Insurance Benefits

Lecture: Weeks 6-7

An introduction

- Central theme: to quantify the value today of a (random) amount to be paid at a random time in the future.
  - main application is in life insurance contracts, but could be applied in other contexts, e.g. warranty contracts.
- Generally computed in two steps:
  1. take the present value (PV) random variable, $b_T v_T$; and
  2. calculate the expected value $\mathsf{E}[b_T v_T]$ for the average value - this value is referred to as the Actuarial Present Value (APV).
- In general, we want to understand the entire distribution of the PV random variable $b_T v_T$:
  - it could be highly skewed, in which case, there is danger to use expectation.
  - other ways of summarizing the distribution such as variances and percentiles/quantiles may be useful.

A simple illustration

Consider the simple illustration of valuing a three-year term insurance policy issued to age 35 where if he dies within the first year, a $1,000 benefit is payable at the end of his year of death.

If he dies within the second year, a $2,000 benefit is payable at the end of his year of death. If he dies within the third year, a $5,000 benefit is payable at the end of his year of death.

Assume a constant interest rate (annual effective) of 5% and the following extract from a mortality table:

x	q_x
35	0.005
36	0.006
37	0.007
38	0.008

Calculate the APV of the benefits.

Chapter summary

- Life insurance
  - benefits payable contingent upon death; payment made to a designated beneficiary
  - actuarial present values (APV)
  - actuarial symbols and notation
- Insurances payable at the moment of death
  - continuous
  - level benefits, varying benefits (e.g. increasing, decreasing)
- Insurances payable at the end of year of death
  - discrete
  - level benefits, varying benefits (e.g. increasing, decreasing)
- Chapter 4, DHW

The present value random variable

- Denote by Z, the present value random variable.
- This gives the value, at policy issue, of the benefit payment. Issue age is usually denoted by x.
- In the case where the benefit is payable at the moment of death, Z clearly depends on the time-until-death T. For simplicity, we drop the subscript x for age-at-issue.
- It is $Z = b_T v_T$ where:
  - b_T is called the benefit payment function
  - v_T is the discount function
- In the case where we have a constant (fixed) interest rate, then $v_T = v^T = (1+i)^{-T} = e^{-\delta T}$.

Fixed term life insurance

- An n-year term life insurance provides payment if the insured dies within n years from issue.
- For a unit of benefit payment, we have

$$b_T = \begin{cases} 1, & T \leq n \\ 0, & T > n \end{cases} \text{ and } v_T = v^T.$$

- The present value random variable is therefore

$$Z = \begin{cases} v^T, & T \leq n \\ 0, & T > n \end{cases} = v^T I(T \leq n)$$

  where $I(\cdot)$ is called indicator function. $\mathsf{E}[Z]$ is called the APV of the insurance.

- Actuarial notation:

$$\bar{A}^{\,1}_{x:\overline{n|}} = \mathsf{E}[Z] = \int_0^n v^t f_x(t)dt = \int_0^n v^t \, {}_tp_x \mu_{x+t} dt.$$

Rule of moments

- The j-th moment of the distribution of Z can be expressed as:

$$\mathsf{E}\left[Z^j\right] = \int_0^n v^{tj} \, {}_tp_x \mu_{x+t} dt = \int_0^n e^{-(j\delta)t} \, {}_tp_x \mu_{x+t} dt.$$

- This is actually equal to the APV but evaluated at the force of interest $j\delta$.
- In general, we have the following rule of moment:

$$\mathsf{E}\left[Z^j\right]_{@\ \delta_t} = \mathsf{E}[Z]_{@\ j\delta_t}.$$

- For example, the variance can be expressed as

$$\mathsf{Var}[Z] = {}^2\bar{A}^1_{x:\overline{n}|} - \left(\bar{A}^1_{x:\overline{n}|}\right)^2.$$

Traditional insurances - continuous

Type	Benefit b_T	PV r.v. Z	APV E$[Z]$	Variance Var$[Z]$
Term life	$I(T \leq n)$	$v^T \cdot I(T \leq n)$	$\bar{A}^{\,1}_{x:\overline{n}\rceil}$	${}^2\bar{A}^{\,1}_{x:\overline{n}\rceil} - \left(\bar{A}^{\,1}_{x:\overline{n}\rceil}\right)^2$
Whole life	1	v^T	$\bar{A}_x$	${}^2\bar{A}_x - \left(\bar{A}_x\right)^2$
Pure endowment	$I(T > n)$	$v^n \cdot I(T > n)$	$A_{x:\overset{1}{\overline{n}\rceil}}$ or ${}_nE_x$	${}^2A_{x:\overset{1}{\overline{n}\rceil}} - \left(A_{x:\overset{1}{\overline{n}\rceil}}\right)^2$
Endowment	1	$v^{\min(T,n)}$	$\bar{A}_{x:\overline{n}\rceil}$	${}^2\bar{A}_{x:\overline{n}\rceil} - \left(\bar{A}_{x:\overline{n}\rceil}\right)^2$
Deferred	$I(T > n)$	$v^T \cdot I(T > n)$	${}_{n\vert}\bar{A}_x$	${}^2_{n\vert}\bar{A}_x - \left({}_{n\vert}\bar{A}_x\right)^2$

Pure endowment insurance

- For an n-year pure endowment insurance, we can also express the PV random variable as:

$$Z = v^n I(T_x > n),$$

where $I(E)$ is 1 if the event E is true, and 0 otherwise.

- The term $I(I(T_x > n))$ is a binary random variable with mean $\mathsf{E}[I(T_x > n)] = {}_n p_x$ and $\mathsf{Var}[I(T_x > n)] = {}_n p_x(1 - {}_n p_x)$.
- APV for pure endowment:

$$A_{x:\overset{\;1}{\overline{n|}}} = {}_n E_x = v^n \, {}_n p_x.$$

- Variance (show also using rule of moments):

$$\mathsf{Var}[Z] = v^{2n} \, {}_n p_x \cdot {}_n q_x = {}^2 A_{x:\overset{\;1}{\overline{n|}}} - \left(A_{x:\overset{\;1}{\overline{n|}}}\right)^2.$$

Endowment insurance

- An n-year endowment insurance is the sum of an n-year term and and n-year pure endowment:

$$Z = Z_1 + Z_2 = v^{\min(T,n)} = v^T \cdot I(T \leq n) + v^n I(T_x > n),$$

where $Z_1 = v^T \cdot I(T \leq n)$ is the term component and $Z_2 = v^n I(T_x > n)$ is the pure endowment component.

- Therefore, it is clear that:

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + {}_nE_x = v^n\, {}_np_x.$$

- One can also use the variance of sums of random variables to get:

$$\mathsf{Var}[Z] = \mathsf{Var}[Z_1] + \mathsf{Var}[Z_2] + 2\mathsf{Cov}[Z_1, Z_2]$$

where one can show that $\mathsf{Cov}[Z_1, Z_2] = \mathsf{E}[Z_1]\mathsf{E}[Z_2]$ since $Z_1 \cdot Z_2 = 0$.

Deferred insurance

- An n-year deferred insurance can be viewed as a discounted (with life) whole life insurance:

$$
{}_{n|}\bar{A}_x = {}_nE_x \cdot \bar{A}_{x+n}
$$

- The pure endowment insurance is used as a discounting with life contingent payments.

Constant force of mortality - all throughout life

Assume mortality is based on a constant force, say μ, and interest is also based on a constant force of interest, say δ.

- Find expressions for the APV for the following types of insurances:
  - whole life insurance;
  - *n*-year term life insurance;
  - *n*-year endowment insurance; and
  - *n*-year deferred life insurance.
- Check out the (corresponding) variances for each of these types of insurance.

[Details in class]

APVs under constant force of mortality

Assume constant force of mortality μ and constant force of interest δ.

Type	APV
Term	$\bar{A}^{\,1}_{x:\overline{n\rvert}} = \dfrac{\mu}{\mu+\delta}\left[1-e^{-(\mu+\delta)n}\right]$
Whole	$\bar{A}_x = \dfrac{\mu}{\mu+\delta}$
Pure	$A_{x:\overline{n\rvert}}^{\;\;\;\,1} = {}_nE_x = e^{-(\mu+\delta)n}$
Endowment	$\bar{A}_{x:\overline{n\rvert}} = \dfrac{\mu}{\mu+\delta}\left[1-e^{-(\mu+\delta)n}\right] + e^{-(\mu+\delta)n}$
Deferred	${}_{n\rvert}\bar{A}_x = \dfrac{\mu}{\mu+\delta}e^{-(\mu+\delta)n}$

De Moivre's law

Find expressions for the APV for the same types of insurances in the case where you have:

- De Moivre's law.

Illustrative example 1

For a whole life insurance of \$1,000 on (x) with benefits payable at the moment of death, you are given:

$$\delta_t = \begin{cases} 0.04, & 0 < t \leq 10 \\ 0.05, & t > 10 \end{cases}$$

and

$$\mu_{x+t} = \begin{cases} 0.006, & 0 < t \leq 10 \\ 0.007, & t > 10 \end{cases}$$

Calculate the actuarial present value for this insurance.

Equivalent probability calculations

We can also compute probabilities of Z as follows. Consider the present value random variable Z for a whole life issued to age x. For $0 < \alpha < 1$, the following is straightforward:

$$\begin{aligned} \text{Pr}[Z \leq \alpha] &= \text{Pr}[e^{-\delta T_x} \leq \alpha = \text{Pr}[-\delta T_x \leq \log(\alpha)] \\ &= \text{Pr}[T_x > -(1/\delta)\log(\alpha)] = {}_u p_x, \end{aligned}$$

where

$$u = (1/\delta)\log(1/\alpha) = \log(1/\alpha)^{1/\delta}.$$

- Consider the case where $\alpha = 0.75$ and $\delta = 0.05$. Then $u = \log(1/0.75)^{1/0.05} = 5.753641$.
- Thus, the probability $\text{Pr}[Z \leq 0.75]$ is equivalent to the probability that (x) will survive for another 5.753641 years.

Insurances with varying benefits

Type	b_T	Z	APV
Increasing whole life	$\lfloor T+1 \rfloor$	$\lfloor T+1 \rfloor v^T$	$(I\bar{A})_x$
Whole life increasing m-thly	$\lfloor Tm+1 \rfloor / m$	$v^T \lfloor Tm+1 \rfloor / m$	$(I^{(m)}\bar{A})_x$
Constant increasing whole life	T	Tv^T	$(\bar{I}\bar{A})_x$
Decreasing n-year term	$\begin{cases} n - \lfloor T \rfloor, & T \leq n \\ 0, & T > n \end{cases}$	$\begin{cases} (n - \lfloor T \rfloor)\, v^T, & T \leq n \\ 0, & T > n \end{cases}$	$(D\bar{A})^1_{x:\overline{n}\rceil}$

* These items will be discussed in class.

Illustrative example 2

For a whole life insurance on (50) with death benefits payable at the moment of death, you are given:

- Mortality follows De Moivre's law with $\omega = 110$.
- $b_t = 10000(1.10)^t$, for $t \geq 0$
- $\delta = 5\%$
- Z denotes the present value random variable for this insurance.

Calculate $\mathsf{E}[Z]$ and $\mathsf{Var}[Z]$.

Can you find an explicit expression for the distribution function of Z, i.e. $\Pr[Z \leq z]$?

Insurances payable at EOY of death

- For insurances payable at the end of the year (EOY) of death, the PV r.v. Z clearly depends on the curtate future lifetime K_x.
- It is $Z = b_{K+1} v_{K+1}$.
- To illustrate, consider an n-year term insurance which pays benefit at the end of year of death:

$$b_{K+1} = \begin{cases} 1, & K = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}, \; v_{K+1} = v^{K+1},$$

and therefore

$$Z = \begin{cases} v^{K+1}, & K = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}.$$

- continued

- APV of n-year term:

$$A^{1}_{x:\overline{n}|} = \mathsf{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1}\,{}_{k}p_x \cdot q_{x+k}$$

- Rule of moments also apply in discrete situations. For example,

$$\mathsf{Var}[Z] = {}^2A^{1}_{x:\overline{n}|} - \left(A^{1}_{x:\overline{n}|}\right)^2,$$

where

$${}^2A^{1}_{x:\overline{n}|} = \mathsf{E}\left[Z^2\right] = \sum_{k=0}^{n-1} e^{-2\delta(k+1)}\,{}_{k}p_x \cdot q_{x+k}.$$

Traditional insurances - discrete

Type	Benefit b_T	PV r.v. Z	APV E$[Z]$	Variance Var$[Z]$
Term life	$I(K < n)$	$v^{K+1} \cdot I(K < n)$	$A^{\,1}_{x:\overline{n}\rvert}$	${}^2A^{1}_{x:\overline{n}\rvert} - \left(A^{1}_{x:\overline{n}\rvert}\right)^2$
Whole life	1	v^{K+1}	A_x	${}^2A_x - (A_x)^2$
Endowment	1	$v^{\min(K+1,n)}$	$A_{x:\overline{n}\rvert}$	${}^2A_{x:\overline{n}\rvert} - \left(A_{x:\overline{n}\rvert}\right)^2$
Deferred	$I(K \geq n)$	$v^{K+1} \cdot I(K \geq n)$	${}_{n\rvert}A_x$	${}^{2}_{n\rvert}A_x - \left({}_{n\rvert}A_x\right)^2$

Recursive relationships

- The following will be derived/discussed in class:
  - whole life insurance: $A_x = vq_x + vp_x\, A_{x+1}$
  - term insurance: $A^{1}_{x:\overline{n|}} = vq_x + vp_x\, A^{\;1}_{x+1:\overline{n-1|}}$
  - endowment insurance: $A_{x:\overline{n|}} = vq_x + vp_x\, A_{x+1:\overline{n-1|}}$

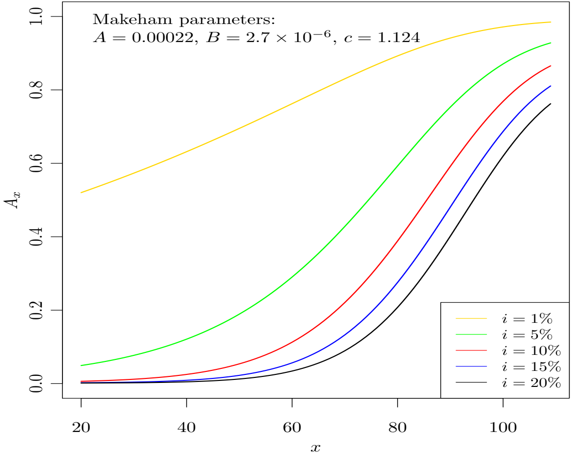

Figure: Actuarial Present Value of a discrete whole life insurance for various interest rate assumptions

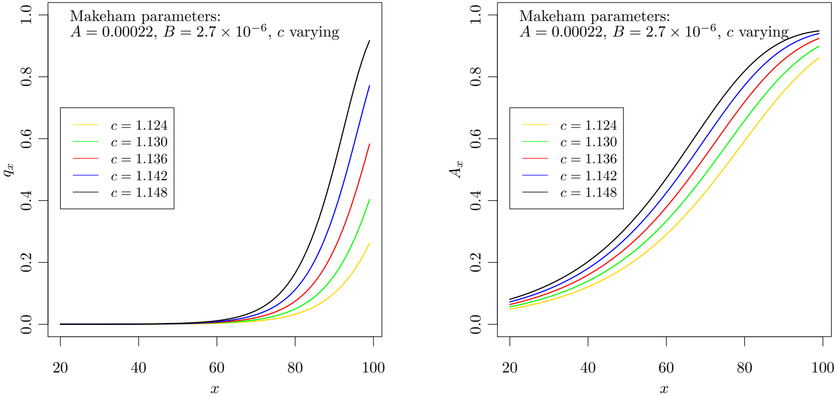

Figure: Actuarial Present Value of a discrete whole life insurance for various mortality rate assumptions with interest rate fixed at 5%

Illustrative example 3

For a whole life insurance of 1 on (41) with death benefit payable at the end of the year of death, let Z be the present value random variable for this insurance.

You are given:

- $i = 0.05$;
- $p_{40} = 0.9972$;
- $A_{41} - A_{40} = 0.00822$; and
- ${}^2A_{41} - {}^2A_{40} = 0.00433$.

Calculate Var$[Z]$.

Other forms of insurance

- Varying benefit insurances
- Very similar to the continuous cases
- You are expected to read and understand these other forms of insurances.
- It is also useful to understand the various (possible) recursion relations resulting from these various forms.

Illustration of varying benefits

For a special life insurance issued to (45), you are given:

- Death benefits are payable at the end of the year of death.
- The benefit amount is \$100,000 in the in the first 10 years of death, decreasing to \$50,000 after that until reaching age 65.
- An endowment benefit of \$100,000 is paid if the insured reaches age 65.
- There are no benefits to be paid past the age of 65.
- Mortality follows the `Standard Ultimate Life Table` at $i = 0.05$.

Calculate the actuarial present value (APV) for this insurance.

Illustrative example 4

For a whole life insurance issued to age 40, you are given:

- Death benefits are payable at the moment of death.
- The benefit amount is \$1,000 in the first year of death, increasing by \$500 each year thereafter for the next 3 years, and then becomes level at \$5,000 thereafter.
- Mortality follows the `Standard Ultimate Life Table` at $i = 0.05$.
- Deaths are uniformly distributed over each year of age.

Calculate the APV for this insurance.

Insurances payable m-thly

- Consider the case where we have just one-year term and the benefit is payable at the end of the m-th of the year of death.
- We thus have

$$A^{(m)}_{\stackrel{1}{x:\overline{1|}}} = \sum_{r=0}^{m-1} v^{(r+1)/m} \cdot {}_{r/m}p_x \cdot {}_{1/m}q_{x+r/m}.$$

- We can show that under the UDD assumption, this leads us to:

$$A^{(m)}_{\stackrel{1}{x:\overline{1|}}} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{1|}}.$$

- In general, we can generalize this to:

$$A^{(m)}_{\stackrel{1}{x:\overline{n|}}} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n|}}.$$

Other types of insurances with m-thly payments

- For other types, we can also similarly derive the following (with the UDD assumption):
  - whole life insurance: $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$
  - deferred life insurance: ${}_{n|}A_x^{(m)} = \frac{i}{i^{(m)}} \, {}_{n|}A_x$
  - endowment insurance: $A_{x:\overline{n|}}^{(m)} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;\;1}$

Relationships - continuous and discrete

- For some forms of insurances, we can get explicit relationships under the UDD assumption:
  - whole life insurance: $\bar{A}_x = \frac{i}{\delta} A_x$
  - term insurance: $\bar{A}^{1}_{x:\overline{n|}} = \frac{i}{\delta} A^{1}_{x:\overline{n|}}$
  - increasing term insurance: $(I\bar{A})^{1}_{x:\overline{n|}} = \frac{i}{\delta} (IA)^{1}_{x:\overline{n|}}$

Illustrative example 5

For a three-year term insurance of 1000 on $[50]$, you are given:

- Death benefits are payable at the end of the quarter of death.
- Mortality follows a select and ultimate life table with a two-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	9706	9687	9661	52
51	9680	9660	9630	53
52	9653	9629	9596	54

- Deaths are uniformly distributed over each year of age.
- $i = 5\%$

Calculate the APV for this insurance.

Illustrative example 6

Each of 100 independent lives purchases a single premium 5-year deferred whole life insurance of 10 payable at the moment of death.

You are given:

- $\mu = 0.004$
- $\delta = 0.006$
- F is the aggregate amount the insurer receives from the 100 lives.
- The 95th percentile of the standard Normal distribution is 1.645.

Using a Normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

Illustrative example 7

Suppose interest rate $i = 6\%$ and mortality is based on the following life table:

x	90	91	92	93	94	95	96	97	98	99	100
ℓ_x	800	740	680	620	560	500	440	380	320	100	0

Calculate the following:

(a) A_{94}

(b) $A^{1}_{90:\overline{5}|}$

(c) ${}_{3|}A^{(4)}_{92}$, assuming UDD between integral ages

(d) $A_{95:\overline{3}|}$

Illustrative example 8

A five-year term insurance policy is issued to (45) with benefit amount of \$10,000 payable at the end of the year of death.

Mortality is based on the following select and ultimate life table:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
45	5282	5105	4856	4600	48
46	4753	4524	4322	4109	49
47	4242	4111	3948	3750	50
48	3816	3628	3480	3233	51

Calculate the APV for this insurance if $i = 5\%$.

Illustrative example 9

Note: This is a modified question from SOA Spring 2016 exam.

A life insurance policy is issued to (35) with present value random variable:

$$Z = \begin{cases} 10v^{T_{35}}, & 0 < T_{35} \leq 25 \\ 40v^{T_{35}}, & 25 < T_{35} \leq 45 \\ 0, & T > 45 \end{cases}$$

You are also given:

- Mortality follows the `Standard Ultimate Life Table`.
- Deaths are uniformly distributed over each year of age.
- $i = 0.05$

Calculate the expected value and the standard deviation of Z.

Other terminologies and notations used

Expression	Other terms/symbols used
Actuarial Present Value (APV)	Expected Present Value (EPV) Net Single Premium (NSP) single benefit premium
basis	assumptions
interest rate (i)	interest per year effective discount rate
benefit amount (b)	sum insured (S) death benefit
Expected value of Z	$\mathsf{E}(Z)$
Variance of Z	$\mathsf{Var}(Z) \quad V[Z]$