_{\overline{K+1}|}$$

$$APV = E[Y] = E[\ddot{a}_{\overline{K+1}|}] = \underbrace{\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \underbrace{\Pr[K=k]}_{k|q_x}}$$

$$\underbrace{q_x + \ddot{a}_{\overline{2}|}\, p_x\, q_{x+1} + \cdots}_{\text{cumbersome.}}$$

$$Y = \ddot{a}_{\overline{K+1}|} = \underbrace{1\,\underbrace{I(T>0)}_{1} + v\,I(T>1) + v^2 I(T>2) + \dots}_{\sum_{k=0}^{\infty} v^k I(T>k)}$$

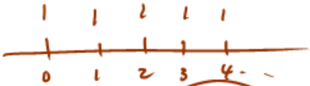
T = future lifetime

$$I(T>k) = 1 \text{ if } T>k, \quad = 0 \text{ if } T \le k$$

$$E[\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} v^k \underbrace{E[I(T>k)]}_{\underbrace{\Pr[T>k]}_{{}_k p_x}}$$

$$= \sum_{k=0}^{\infty} v^k \, {}_k p_x$$

$$= 1 + v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \dots$$



intuitive

$$\ddot{a}_x$$

APV of a (discrete) WL annuity due to (x)

$$= \ddot{a}_x = E[\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} v^k {}_kp_x$$

VIF

Y

sums of pure endowments

$$= \sum_{k=0}^{\infty} {}_kE_x$$

$\frac{l_{x+k}}{l_x}$

"current payment techniques"

$$\frac{1}{l_x} \sum_{k=0}^{\infty} v^k l_{x+k}$$

$$Var[Y] = E[Y^2] - \underbrace{(E[Y])^2}_{(\ddot{a}_x)^2}$$

$E[Y^2] \neq @2\delta$

${}^2\ddot{a}_x$

$$Y = \ddot{a}_{\overline{K+1}|} = \frac{1 - v^{K+1}}{d}$$

PV of a WL insurance

$$= \frac{1 - Z}{d}$$

$$Var\left(\frac{Z}{d}\right)$$

Relationship between annuities and insurance

$$E[Y] = \frac{1 - E[Z]}{d}$$

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

$$\boxed{A_x = 1 - d\,\ddot{a}_x}$$ VIF

$$Var[Y] = Var\left[\frac{1-Z}{d}\right]$$

$$= \frac{1}{d^2} Var[Z]$$

$$= \underbrace{\frac{1}{d^2}\left[{}^2A_x - (A_x)^2 \right]}$$

$A_x = 1 - d\ddot{a}_x \implies A_x + d\ddot{a}_x = 1$

i

d

true,

1

today

interest payment at b.o.y. while alive

back your investment of 1 when you die

Current payment technique

By writing the PV random variable as

$$Y = I(T>0) + vI(T>1) + v^2 I(T>2) + \cdots = \sum_{k=0}^{\infty} v^k I(T>k),$$

one can immediately deduce that

$$\begin{aligned}
\ddot{a}_x &= \mathsf{E}[Y] = \mathsf{E}\left[\sum_{k=0}^{\infty} v^k I(T>k)\right] \\
&= \sum_{k=0}^{\infty} v^k \mathsf{E}[I(T>k)] = \sum_{k=0}^{\infty} v^k \mathsf{Pr}[T>k] \\
&= \sum_{k=0}^{\infty} v^k {}_k p_x = \sum_{k=0}^{\infty} {}_k E_x = \sum_{k=0}^{\infty} A_{x:\overset{1}{\overline{k|}}}.
\end{aligned}$$

A straightforward proof of $\underbrace{\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \cdot {}_{k|}q_x}$ $= \sum_{k=0}^{\infty} v^k {}_k p_x$ is in **Example 5.1**.

Current payment technique - continued

$$\sum \ddot{a}_{\overline{k+1}|} \Pr(K \ge k)$$

vs

$$\sum v^k \, {}_kp_x$$

- The commonly used formula $\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$ is the so-called current payment technique for evaluating life annuities.
- Indeed, this formula gives us another intuitive interpretation of what life annuities are: they are nothing but sums of pure endowments (you get a benefit each time you survive).
- The primary difference lies in when you view the payments: one gives the series of payments made upon death, the other gives the payment made each time you survive.

Some useful formulas

relates insurance to annuity

By recalling that $\ddot{a}_{\overline{K+1}|} = \dfrac{1 - v^{K+1}}{d}$, we can use this to derive:

- relationship to whole life insurance

$$\ddot{a}_x = \mathsf{E}\left[\frac{1 - v^{K+1}}{d}\right] = \frac{1}{d}\left(1 - A_x\right).$$

Alternatively, we write: $A_x = 1 - d\ddot{a}_x$. very important formula!

- the variance formula

$$\mathsf{Var}[Y] = \mathsf{Var}\left[\ddot{a}_{\overline{K+1}|}\right] = \frac{1}{d^2}\mathsf{Var}\left[v^{K+1}\right] = \boxed{\frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right].}$$

Illustrative example 1

Suppose you are interested in valuing a whole life annuity-due issued to (95). You are given:

- $i = 5\%$, and
- the following extract from a life table:

x	95	96	97	98	99	100
ℓ_x	100	70	40	20	4	0

1. Express the present value random variable for a whole life annuity-due to (95). $Y = \ddot{a}_{\overline{K+1}|}$ where K is the curtate lifetime of (95)
2. Calculate the expected value of this random variable.
3. Calculate the variance of this random variable.

①

x	ℓ_x	
95	100	30
96	70	30
97	40	20
98	20	16
99	4	4
100	0	

CPT $\quad APV = \ddot{a}_{95} = 1 + v\frac{70}{100} + v^2\frac{40}{100} + v^3\frac{20}{100} + v^4\frac{4}{100} + \phi$ ✓

$v = \frac{1}{1.05}$

$= 2.2351549$ ✓

$Var[Y] = \frac{1}{d^2}\left[{}^2A_{95} - (A_{95})^2\right]$

$d = \frac{.05}{1.05}$

$A_{95} = 1 - d\,\ddot{a}_{95} = 1 - \frac{.05}{1.05}(2.2351549) = .8935641$ ✓

$d = \frac{i}{1+i} = 1 - v$

$d@2i = 1 - v^2$

${}^2A_{95} = 1 - \underbrace{(d@2i)}_{(1-v^2)}\,{}^2\ddot{a}_{95}$

${}^2\ddot{a}_{95} = 1 + v^2\frac{70}{100} + v^4\frac{40}{100} + v^6\frac{20}{100} + v^8\frac{4}{100} + \phi = 2.140318$ ✓

$= *$

$Var[Y] = \frac{1}{1-v^2}\left[* - .8935641\right] = 1.127508$ ✓

${}^2A_x$

${}^2\ddot{a}_x$

(2)

k	$\Pr[K=k]$	$\ddot{a}_{\overline{k+1}\rceil}$	$\ddot{a}_{\overline{k+1}\rceil}\Pr(K=k)$	$\ddot{a}^2_{\overline{k+1}\rceil}\Pr(K=k)$
0	$30/100$	1	$30/100$	$30/100$
1	$30/100$	$\ddot{a}_{\overline{2}\rceil} = 1+v = 1.952381$	$\vdots$	$\vdots$
2	$20/100$	$\ddot{a}_{\overline{3}\rceil} = 1+v+v^2$		
3	$16/100$	$\ddot{a}_{\overline{4}\rceil} = 1+v+v^2+v^3$		
4	$4/100$	$\ddot{a}_{\overline{5}\rceil} = 1+v+v^2+v^3+v^4$		
5	0			

$$\sum = \ddot{a}_{95} = 2.2351549$$

$$\sum = E[Y^2] = 6.123433$$

$$Var[Y] = E[Y^2] - (E[Y])^2$$

$$= \underbrace{6.123433 - (2.2351549)^2}$$

$$= 1.127508$$

Traditional life annuities - discrete

Annuity Type	PV r.v. Y	APV $\mathsf{E}[Y]$	Current payment technique	Variance $\mathsf{Var}[Y]$	Relationship to insurance
whole life	$\ddot{a}_{\overline{K+1}\rvert}$	$\ddot{a}_x$	$\sum_{k=0}^{\infty} v^k {}_kp_x$	$\frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$	$\frac{1}{d}(1-A_x)$
temporary life	$\ddot{a}_{\overline{\min(K+1,n)}\rvert}$	$\ddot{a}_{x:\overline{n}\rvert}$	$\sum_{k=0}^{n-1} v^k {}_kp_x$	$\frac{1}{d^2}\left[{}^2A_{x:\overline{n}\rvert} - \left(A_{x:\overline{n}\rvert}\right)^2\right]$	$\frac{1}{d}\left(1-A_{x:\overline{n}\rvert}\right)$
deferred life	${}_{n\rvert}\ddot{a}_{\overline{K+1-n}\rvert} I(K \geq n)$	${}_{n\rvert}\ddot{a}_x$	$\sum_{k=n}^{\infty} v^k {}_kp_x$	complicated	$\frac{1}{d}\left({}_nE_x - {}_{n\rvert}A_x\right)$

varying payments to (x)

whole life annuity due with payments of

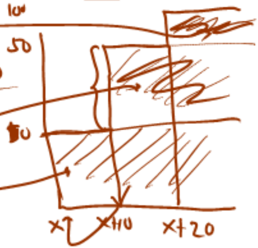
10 in the first 10 years

50 in the following 10 years

100 thereafter

Write the APV of this life annuity due in terms of whole life annuity due

$$APV = 10\,\ddot{a}_x + 40\,{}_{10}E_x\,\ddot{a}_{x+10} + 50\,{}_{20}E_x\,\ddot{a}_{x+20}$$



SULT A_x $A_{x:\overline{10}|}$ $A_{x:\overline{20}|}$ ${}_5E_x$ ${}_{10}E_x$ ${}_{20}E_x$

$\ddot{a}_x$ $\ddot{a}_{x:\overline{10}|}$ $\ddot{a}_{x:\overline{20}|}$ x l_x q_x

n-year temporary life annuity due to (x)

$$\text{P.V. r.v.} = Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K < n \\ \ddot{a}_{\overline{n}|}, & K \geq n \end{cases}$$

$$= \ddot{a}_{\overline{\min(K+1,\,n)}|}$$

$$E[Y] = \ddot{a}_{x:\overline{n}|} = \underbrace{\sum_{K=0}^{\infty} \ddot{a}_{\overline{\min(K+1,\,n)}|}\ {}_{K|}q_x}$$

current payment technique

$$= \sum_{K=0}^{n-1} v^K\, {}_K p_x \longrightarrow \underline{\text{n payments}}$$

✓ $Y = \ddot{a}_{\overline{\min(K+1,n)}|} = \dfrac{1 - v^{\min(K+1,\,n)}}{d} \longrightarrow$ PV of an endowment

$$A_x = 1 - d\,\ddot{a}_x$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}$$

$$\boxed{A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}}$$

n-year deferred ~~[illegible]~~ life annuity-due

$$Y = \begin{cases} 0, & K < n \\ v^n \ddot{a}_{\overline{K-n+1}|}, & K \geq n \end{cases}$$

$$E[Y] = {}_{n|}\ddot{a}_x = {}_nE_x\,\ddot{a}_{x+n}$$

$$ {}_{n|}\ddot{a}_x = {}_nE_x \underbrace{\ddot{a}_{x+n}}_{\frac{1-A_{x+n}}{d}} = \frac{{}_nE_x - \overbrace{{}_nE_x A_{x+n}}^{{}_{n|}A_x}}{d} $$

$$ d\,{}_{n|}\ddot{a}_x = {}_nE_x - {}_{n|}A_x $$

$$ {}_{n|}A_x = {}_nE_x - d \cdot {}_{n|}\ddot{a}_x $$

$\Downarrow$

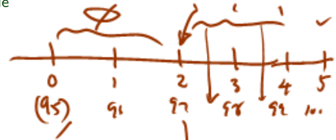
why?

$$ A_x + d\ddot{a}_x = 1 $$

$$ {}_{n|}A_x + d\,{}_{n|}\ddot{a}_x = {}_nE_x $$

$n|$

Illustrative example 2



Suppose you are interested in valuing a 2-year deferred whole life annuity-due issued to (95). You are given:

- $i = 6\%$ and
- the following extract from a life table:

x	95	96	97	98	99	100
ℓ_x	1000	750	400	225	75	0

175 150 75

$Y = \begin{cases} 0, & K<2 \\ {}_{2|}\ddot{a}_{\overline{K-2+1}|}, & K \geq 2 \end{cases}$

1. Express the present value random variable for this annuity.
2. Calculate the expected value of this random variable.
3. Calculate the variance of this random variable.

$$APV = {}_{2|}\ddot{a}_{95} = \sum_{k=2}^{\infty} v^k \, l_{95+k}/l_{95} = \underbrace{v^2 \frac{\overset{400}{l_{97}}}{\underset{1000}{l_{95}}} + v^3 \frac{\overset{225}{l_{98}}}{\underset{1000}{l_{95}}} + v^4 \frac{\overset{75}{l_{99}}}{\underset{1000}{l_{95}}}}$$

$$v = \frac{1}{1.06}$$

$$= 0.6043199$$

K	$Pr[K=k]$	Y	Y^2	$Y^2 \times Pr[K=k]$
0	—	—	—	—
1	—	—	—	—
2	175/1000	v^2	$(\ )^2$	$\longrightarrow$
3	150/1000	v^2+v^3	$(\ )^2$	$\longrightarrow$
4	75/1000	$v^2+v^3+v^4$	$(\ )^2$	$\longrightarrow$

$$E[Y^2] = 1.064278$$

$$Var[Y] = E[Y^2] - (E[Y])^2 = 1.064278 - (.6043199)^2 = \underline{0.6990758}$$

Recursive relationships

- The following relationships are easy to show:

$$\begin{aligned} \ddot{a}_x &= 1 + vp_x\ddot{a}_{x+1} = 1 + {}_1E_x\ddot{a}_{x+1} \\ &= 1 + vp_x + v^2{}_2p_x\ddot{a}_{x+2} = 1 + {}_1E_x + {}_2E_x\ddot{a}_{x+2} \end{aligned}$$

- In general, because E's are multiplicative, we can generalized this recursions to

$$\begin{aligned} \ddot{a}_x &= \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{n-1} {}_kE_x + \sum_{k=n}^{\infty} {}_kE_x \\ & \quad \text{apply change of variable } k^* = k - n \\ &= \ddot{a}_{x:\overline{n}|} + \sum_{k^*=0}^{\infty} {}_nE_x \, {}_{k^*}E_{x+n} = \ddot{a}_{x:\overline{n}|} + {}_nE_x \sum_{k^*=0}^{\infty} {}_{k^*}E_{x+n} \\ &= \ddot{a}_{x:\overline{n}|} + {}_nE_x \, \ddot{a}_{x+n} = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x \end{aligned}$$

- The last term shows that a whole life annuity is the sum of a term life annuity and a deferred life annuity.

Insurance $A_x = v q_x + v p_x A_{x+1}$ $\quad x \longrightarrow x+1$

annuities $\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$

$= 1 + v p_x + v^2 {}_2p_x \ddot{a}_{x+2}$

$$\ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = 1 + \underbrace{\sum_{k=1}^{\infty} {}_kE_x}_{\substack{k^* = k-1 \\ k = k^*+1}} = 1 + \sum_{k^*=0}^{\infty} \underbrace{{}_{k^*+1}E_x}_{{}_1E_x \cdot {}_{k^*}E_{x+1}}$$

$$= 1 + \underbrace{{}_1E_x}_{v p_x} \underbrace{\sum_{k=0}^{\infty} {}_kE_{x+1}}_{\ddot{a}_{x+1}}$$

VIF

$$\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$$

$$= 1 + v p_x + v^2 {}_2p_x \ddot{a}_{x+2}$$

$$\vdots$$

$$= \underbrace{1 + v p_x + v^2 {}_2p_x + \dots + v^{n-1} {}_{n-1}p_x}_{\text{n-year temporary}} + \underbrace{\underline{v^n {}_np_x \ddot{a}_{x+n}}}_{{}_nE_x \ddot{a}_{x+n} \text{ n-year deferred}}$$

whole life annuity = n-year temporary + n-year deferred

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x$$

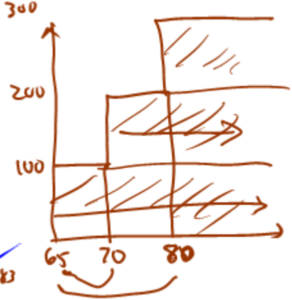
$$A_x = A^1_{x:\overline{n}|} + {}_{n|}A_x$$

varying benefits whole life annuity due with payments

x = issue age 65 — 100 for the first 5 years

200 for the following 10 years

300 thereafter



Assume: Mortality follows SULT

$i = 5\%$

Calculate APV of these payments.

$$APV(\text{annuity}) = 100\,\underset{13.5498}{\ddot{a}_{65}} + 100\,\underset{.75455}{{}_5E_{65}}\,\overset{12.0083}{\ddot{a}_{70}} + 100\,\underbrace{{}_{15}E_{65}}_{{}_5E_{65}\;{}_{10}E_{70}}\,\overset{8.5484}{\ddot{a}_{80}}$$

${}_5E_{65} = .75455$ ${}_{10}E_{70} = .50994$

$= 2{,}589.988 \approx \underline{\underline{2590}}$

post practice
problems

– book

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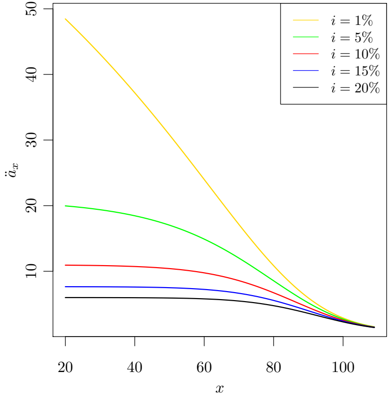
no classes next
week



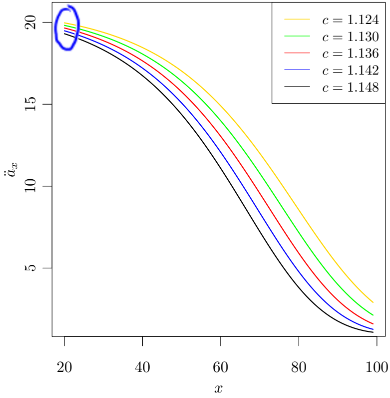
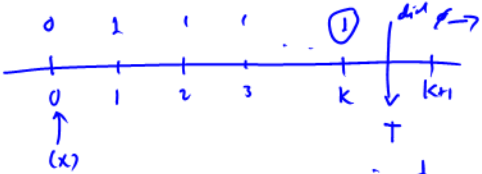


Figure: Comparing APV of a whole life annuity-due for based on the Standard Ultimate Life Table (Makeham with $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $c = 1.124$). Left figure: varying i. Right figure: varying c with $i = 5\%$

whole life annuity immediate



$$Y = PV = a_{\overline{K}|} = \frac{1-v^K}{i}$$

$K = K_x$ = curtate lifetime of (x)

$iv = d$

$E[Y] = EPV = APV$(annuity immediate) $= a_x = E[a_{\overline{K}|}] = \sum a_{\overline{k}|} P(K=k)$

$$Var[Y] = Var[a_{\overline{K}|}] = Var\left[\frac{1-v^K}{i}\right] = \frac{1}{i^2v^2} Var\left[\frac{v^{K+1}}{\cancel{v}}\right] = \frac{1}{(iv)^2} Var[v^{K+1}]$$

$$= \underbrace{\frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]}_{Var[\ddot{a}_{\overline{K+1}|}]}.$$

$$a_x = \sum_{k=1}^{\infty} v^k {}_kp_x$$

$E[a_{\overline{K}|}]$

CPT

$$\ddot{a}_x = \underbrace{\sum_{k=0}^{\infty}}\, v^k {}_kp_x = 1 + a_x$$

$E[a_{\overline{k|}}] = E\left[\frac{1-v^k}{i}\right] \Rightarrow E\left[\frac{v - v^{k+1}}{d}\right] = E\left[\underbrace{\frac{1+v-v^{k+1}-1}{d}}\right]$

$E[\ddot{a}_{\overline{k+1|}}] = E\left[\frac{1-v^{k+1}}{d}\right] =$

$= E\left[\frac{1-v^{k+1}}{d} - \frac{\overbrace{(1-v)}^{1}}{d}\right]$

$\ddot{a}_x > a_x$

$d = iv$
$= 1-v$

$= \ddot{a}_x - 1$

$a_x = \frac{1}{i}\left[1 - E\left[\frac{v^{k+1}}{v}\right]\right]$

$= \frac{1}{i}\left[1 - \frac{1}{v}A_x\right] = \frac{1-(1+i)A_x}{i}$

$\boxed{1 = i\,a_x + (1+i)A_x}$

$\boxed{1 = d\ddot{a}_x + A_x}$ VIF

equal

Whole life annuity-immediate

- Procedures and principles for annuity-due can be adapted for annuity-immediate.
- Consider the whole life annuity-immediate, the PV random variable is clearly $Y = a_{\overline{K|}}$ so that APV is given by

$$a_x = \mathsf{E}[Y] = \sum_{k=0}^{\infty} a_{\overline{K|}} \cdot {}_kp_x \, q_{x+k} = \sum_{k=1}^{\infty} v^k \, {}_kp_x.$$

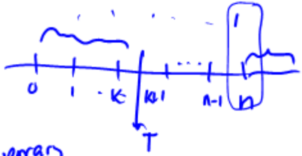
- Relationship to life insurance:

$$Y = \frac{1}{i}\left(1 - v^K\right) = \frac{1}{i}\left[1 - (1+i)v^{K+1}\right]$$

leads to $1 = ia_x + (1+i)A_x$.

- Interpretation of this equation - to be discussed in class.

Other types of life annuity-immediate



temporary

- For an n-year life annuity-immediate:
  - Find expression for the present value random variable.
  - Express formulas for its actuarial present value or expectation.
  - Find expression for the variance of the present value random variable.

deferred

- For an n-year deferred whole life annuity-immediate:
  - Find expression for the present value random variable.
  - Give expressions for the actuarial present value.
- Details to be discussed in lecture.

Life annuities with m-thly payments

m = 1
m = 12 mnthly
m = 4
m = 6

In practice, life annuities are often payable more frequently than once a year, e.g. monthly ($m = 12$), quarterly ($m = 4$), or semi-annually ($m = 2$).

Here, we define the random variable $K_x^{(m)}$, or simply $K^{(m)}$, to be the complete future lifetime rounded down to the nearest $1/m$-th of a year.

For example, if the observed $T = 45.86$ for a life (x) and $m = 4$, then the observed $K^{(4)}$ is $45\frac{3}{4}$.

Indeed, we can write

$$K^{(m)} = \frac{1}{m} \lfloor mT \rfloor,$$

where $\lfloor\ \rfloor$ is greatest integer (or floor) function.

UCONN

m-thly dues

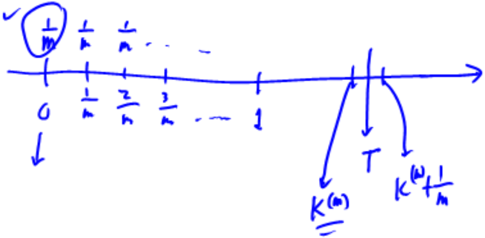
K = curtate lifetime

$$K^{(m)} = K_x^{(m)} = \frac{\lfloor mT \rfloor}{m}$$

$T = 45.86 \quad m = 4$

$$K^{(4)} = 45.75 = 45\tfrac{3}{4}$$

$$K^{(12)} = \frac{\lfloor 12 \times 45.86 \rfloor}{12} = \frac{\lfloor 550.32 \rfloor}{12} = \frac{550}{12} = \cancel{45.8333} = 45\tfrac{10}{12}$$



Whole life annuity-due payable m times a year

- Consider a whole life annuity-due with payments made m times a year. Its PV random variable can be expressed as

$$Y = \ddot{a}^{(m)}_{\overline{K^{(m)}+(1/m)}|} = \frac{1 - v^{K^{(m)}+(1/m)}}{d^{(m)}}.$$

- The APV of this annuity is

$$\mathsf{E}[Y] = \ddot{a}_x^{(m)} = \frac{1}{m}\sum_{h=0}^{\infty} v^{h/m} \cdot {}_{h/m}p_x = \frac{1 - A_x^{(m)}}{d^{(m)}}.$$

- Variance is

$$\mathsf{Var}[Y] = \frac{\mathsf{Var}\left[v^{K^{(m)}+(1/m)}\right]}{\left(d^{(m)}\right)^2} = \frac{{}^2A_x^{(m)} - \left(A_x^{(m)}\right)^2}{\left(d^{(m)}\right)^2}.$$

PV of a whole life annuity due payable m-thly is $Y = \ddot{a}^{(m)}_{\overline{K^{(m)}+\frac{1}{m}}|}$

$$E\left[\ddot{a}^{(m)}_{\overline{K^{(m)}+\frac{1}{m}}|}\right] = E\left[\frac{1-v^{K^{(m)}+\frac{1}{m}}}{d^{(m)}}\right]$$

$$\boxed{\ddot{a}^{(m)}_x = \frac{1-A^{(m)}_x}{d}}$$ VIF

Using CPT

$$\ddot{a}^{(m)}_x = \sum_{j=0}^{\infty} \frac{1}{m} v^{j/m} \, {}_{j/m}p_x = \frac{1}{m} + \frac{1}{m} v^{1/m} \, {}_{\frac{1}{m}}p_x + \frac{1}{m} v^{2/m} \, {}_{\frac{2}{m}}p_x + \dots$$

impossible task we approximate

$m=1$

$$\ddot{a}_x = E[\ddot{a}_{\overline{K+1}|}]$$

$$\frac{1-v^{K+1}}{d}$$

✓ $A_x = 1 - d\ddot{a}_x$

$$\underbrace{Var[Y] = \frac{1}{d^2}[{}^2A_x - (A_x)^2]}$$

m other than 1

$$\ddot{a}_x^{(m)} = E[\ddot{a}^{(m)}_{\overline{K^{(m)}+\frac{1}{m}}|}]$$

✓ $A_x^{(m)} = 1 - d^{(m)}\ddot{a}_x^{(m)}$

✓ $$\underbrace{Var[Y] = \frac{1}{d^{(m)2}}\left[{}^2A_x^{(m)} - (A_x^{(m)})^2\right]}$$

$i = 5\%$

how to approximate $\ddot{a}_x^{(m)}$?

UDD

—

—

UDD $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$

$$A_x^{(m)} = 1 - d^{(m)} \ddot{a}_x^{(m)}$$

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1 - \frac{i}{i^{(m)}} \overset{(1 - d\ddot{a}_x)}{A_x}}{d^{(m)}}$$

UDD

$$\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}}}{d^{(m)}} + \frac{\frac{i}{i^{(m)}} d}{d^{(m)}} \ddot{a}_x$$

$$= \frac{i^{(m)} - i}{i^{(m)} d^{(m)}} + \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x = \overset{\alpha(m)}{\frac{id}{i^{(m)} d^{(m)}}} \ddot{a}_x - \overbrace{\frac{i - i^{(m)}}{i^{(m)} d^{(m)}}}^{\beta(m)}$$

$$= \alpha(m)\, \ddot{a}_x - \beta(m)$$

$$\boxed{\ddot{a}_x^{(m)} = \alpha(m)\, \ddot{a}_x - \beta(m)}$$

VIF UDD

$$\alpha(m) = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}$$

$$\beta(m) = \frac{i\,d}{i^{(m)} d^{(m)}}$$

$i = 5\%$

table

m	$\alpha(m)$	$\beta(m)$
1	0	1
2	1.00015	0.25617
4	1.00019	0.38272
12	1.00020	0.46651
∞	1.00020	.58823

Some useful relationships

Here we list some important relationships regarding the life annuity-due with m-thly payments (Note - these are exact formulas):

- $1 = d\ddot{a}_x + A_x = d^{(m)}\ddot{a}_x^{(m)} + A_x^{(m)}$ most important

derive

- $\ddot{a}_x^{(m)} = \dfrac{d}{d^{(m)}}\ddot{a}_x - \dfrac{1}{d^{(m)}}\left(A_x^{(m)} - A_x\right) = \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x - \ddot{a}_{\overline{\infty}|}^{(m)}\left(A_x^{(m)} - A_x\right)$

- $\ddot{a}_x^{(m)} = \dfrac{1 - A_x^{(m)}}{d^{(m)}} = \ddot{a}_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)} A_x^{(m)}$ derive

Other types of life annuity-due payable m-thly

m=1

n-year term	PV random variable	Y	$= \ddot{a}^{(m)}_{\overline{\min(K^{(m)}+(1/m),n)}\rceil}$
	APV symbol	$\mathrm{E}[Y]$	$= \ddot{a}^{(m)}_{x:\overline{n}\rceil}$
	current payment technique		$= \dfrac{1}{m} \displaystyle\sum_{h=0}^{mn-1} v^{h/m} \cdot {}_{h/m}p_x$
	other relationships		$= \ddot{a}^{(m)}_x - {}_nE_x\, \ddot{a}^{(m)}_{x+n}$
	relation to life insurance		$= \dfrac{1}{d^{(m)}} \left[1 - A^{(m)}_{x:\overline{n}\rceil}\right]$
n-year deferred	PV random variable	Y	$= v^n \ddot{a}^{(m)}_{\overline{K^{(m)}+(1/m)-n}\rceil} I(K \geq n)$
	APV symbol	$\mathrm{E}[Y]$	$= {}_{n\vert}\ddot{a}^{(m)}_x$
	current payment technique		$= \dfrac{1}{m} \displaystyle\sum_{h=mn}^{\infty} v^{h/m} \cdot {}_{h/m}p_x$
	other relationships		$= {}_nE_x\, \ddot{a}^{(m)}_{x+n} = \ddot{a}^{(m)}_x - \ddot{a}^{(m)}_{x:\overline{n}\rceil}$
	relation to life insurance		$= \dfrac{1}{d^{(m)}} \left[{}_nE_x - {}_{n\vert}A^{(m)}_x\right]$

n-year term

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - \underbrace{{}_{n|}\ddot{a}_x}_{{}_nE_x\, \ddot{a}_{x+n}}$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x - \underbrace{{}_{n|}\ddot{a}^{(m)}_x}$$

$$= \ddot{a}^{(m)}_x - \underbrace{{}_nE_x}\, \ddot{a}^{(m)}_{x+n}$$

n-deferred

$${}_{n|}\ddot{a}^{(m)}_x = {}_nE_x\, \ddot{a}^{(m)}_{x+n}$$

extn to immediate

Illustrative example 3

varying payments

m=12

Professor Balducci is currently age 60 and will retire immediately. He purchased a whole life annuity-due contract which will pay him on a monthly basis the following benefits:

- \$12,000 each year for the next 10 years;
- \$24,000 each year for the following 5 years after that; and finally,
- \$48,000 each year thereafter.

(60)

You are given:

- $i = 3\%$ and the following table:

x	$1000A_x^{(12)}$	${}_5p_x$
60	661.11	0.8504
65	712.33	0.7926
70	760.65	0.7164
75	804.93	0.6196

Calculate the APV of Professor Balducci's life annuity benefits.

whole life, $m=12$

$$APV = 12000\, \ddot{a}_{60}^{(12)}$$

$$+\ 12000\ {}_{10}E_{60}\, \ddot{a}_{70}^{(12)}$$

$$+\ 24000\ {}_{15}E_{60}\, \ddot{a}_{75}^{(12)}$$

$$\approx \ 235{,}693.10$$

$${}_{10}E_{60} = v^{10}\ \underbrace{{}_5p_{60}}_{.8504}\ \underbrace{{}_5p_{65}}_{.7926}$$

$$v^{10} = \left(\frac{1}{1.03}\right)^{10}$$

$${}_{15}E_{60} = \left(\frac{1}{1.03}\right)^{15} {}_5p_{60}\ {}_5p_{65}\ \underbrace{{}_5p_{70}}_{.7164}$$

$$\ddot{a}_{60}^{(12)} = \frac{1-A_{60}^{(12)}}{d^{(12)}} \qquad A_{60}^{(12)} = .66111$$

$$\ddot{a}_{70}^{(12)} = \frac{1-A_{70}^{(12)}}{d^{(12)}} \qquad A_{70}^{(12)} = .76065$$

$$\ddot{a}_{75}^{(12)} = \frac{1-A_{75}^{(12)}}{d^{(12)}} \qquad A_{75}^{(12)} = .80493$$

$A_x^{(12)}$ ${}_5p_x$

$i = 3\%$

$d^{(12)} = \dots$

$d^{(12)} = .02952243$

$m \to \infty$ continuous

(Continuous) whole life annuity

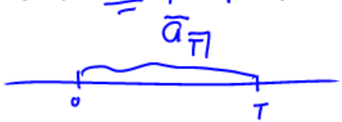
- A life annuity payable continuously at the rate of one unit per year.
- One can think of it as life annuity payable m-thly per year, with $m \to \infty$.
- The PV random variable is $Y = \bar{a}_{\overline{T|}}$ where T is the future lifetime of (x).
- The APV of the annuity:

$$\begin{aligned} \bar{a}_x &= \mathsf{E}[Y] = \mathsf{E}\left[\bar{a}_{\overline{T|}}\right] = \int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} dt \\ & \quad \text{use integration by parts - see page 117 for proof} \\ &= \int_0^\infty v^t {}_tp_x dt = \int_0^\infty {}_tE_x dt \end{aligned}$$

continuous whole life annuities -

$m \to \infty$

at the rate of 1 per year



$PV \text{ r.v.} = 1 \cdot \bar{a}_{\overline{T|}}$

$E[\bar{a}_{\overline{T|}}] = APV = \bar{a}_x$

$Var[\bar{a}_{\overline{T|}}] = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right]$

$E\left[\frac{1-v^T}{\delta}\right] = \frac{1-\bar{A}_x}{\delta} = \bar{a}_x \Rightarrow \boxed{\bar{A}_x = 1 - \delta\,\bar{a}_x}$

CPT $\quad \ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x \qquad \ddot{a}_x^{(m)} = \underbrace{\sum_{j=0}^{\infty} \frac{1}{m} v^{j/m}\,{}_{j/m}p_x}$

$m \to \infty \quad \lim_{m\to\infty} \ddot{a}_x^{(m)} = \int_0^{\infty} \underbrace{v^t\,{}_tp_x}\, dt$ -

$Z = v^T$

$\underbrace{Pr[Z \leq a]}$

equivalent to probability of dying later -

$Y = \bar{a}_{\overline{T|}}$ $\quad \underbrace{\Pr[Y \le a]}$ equivalent to a probability of dying early

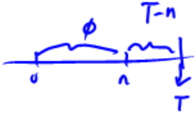
$$(Y \le a) \iff \frac{1-v^T}{\delta} \le a \iff 1-v^T \le \delta a$$

$$\iff v^T \ge 1-\delta a$$

$$\iff T \underbrace{\log v}_{-\delta} \ge \log(1-\delta a)$$

$$\iff T \le \frac{\log(1-\delta a)}{-\delta} = \underbrace{\log(1-\delta a)^{-1/\delta}}_{\text{how early}}$$

$\bar{a}_{x:\overline{n|}}$ n-year term $\quad Y = \begin{cases} \bar{a}_{\overline{T|}}, & T \le n \\ \bar{a}_{\overline{n|}}, & T > n \end{cases}$



${}_{n|}\bar{a}_x$ n-year deferred $\quad Y = \begin{cases} 0, & T \le n \\ {}_{n|}\bar{a}_{\overline{T-n|}}, & T > n \end{cases}$

constant force μ

de Moivre's

constant force $\mu \Rightarrow$ Exponential $\quad {}_tp_x = e^{-\mu t}$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \underbrace{{}_kp_x} = \sum_{k=0}^{\infty} \underbrace{e^{-\delta k} e^{-\mu k}}_{e^{-(\mu+\delta)k}} = \frac{1}{1-e^{-(\mu+\delta)}}$$

$$\ddot{a}_x^{(m)} = \ldots$$

$$\bar{a}_x = \int_0^{\infty} v^t \, {}_tp_x = \int_0^{\infty} e^{-\delta t} e^{-\mu t} \, dt$$

$$= \underbrace{\int_0^{\infty} e^{-(\mu+\delta)t} \, dt}$$

$$= \left. \frac{e^{-(\mu+\delta)t}}{\mu+\delta} \right|_0^{\infty} = \frac{1}{\mu+\delta}$$

$$\bar{A}_x = \frac{\mu}{\mu+\delta}$$

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1-\frac{\mu}{\mu+\delta}}{\delta} = \frac{\cancel{\mu}+\delta-\cancel{\mu}}{\delta(\mu+\delta)} = \frac{1}{\mu+\delta}$$

de Moivre's

$X \sim (0, \omega)$ uniform

$T_x \sim (0, \omega - x)$ uniform $\Rightarrow$ $\underbrace{f_T(t) = \frac{1}{\omega - x}}$, $\underbrace{0 \le t \le \omega - x}$

$F_T(t) = \int_0^t \frac{1}{\omega - x} dz = \frac{t}{\omega - x}$

$$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$$

${}_tp_x = \Pr[T > t] = \left(1 - \frac{t}{\omega - x}\right)$

$$\bar{A}_x = E[v^T] = \int_0^\infty v^t f_T(t) \, dt = \int_0^{\omega - x} v^t \frac{1}{\omega - x} dt = \frac{1}{\omega - x} \underbrace{\int_0^{\omega - x} v^t \, dt}_{\bar{a}_{\overline{\omega - x}|}}$$

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta} = \frac{1 - \frac{1}{\omega - x} \bar{a}_{\overline{\omega - x}|}}{\delta}$$

- continued

$\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$

- One can also write expressions for the cdf and pdf of Y in terms of the cdf and pdf of T. For example,

$$\Pr[Y \leq y] = \Pr\left[1 - v^T \leq \delta y\right] = \Pr\left[T \leq \frac{\log(1-\delta y)}{\log v}\right]$$

- Recursive relation: $\bar{a}_x = \bar{a}_{x:\overline{1}|} + v p_x \bar{a}_{x+1}$ study → intuition

- Variance expression: $\text{Var}\left[\bar{a}_{\overline{T}|}\right] = \text{Var}\left[\dfrac{1-v^T}{\delta}\right] = \dfrac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$

- Relationship to whole life insurance: $\bar{A}_x = 1 - \delta \bar{a}_x$

- Try writing explicit expressions for the APV and variance where we have constant force of mortality and constant force of interest.

Temporary life annuity

READ

- A (continuous) n-year temporary life annuity pays 1 per year continuously while (x) survives during the next n years.
- The PV random variable is $Y = \begin{cases} \bar{a}_{\overline{T}|}, & 0 \leq T < n \\ \bar{a}_{\overline{n}|}, & T \geq n \end{cases} = \bar{a}_{\overline{\min(T,n)}|}$
- The APV of the annuity:

$$\begin{aligned} \bar{a}_{x:\overline{n}|} &= \mathsf{E}[Y] = \int_0^n \bar{a}_{\overline{t}|} \cdot {}_tp_x \mu_{x+t} dt \\ &\quad + \int_n^\infty \bar{a}_{\overline{n}|} \cdot {}_tp_x \mu_{x+t} dt = \int_0^n v^t \, {}_tp_x dt. \end{aligned}$$

- Recursive formula: $\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{1}|} + vp_x \bar{a}_{x+1:\overline{n-1}|}$.
- To derive variance, one way to get explicit form is to note that $Y = (1 - Z)/\delta$ where Z is the PV r.v. for an n-year endowment ins. [details in class.]

Deferred whole life annuity

READ

- Pays a benefit of a unit $\$1$ each year continuously while the annuitant (x) survives from $x+n$ onward.
- The PV random variable is

$$Y = \begin{cases} 0, & 0 \leq T < n \\ v^n \, \bar{a}_{\overline{T-n}|}, & T \geq n \end{cases} = \begin{cases} 0, & 0 \leq T < n \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|}, & T \geq n \end{cases}.$$

- The APV [expected value of Y] of the annuity is

$${}_{n|}\bar{a}_x = {}_nE_x \, \bar{a}_{x+n} = \bar{a}_x - \bar{a}_{x:\overline{n}|} = \int_n^\infty v^t \, {}_tp_x dt.$$

- The variance of Y is given by

$$\mathsf{Var}[Y] = \frac{2}{\delta} v^{2n} \, {}_np_x \left(\bar{a}_{x+n} - {}^2\bar{a}_{x+n} \right) - \left({}_{n|}\bar{a}_x \right)^2$$

Special mortality laws

- Just as in the case of life insurance valuation, we can derive nice explicit forms for "life annuity" formulas in the case where mortality follows:
  - constant force (or Exponential distribution); or
  - De Moivre's law (or Uniform distribution).
- Try deriving some of these formulas. You can approach them in a couple of ways:
  - Know the results for the "life insurance" case, and then use the relationships between annuities and insurances.
  - You can always derive it from first principles, usually working with the current payment technique.
- In the continuous case, one can use numerical approximations to evaluate the integral:
  - trapezium (trapezoidal) rule
  - repeated Simpson's rule

Illustrative example 4

very nice

${}_tp_x = e^{-\mu t}$

For a whole life annuity-due on (40), you are given:

- Before age 65, mortality follows a constant force $\mu = 0.005$.
- $\delta = 0.03$
- $A_{65} = 0.425$

Calculate $\ddot{a}_{40}$.

$A_{40} = A^{1}_{40:\overline{25}|} + {}_{25}E_{40} A_{65} \Rightarrow 25.06466$

$.0820607$

${}_{25}E_{40} = v^{25} e^{-.005(25)}$, where $v^{25} = e^{-.03(25)}$

$A_{65} = .425$

A_{40}

try this!

$\ddot{a}_{40} = \ddot{a}_{40:\overline{25}|} + {}_{25}E_{40}\, \ddot{a}_{65}$

$\ddot{a}_{40:\overline{25}|} = \sum_{k=0}^{24} v^k\, {}_kp_{40}$

$\ddot{a}_{65} = \frac{1-A_{65}}{d}$

$$A^{1}_{40:\overline{25}|} = \sum_{k=0}^{24} v^{k+1} \underbrace{{}_{k|}q_{40}}_{{}_kp_{40}\; q_{40+k}}$$

$\underline{\underline{T_{40}}} \sim$ constant fm $\mu = .005$

$$e^{-.03(k+1)} \quad e^{-.005(k)} \quad \underline{(1-e^{-.005})}$$

$$= e^{-.03}(1-e^{-.005}) \sum_{k=0}^{24} e^{-.035k}$$

$$\sum v^{k+1} \, {}_kp_{40} \, q_{40+k}$$

$$\sum_{k=0}^{24} e^{-.03(k+1)} e^{-.005(k)} (1-e^{-.005}) = e^{-.03}(1-e^{-.005}) \underbrace{\sum_{k=0}^{24} e^{-.035k}}_{\frac{1-e^{-.035(25)}}{1-e^{-.035}}}$$

$$= .08206107$$

1 sheet -

+

1 sheet -

+

Calculator -

12 questions -

Life annuities with varying benefits

- Some of these are discussed in details in Section 5.10.
- You may try to remember the special symbols used, especially if the variation is a fixed unit of \$1 (either increasing or decreasing).
- The most important thing to remember is to apply similar concept of "discounting with life" taught in the life insurance case (note: this works only for valuing actuarial present values):
  - work with drawing the benefit payments as a function of time; and
  - use then your intuition to derive the desired results.

Methods for evaluating annuity functions

$\ddot{a}_x^{(m)}$ ✓ → UDD

- Section 5.11
- Recursions:
  - For example, in the case of a whole life annuity-due on (x), recall $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$. Given a set of mortality assumptions, start with

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x$$

    and then use the recursion to evaluate values for subsequent ages.
- UDD: deaths are uniformly distribution between integral ages.
- Woolhouse's approximations

UDD $\ddot{a}_x^{(m)} = \alpha(m)\,\ddot{a}_x - \beta(m)$

$\beta(m) \to \dfrac{i - i^{(m)}}{i^{(m)}\, d^{(m)}}$

$\ddot{a}_x \to$ yearly values

$\alpha(m) \to \dfrac{i}{i^{(m)}} \dfrac{d}{d^{(m)}}$

α, β are function of i

temporary $\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_nE_x\, \ddot{a}_{x+n}^{(m)}$

$\ddot{a}_x^{(m)} \to \alpha(m)\,\ddot{a}_x - \beta(m)$

$\ddot{a}_{x+n}^{(m)} \to \alpha(m)\,\ddot{a}_{x+n} - \beta(m)$

$$= \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)\left[1 - {}_nE_x\right]$$

n-deferred ${}_{n|}\ddot{a}_x^{(m)} = {}_nE_x\, \ddot{a}_{x+n}^{(m)}$

$$= {}_nE_x\left(\alpha(m)\,\ddot{a}_{x+n} - \beta(m)\right)$$

$$= \alpha(m)\, {}_{n|}\ddot{a}_x - \beta(m)\, {}_nE_x$$

Uniform Distribution of Deaths (UDD)

Under the UDD assumption, we have derived in the previous chapter that the following holds:

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$$

Then use the relationship between annuities and insurance:

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}$$

This leads us to the following result when UDD holds:

$$\ddot{a}_x^{(m)} = \alpha(m)\, \ddot{a}_x - \beta(m),$$

VIP

where

$$\begin{aligned} \alpha(m) &= s_{\overline{1}|}^{(m)} \ddot{a}_{\overline{1}|}^{(m)} = \frac{i}{i^{(m)}} \cdot \frac{d}{d^{(m)}} \\ \beta(m) &= \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} \end{aligned}$$

Woolhouse's approximate fomulas

Woolhouse formulas

The Woolhouse's approximate formulas for evaluating annuities are based on the Euler-Maclaurin formula for numerical integration:

details

$$\int_0^\infty g(t)dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g''(0) + \cdots$$

for some positive constant h. This formula is then applied to $g(t) = v^t\,{}_tp_x$ which leads us to

$$g'(t) = -v^t\,{}_tp_x\,(\delta - \mu_{x+t}).$$

$$\underbrace{\mu_x = \frac{1}{2}\left[\log p_x + \log p_{x-1}\right]}_{W3^*}$$

We can obtain the following Woolhouse's approximate formula:

$$\boxed{\underbrace{\underbrace{\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}}_{W2} - \frac{m^2-1}{12m^2}(\delta + \mu_x)}_{W3}}$$

final

$\ddot{a}_x^{(m)}$

UDD $\alpha(m)\,\ddot{a}_x - \beta(m)$

W2 $\ddot{a}_x - \frac{m-1}{2m}$

W3 $\ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$

W3* $\mu_x \approx \frac{1}{2}\left[\underbrace{\log p_{x-1} + \log p_x}_{\underbrace{\log(p_{x-1} \cdot p_x)^{1/2}}}\right]$

derive temporary annuity ✓

n-deferred ✓

varying benefits → all in whole WL ✓

Approximating an n-year temporary life annuity-due with m-thly payments

Apply the Woolhouse's approximate formula to

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_nE_x \ddot{a}_{x+n}^{(m)}$$

This leads us to the following Woolhouse's approximate formulas:

Use 2 terms (W2)	$\ddot{a}_{x:\overline{n}\|}^{(m)} \approx \ddot{a}_{x:\overline{n}\|} - \frac{m-1}{2m}\left(1 - {}_nE_x\right)$
Use 3 terms (W3)	$\ddot{a}_{x:\overline{n}\|}^{(m)} \approx \ddot{a}_{x:\overline{n}\|} - \frac{m-1}{2m}\left(1 - {}_nE_x\right)$
	$-\frac{m^2-1}{12m^2}\left[\delta + \mu_x - {}_nE_x\left(\delta + \mu_{x+n}\right)\right]$
Use 3 terms (W3*)	use approximation for force of mortality
(modified)	$\mu_x \approx -\frac{1}{2}\left[\log(p_{x-1}) + \log(p_x)\right]$

Numerical illustrations

$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x} - {}_nE_x \, \ddot{a}^{(m)}_{x+n}$

We compare the various approximations: UDD, W2, W3, W3* based on the Standard Ultimate Survival Model with Makeham's law

$$\mu_x = A + Bc^x,$$

where $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$.

The results for comparing the values for:

- $\ddot{a}^{(12)}_{x:\overline{10}|}$ with $i = 10\%$
- $\ddot{a}^{(2)}_{x:\overline{25}|}$ with $i = 5\%$

are summarized in the following slides.

Values of $\ddot{a}^{(12)}_{x:\overline{10}|}$ with $i = 10\%$

$\sum_{j=0}^{119} v^{j/12} \, {}_{j/12}p_x$

$\ddot{a}^{(12)}_{x:\overline{10}|}$ $\alpha(m)$, $\beta(m)$ worse $\frac{m-1}{2m}$ $\frac{m^2-1}{12m^2}$ μ

x	$\ddot{a}_x$	$\ddot{a}_x^{(12)}$	${}_{10}E_x$	Exact	UDD	W2	W3	W3*
20	10.9315	10.4653	0.384492	6.4655	6.4655	6.4704	6.4655	6.4655
30	10.8690	10.4027	0.384039	6.4630	6.4630	6.4679	6.4630	6.4630
40	10.7249	10.2586	0.382586	6.4550	6.4550	6.4599	6.4550	6.4550
50	10.4081	9.9418	0.377947	6.4295	6.4294	6.4344	6.4295	6.4295
60	9.7594	9.2929	0.363394	6.3485	6.3482	6.3535	6.3485	6.3485
70	8.5697	8.1027	0.320250	6.0991	6.0982	6.1044	6.0990	6.0990
80	6.7253	6.2565	0.213219	5.4003	5.3989	5.4073	5.4003	5.4003
90	4.4901	4.0155	0.057574	3.8975	3.8997	3.9117	3.8975	3.8975
100	2.5433	2.0505	0.000851	2.0497	2.0699	2.0842	2.0497	2.0496

Values of $\ddot{a}^{(2)}_{x:\overline{25}|}$ with $i = 5\%$

x	$\ddot{a}_x$	$\ddot{a}_x^{(2)}$	${}_{25}E_x$	Exact	UDD	W2	W3	W3*
20	19.9664	19.7133	0.292450	14.5770	14.5770	14.5792	14.5770	14.5770
30	19.3834	19.1303	0.289733	14.5506	14.5505	14.5527	14.5506	14.5506
40	18.4578	18.2047	0.281157	14.4663	14.4662	14.4684	14.4663	14.4663
50	17.0245	16.7714	0.255242	14.2028	14.2024	14.2048	14.2028	14.2028
60	14.9041	14.6508	0.186974	13.4275	13.4265	13.4295	13.4275	13.4275
70	12.0083	11.7546	0.068663	11.5117	11.5104	11.5144	11.5117	11.5117
80	8.5484	8.2934	0.002732	8.2889	8.2889	8.2938	8.2889	8.2889
90	5.1835	4.9242	0.000000	4.9242	4.9281	4.9335	4.9242	4.9242
100	2.7156	2.4425	0.000000	2.4425	2.4599	2.4656	2.4424	2.4424

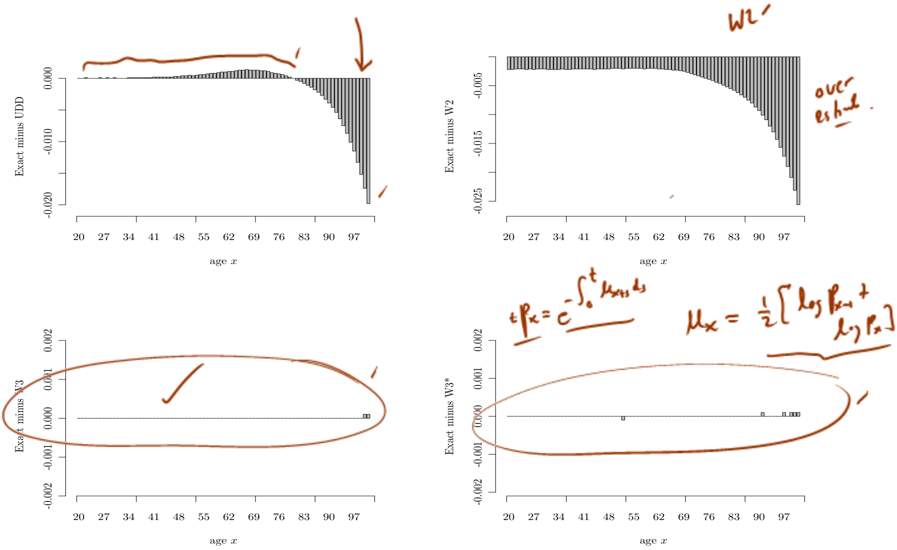


Figure: Visualizing the different approximations for $\ddot{a}^{(2)}_{x:\overline{25|}}$

Illustrative example 5

$\ddot{a}^{(12)}_{50:\overline{3}|} = \ddot{a}^{(12)}_{50} - {}_3E_{50}\, \ddot{a}^{(12)}_{53}$

You are given:

- $i = 5\%$ and the following table:

$v = \frac{1}{1.05}$

x	ℓ_x	μ_x
49	811	0.0213
50	793	0.0235
51	773	0.0258
52	753	0.0284
53	731	0.0312
54	707	0.0344

$\ddot{a}_{50:\overline{3}|} = 1 + v\,p_{50} + v^2\,p_{50}\,p_{51}$

$= \underbrace{1 + \frac{1}{1.05}\frac{773}{793} + \frac{1}{1.05^2}\frac{753}{793}}$

$= 2.789637$

$\alpha(12) = 1.000197$

$\beta(12) = .466508$

Approximate $\ddot{a}^{(12)}_{50:\overline{3}|}$ based on the following methods:

1. UDD assumptions
2. Woolhouse's formula using the first two terms only
3. Woolhouse's formula using all three terms
4. Woolhouse's formula using all three terms but approximating the force of mortality

UCONN

UDD: $\ddot{a}^{(12)}_{50:\overline{3}|} = \alpha(12)\,\ddot{a}_{50:\overline{3}|} - \beta(12)\left[1 - {}_3E_{50}\right]$

$\alpha(12) = 1.000197$, $\ddot{a}_{50:\overline{3}|} = 2.789637$, $\beta(12) = .466508$

${}_3E_{50} = v^3\,{}_3p_{50} = \frac{1}{1.05^3}\,\frac{731}{793}$

$= 2.69516$

W2: $\ddot{a}^{(12)}_{50:\overline{3}|} = \ddot{a}_{50:\overline{3}|} - \frac{11}{24}\left(1 - {}_3E_{50}\right) = 2.694516$

$\frac{m-1}{2m}$ $\frac{11}{24}$

$i = 5\%$

$\delta = \log 1.05$

W3: $\ddot{a}^{(12)}_{50:\overline{3}|} = W2 - \frac{12^2-1}{12(12)^2}\left((\delta + \mu_{50}) - {}_3E_{50}(\delta + \mu_{53})\right)$

$\mu_{50} = .0235$, $\mu_{53} = .0312$

$= 2.695565$

$\mu_{50} = \frac{1}{2}\left[\log p_{49} + \log p_{50}\right] = .0239945$

$\mu_{53} = \frac{1}{2}\left[\log p_{52} + \log p_{53}\right] = .03151728$

Finish Chap. 5

W3*: $\ddot{a}^{(12)}_{50:\overline{12}|} = 2.695143$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x - {}_nE_x \left[\ddot{a}^{(m)}_{x+n}\right] \rightarrow \left[\ddot{a}_{x+n} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_{x+n})\right]$$

$$= \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

$$= \underbrace{\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)}_{W2} - \frac{m^2-1}{12m^2}\left[(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\right]$$

Practice problem 1

$d = i \cdot v = 1 - v$

$v = e^{-\delta}$

$i = e^{\delta} - 1$

You are given: linear

- $\ell_x = 115 - x$, for $0 \le x \le 115$
- $\delta = 4\%$

Calculate $\ddot{a}_{65:\overline{20}|}$.

$A_{65:\overline{20}|} = 1 - d\,\ddot{a}_{65:\overline{20}|} \Rightarrow \ddot{a}_{65:\overline{20}|} = \dfrac{1 - A_{65:\overline{20}|}}{d}$

$= A^{1}_{65:\overline{20}|} + \underbrace{A_{65:\overline{20}|}^{\ \ 1}}_{\underbrace{v^{20}\,{}_{20}p_{65}}} \to \left(\frac{1}{1+i}\right)^{20} \frac{\ell_{85}}{\ell_{65}} \quad {}^{30}_{50}$

$A^{1}_{65:\overline{20}|} = \sum_{k=0}^{19} v^{k+1}\, \underbrace{{}_{k|}q_{65}}_{\frac{d_{65+k}}{\ell_{65}} \, 50} \quad 1$

$= \frac{1}{50} \underbrace{\sum_{k=0}^{19} e^{-.04(k+1)}}_{\frac{e^{-.04}(1-e^{-.04(20)})}{1-e^{-.04}}}$

term + pure

$.5394629$

$\sum_{k=0}^{19} v^k\, {}_kp_{65} = \sum_{k=0}^{19} v^k \frac{\ell_{65+k}}{\ell_{65}} \quad {}^{115-(65+k)}_{115-65} = \underbrace{\sum_{k=0}^{19} v^k \left(\frac{50-k}{50}\right)}$ $\quad 1 - \frac{k}{50}$

$\dfrac{1 - .5394629}{1 - v} = \dfrac{1 - .5394629}{1 - e^{-.04}} = 11.74523$

Practice problem 2

$\bar{A}_x = \frac{\mu}{\mu+\delta}$

X = does not matter because of constant

$E[Y] = \bar{a}_x = \frac{1}{\mu+\delta} = \frac{1}{.08}$

$Var[Y] = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right] = 36.05769$

$\delta^2 \to (.05)^2$, ${}^2\bar{A}_x \to \frac{\mu}{\mu+2\delta} \to \frac{3}{13}$, $(\bar{A}_x)^2 \to \left(\frac{\mu}{\mu+\delta}\right)^2 \to \left(\frac{3}{8}\right)^2$

You are given:

- $\mu_{x+t} = 0.03$ for $t \geq 0$
- $\delta = 5\%$
- Y is the present value random variable for a continuous whole life annuity of \$1 issued to (x).

Calculate $\Pr\left[Y \geq \mathrm{E}[Y] - \sqrt{\mathrm{Var}[Y]}\right]$. $=$ $\Pr[Y \geq 6.495144]$

$Y = \frac{1-v^T}{\delta}$

$\underbrace{\frac{1}{.08} - \sqrt{36.05769}}_{6.495144}$

$$\Pr[Y \geq 6.495144] = \Pr\left[\underbrace{\frac{1-v^T}{\delta} \geq 6.495144}\right]$$

$\log v^T = T \cdot \log v$

$\qquad\qquad\quad -\delta$

$$= \Pr\left[T > \underbrace{\frac{\log(1-6.495144 \cdot \delta)}{-\delta}}_{7.853733 \text{ years}}\right]$$

Constant μ

$\downarrow$

Exponential

$${}_tp_x = \underbrace{\Pr[T > t]}_{e^{-\mu t}}$$

$$= \underbrace{e^{-.03(7.853733)}}_{0.7900872}$$

Practice problem 3 - modified SOA MLC Spring 2012

For a whole life annuity-due of \$1,000 per year on (65), you are given:

- Mortality follows Gompertz law with

$$\mu_x = Bc^x, \quad \text{for } x \geq 0,$$

  where $B = 5 \times 10^{-5}$ and $c = 1.1$.
- $i = 4\%$
- Y is the present value random variable for this annuity.

$Y = 1000\, \ddot{a}_{\overline{K+1|}}$

Calculate the probability that Y is less than \$11,500.

$\Pr[Y < 11{,}500] \longrightarrow \Pr[\underbrace{K \leq \text{something}}]$

$$Y < 11500 \iff \frac{1000\,\ddot{a}_{\overline{K+1}|}}{1000} < \frac{11500}{1000}$$

$$\frac{1-v^{K+1}}{d} < 11.5 \Rightarrow v^{K+1} > 1-11.5d$$

$$(K+1)\cancel{\log v}\ \cancel{>}\ < \frac{\log(1-11.5d)}{-\delta}$$

$i = 4\% = .04$

$d = iv = \frac{.04}{1.04}$

$\delta = \log 1.04$

$$\iff K < \underbrace{\frac{\log(1-11.5d)}{-\delta} - 1}_{13.88876}$$

$$\iff K \le 13$$

$$\Pr[Y<11500]$$

$$= \Pr[K \le 13] = {}_{14}q_{65} = 1 - {}_{14}p_{65}$$

$$\mu_x = Bc^x \qquad {}_tp_x = e^{-\int_0^t \mu_{x+s}\,ds}$$

$$= e^{-\int_0^t Bc^x c^s\,ds}$$

$$= e^{-\frac{Bc^x}{\log c}(c^t-1)}$$

$$= e^{-\frac{Bc^x}{\log c}(c^t-1)}$$

$$ {}_tp_x = e^{-\frac{Bc^x}{\log c}(c^t-1)}$$

$$\int c^s\,ds = \frac{c^s}{\log c}$$

$$c^s \to e^{s\cdot\log c}$$

$x=65 \qquad B = 5\times 10^{-5}$

$c = 1.1$

$$ {}_{14}p_{65} = \underbrace{e^{-\frac{Bc^{65}}{\log c}(c^{14}-1)}}_{e^{-.7196562}}$$

$$\underbrace{\Pr\{Y<11500\}}_{} = \underbrace{1-e^{-.7196562}}_{} = \underline{\underline{.5130804}}$$

Practice problem 4 - SOA MLC Spring 2014

For a group of 100 lives age x with independent future lifetimes, you are given:

whole life annuity

- Each life is to be paid \$1 at the beginning of each year, if alive.
- $A_x = 0.45$
- ${}^2A_x = 0.22$
- $i = 0.05$

$$\underbrace{Y = Y_1 + Y_2 + \cdots + Y_{100}}_{\text{are independent, identically}}$$

$\sim$ approximately Normal

$E(Y)$ $Var(Y)$

Y is the present value random variable of the aggregate payments.

Using the Normal approximation to Y, calculate the initial size of the fund needed in order to be 95% certain of being able to make the payments for these life annuities. Find F today so that $P_r(F > Y) = 0.95 \Rightarrow P_r(Y < F) = 0.95$

1st policyholder $E(Y_1) = 1 \cdot \ddot{a}_x = \dfrac{1-A_x}{d} = \dfrac{1-.45}{.05/1.05} = 11.55$

$^2A_x = .22$

$A_x = .45$

$$Var(Y_1) = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right] = \underbrace{\frac{1}{(.05/1.05)^2}\left[.22 - (.45)^2\right]}_{7.7175}$$

aggregate $E(Y) = 100\,E(Y_1) = 100(11.55) = 1155$

$Var(Y) = 100\,Var(Y_1) = 100(7.7175) = 771.75$

$$Pr[Y \le F] = .95 \Rightarrow Pr\left[\underbrace{\frac{Y - E(Y)}{\sqrt{Var(Y)}}}_{\approx N} \le \underbrace{\frac{F - 1155}{\sqrt{771.75}}}_{\alpha = 1.645}\right] = 0.95$$

$N \sim N(0,1)$

95th percentile of a standard normal is 1.645

$$F = 1.645\sqrt{771.75} + 1155 = 1200.699$$

$= Z_\alpha \cdot SD(Y) + E(Y)$

12 each

Other terminologies and notations used

Expression	Other terms/symbols used
temporary life annuity-due	term annuity-due n-year term life annuity-due
annuity-immediate	immediate annuity annuity immediate