$v = \frac{1}{1+i}$

$PV = B\, v^{T_x}$

random variable

Insurance Benefits

financial mathematics + statistics = life insurance mathematics

$E[PV]$ = expected Present Value of benefits

Lecture: Weeks 6-7

= Actuarial Present Value

UCONN

$Var(b_T v_T)$

An introduction

$APV(\text{benefit}) = E(b_T v_T)$ -

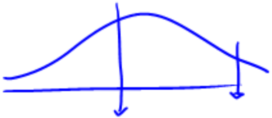
- Central theme: to quantify the value today of a (random) amount to be paid at a random time in the future.
  - main application is in life insurance contracts, but could be applied in other contexts, e.g. warranty contracts.
- Generally computed in two steps:

  benefit discount $T =$ random variable

  EPV ❶ take the present value (PV) random variable, $b_T v_T$; and

  APV ❷ calculate the expected value $\mathrm{E}[b_T v_T]$ for the average value - this value is referred to as the Actuarial Present Value (APV).
- In general, we want to understand the entire distribution of the PV random variable $b_T v_T$:
  - it could be highly skewed, in which case, there is danger to use expectation.
  - other ways of summarizing the distribution such as variances and percentiles/quantiles may be useful.

$b_T V_T$ is a random variable



$E(b_T V_T)$

$Var(b_T V_T)$

quantile of $b_T V_T$ e.g. 95th quantile policy expires

A simple illustration

at the end of 3 years, no more benefits

Consider the simple illustration of valuing a three-year term insurance policy issued to age 35 where if he dies within the first year, a \$1,000 benefit is payable at the end of his year of death.

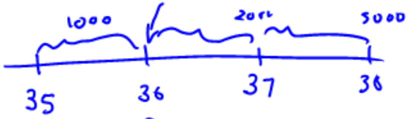
If he dies within the second year, a \$2,000 benefit is payable at the end of his year of death. If he dies within the third year, a \$5,000 benefit is payable at the end of his year of death.

Assume a constant interest rate (annual effective) of 5% and the following extract from a mortality table:

x	q_x
35	0.005
36	0.006
37	0.007
38	0.008

Calculate the APV of the benefits.

$i = 5\%$ $v = \frac{1}{1.05}$



$$\underbrace{1000 * q_{35} \cdot v}+\underbrace{2000 * p_{35} * q_{36} \cdot v^2}$$
$$\underbrace{+\, 5000 * p_{35} * p_{36} * q_{37} \cdot v^3}$$

discrete -

APV(benefit) using intuition

$$1000\left[.005\left(\frac{1}{1.05}\right)+2*(1-.005)\,.006\left(\frac{1}{1.05}\right)^2 + 5*(1-.005)(1-.006)\,.007\left(\frac{1}{1.05}\right)^3\right]$$

$$\underbrace{}$$

45.49448

probability you die!

Chapter summary

life contingencies

- Life insurance
  - benefits payable contingent upon death; payment made to a designated beneficiary
  - actuarial present values (APV)
  - actuarial symbols and notation
- Insurances payable at the moment of death
  T_x
  - continuous
  - level benefits, varying benefits (e.g. increasing, decreasing)
- Insurances payable at the end of year of death
  K_x
  - discrete
  - level benefits, varying benefits (e.g. increasing, decreasing)
- Chapter 4, DHW

The present value random variable

- Denote by Z, the present value random variable.
- This gives the value, at policy issue, of the benefit payment. Issue age is usually denoted by x.
- In the case where the benefit is payable at the moment of death, Z clearly depends on the time-until-death T. For simplicity, we drop the subscript x for age-at-issue.
- It is $Z = b_T v_T$ where:
    - b_T is called the benefit payment function
    - v_T is the discount function

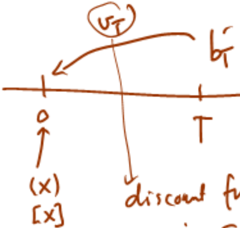
$v_T = v^T = \left(\frac{1}{1+i}\right)^t$

- In the case where we have a constant (fixed) interest rate, then $v_T = v^T = (1+i)^{-T} = e^{-\delta T}$.

$1+i = e^{\delta}$

Z = present value random variable

$\underbrace{\hphantom{present value random variable}}$ value of your insurance benefit upon death



discount function

$i,\ \delta$

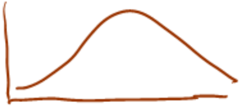
$v_T = v^T = \left(\frac{1}{1+i}\right)^T$

$= e^{-\delta T}$

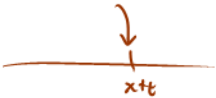
$Z = b_T v_T$

T is random

$E[Z] =$ APV (benefits) actuarial present value (expected) present value EPV



T → density



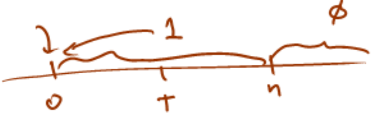
$$Var(Z) = E[Z^2] - (E[Z])^2$$

$$E[g(T)] = \int_0^\infty g(t)\underbrace{f_T(t)}_{{}_tp_x\,\mu_{x+t}}\,dt$$

n-year term insurance policy

- provides you benefit upon death, so long as you die with n years
  at the moment of death

$$Z = \begin{cases} v^T, & T \le n \\ 0, & T > n \end{cases}$$



$$= v^T I(\underbrace{T \le n})$$

$I(\cdot)$ = indicator function

$$I(A) = \begin{cases} 1, & \text{if } A \text{ is true} \\ 0, & \text{if } A \text{ is false} \end{cases}$$

$$APV(n\text{ year term}) = E[v^T I(T \le n)]$$

$$= \int_0^\infty v^t \underbrace{I(t \le n)}\, {}_tp_x \mu_{x+t}\, dt$$

APV of an n-year term of \$1 of benefit.

$$E[Z] = \underbrace{\int_0^n v^t\, {}_tp_x \mu_{x+t}\, dt} = \bar{A}^{\,1}_{x:\overline{n}|}$$

$$Var(Z) = \underbrace{E[Z^2]}_{\text{rule of moment}} - \left(E[Z]\right)^2$$

$$\left(E[Z]\right)^2 \rightarrow \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2$$

$$v = e^{-\delta T}$$

$$v^2 = e^{-2\delta T} = \left(e^{-\delta T}\right)^2$$

$$= \, {}^{2}\bar{A}^{1}_{x:\overline{n}|} - \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2$$

${}^{2}\bar{A}^{1}_{x:\overline{n}|}$ @ 2δ

$\left(\bar{A}^{1}_{x:\overline{n}|}\right)^2$ @ δ but ^{2}

$$E[Z^2] = \underbrace{E[v^{2T} I(T \le n)]}$$

$$Z = v^T I(T \le n)$$

$$= \underbrace{E[(v^2)^T I(T \le n)]}_{\text{APV of an n-year term insurance evaluated at } 2\delta}$$

$j^{\underline{th}}$ rule of moment $\quad E[Z^j] = E[Z]\ @\ j\delta$

evaluate at $\underline{j\delta}$

$n \to \infty$ whole life insurance

an insurance policy that pays at the moment of death

$b_T = 1$

$$Z = v^T \underbrace{I(T \leq \infty)}_{1} = v^T$$

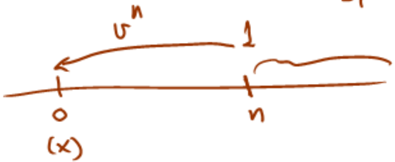
$$E[Z] = \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, dt = \bar{A}_x$$

$$Var(Z) = {}^2\bar{A}_x - (\bar{A}_x)^2$$

Pure Endowment is an insurance policy that pays a benefit if you are alive at the end of n years

n-year

$b_T = 1$



$$Z = \begin{cases} 0, & T \le n \\ v^n, & T > n \end{cases}$$

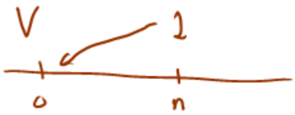
$$Z = v^n \underbrace{I(T>n)}_{\text{binary random variable}}$$

$I(T>n) \sim \text{Bernoulli}(p)$

$$p = \underbrace{P[T>n]} = {}_n p_x$$

$$E[Z] = E[v^n \underbrace{I(T>n)}] = v^n \cdot {}_n p_x \longrightarrow \text{APV of an n-year pure endowment of \$1}$$

$$Var(Z) = Var(v^n I(T>n)) = v^{2n} \cdot {}_n p_x (1 - {}_n p_x)$$



$v^n\,{}_np_x = \underline{\underline{EPV}}, = {}_nE_x$

$\underline{v^n\,{}_np_x = APV}$,

Symbols

① $\underbrace{APV(\text{n-year pure endowment})}$ $= A_{x:\overset{1}{\overline{n|}}} = v^n\,{}_np_x$

② ${}_nE_x = \underline{A_{x:\overset{1}{\overline{n|}}}} = v^n\,{}_np_x$

discount factor with life.

$\underset{FV}{Y} \cdot \underbrace{v^n\,{}_np_x}_{{}_nE_x} = \underset{PV}{V}$

$Y = V / {}_nE_x$

pure + term

$\underbrace{\qquad\qquad\qquad}$

endowment insurance

$$Z = \begin{cases} v^T, & T \le n \\ v^n, & T > n \end{cases}$$

$$E[Z] = E[\underbrace{v^T \underbrace{I(T \le n)}}_{} + \underbrace{v^n \underbrace{I(T > n)}}_{}]$$

$$\bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;1}$$

$$Var(Z) = \underbrace{{}^2\bar{A}_{x:\overline{n|}}}_{@ 2\delta} - \left(\bar{A}_{x:\overline{n|}}\right)^2$$

$$Y = V / {}_nE_x$$

$$= V / v^n \, {}_np_x$$

FV

$$= \underbrace{V}_{PV} \cdot (1+i)^n \left(\frac{1}{{}_np_x}\right)$$

${}_nE_x$ is a discount factor of life

n-year pure endowment variance @ 2δ

$$Z = v^n I(T>n) = \begin{cases} 0, & T \le n \\ v^n, & T > n \end{cases}$$

$$E[Z] = A_{x:\overset{1}{\overline{n|}}} = \underbrace{v^n}\cdot\underbrace{{}_n p_x}$$

$$Var(Z) = \underbrace{E[Z^2]} - (E[Z])^2$$

$$= v^{2n}\cdot {}_n p_x - (v^n\, {}_n p_x)^2$$

$$= v^{2n}\, \underline{{}_n p_x} - v^{2n} (\underline{{}_n p_x})^2$$

$$= v^{2n}\, \underbrace{{}_n p_x (1 - {}_n p_x)}$$

$$\underbrace{{}^2A_{x:\overset{1}{\overline{n|}}}}_{@\, 2\delta}$$

2δ rule <u>holds</u>.

4 conventional policies

Types	PV r.v. $= Z$	$E[Z] \to$ APV of benefits	$Var(Z)$
n-year term	$v^T I(T \le n)$	$\bar{A}^{1}_{x:\overline{n}\rceil}$	${}^2\bar{A}^{1}_{x:\overline{n}\rceil} - \left(\bar{A}^{1}_{x:\overline{n}\rceil}\right)^2$
whole life $n \to \infty$	v^T	$\bar{A}_x$	${}^2\bar{A}_x - \left(\bar{A}_x\right)^2$
n-year pure endowment	$v^n \underbrace{I(T>n)}_{\text{binary}}$	$\underbrace{A_{x:\overline{n}\rceil}^{\ \ 1},\ {}_nE_x}_{v^n \cdot {}_np_x}$	$\underbrace{{}^2A_{x:\overline{n}\rceil}^{\ \ 1} - \left(A_{x:\overline{n}\rceil}^{\ \ 1}\right)^2}_{v^{2n} \cdot {}_np_x (1 - {}_np_x)}$
n-year endowment = pure + term	$\begin{cases} v^T, & T \le n \\ v^n, & T > n \end{cases}$	$\bar{A}_{x:\overline{n}\rceil}$	${}^2\bar{A}_{x:\overline{n}\rceil} - \left(\bar{A}_{x:\overline{n}\rceil}\right)^2$

Lifetime is exponential, (constant force μ)

$$f_T(t) = {}_t p_x \, \mu_{x+t} = \underline{\underline{\mu e^{-\mu t}}}$$

$$v = e^{-\delta}$$

n-year term

$$\bar{A}^{1}_{x:\overline{n}|} = \int_0^n v^t \cdot \mu e^{-\mu t} \, dt$$

$$= \mu \int_0^n \underbrace{e^{-\delta t} e^{-\mu t}}_{e^{-(\mu+\delta)t}} dt = \underbrace{\frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)n}\right)}$$

$n \to \infty$ whole life

$$\bar{A}_x = \frac{\mu}{\mu+\delta}$$

$$\lim_{n \to \infty} \bar{A}^{1}_{x:\overline{n}|}$$

n-year pure

$$A_{x:\overline{n}|}^{\;\;1} = v^n \, {}_n p_x = v^n e^{-\mu n} = e^{-(\mu+\delta)n}$$

n-year endowment

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1} = \underbrace{\frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)n}\right)}_{\text{term}} + \underbrace{e^{-(\mu+\delta)n}}_{\text{pure endowment}}$$

$\mu = .05$, $\delta = .03$

(a) APV of a whole life of 1, payable at the moment of death

(b) variance of this policy

$$\bar{A}_x = \frac{\mu}{\mu+\delta} = \frac{.05}{.05+.03} = 5/8 = .625$$

$$Var(Z) = \underbrace{{}^2\bar{A}_x} - (\bar{A}_x)^2$$

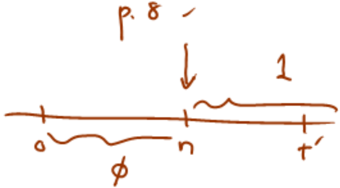
$$= \frac{\mu}{\mu+2\delta} - (.625)^2$$

$$= \underbrace{\frac{.05}{.05+.06}}_{\frac{5}{11}} - (.625)^2 = \underline{.454545}$$

n-year deferred insurance

$$Z = \begin{cases} 0, & T \le n \\ v^T, & T > n \end{cases}$$

$$= v^T I(T > n)$$



$$E[Z] = {}_{n|}\bar{A}_x$$

deferred

$$E[Z^2] = {}_{n|}^{2}\bar{A}_x$$

$$\int_n^\infty v^t \cdot {}_tp_x \, \mu_{x+t} \, dt$$

Exponential lifetime

$$_{n|}\bar{A}_x = \int_n^\infty v^t \cdot \mu \cdot e^{-\mu t} \, dt$$

$$= \mu \int_n^\infty e^{-(\mu+\delta)t} \, dt$$

$$= \frac{\mu}{\mu+\delta} e^{-(\mu+\delta)n}$$

Fixed term life insurance

- An n-year term life insurance provides payment if the insured dies within n years from issue.
- For a unit of benefit payment, we have

$$b_T = \begin{cases} 1, & T \leq n \\ 0, & T > n \end{cases} \text{ and } v_T = v^T.$$

- The present value random variable is therefore

$$Z = \begin{cases} v^T, & T \leq n \\ 0, & T > n \end{cases} = v^T I(T \leq n)$$

  where $I(\cdot)$ is called indicator function. $\mathsf{E}[Z]$ is called the APV of the insurance.

- Actuarial notation:

$$\bar{A}^{\,1}_{x:\overline{n}|} = \mathsf{E}[Z] = \int_0^n v^t f_x(t) dt = \int_0^n v^t \, {}_tp_x \mu_{x+t} dt.$$

Rule of moments

- The j-th moment of the distribution of Z can be expressed as:

$$\mathsf{E}\left[Z^j\right] = \int_0^n v^{tj} \, {}_tp_x \mu_{x+t} dt = \int_0^n e^{-(j\delta)t} \, {}_tp_x \mu_{x+t} dt.$$

- This is actually equal to the APV but evaluated at the force of interest $j\delta$.
- In general, we have the following rule of moment:

$$\mathsf{E}\left[Z^j\right]_{@\ \delta_t} = \mathsf{E}[Z]_{@\ j\delta_t}.$$

- For example, the variance can be expressed as

$$\mathsf{Var}[Z] = {}^2\bar{A}^{1}_{x:\overline{n}|} - \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2.$$

Traditional insurances - continuous

Type	Benefit b_T	PV r.v. Z	APV E$[Z]$	Variance Var$[Z]$
Term life	$I(T \leq n)$	$v^T \cdot I(T \leq n)$	$\bar{A}^{1}_{x:\overline{n}\rceil}$	${}^2\bar{A}^{1}_{x:\overline{n}\rceil} - \left(\bar{A}^{1}_{x:\overline{n}\rceil}\right)^2$
Whole life	1	v^T	$\bar{A}_x$	${}^2\bar{A}_x - \left(\bar{A}_x\right)^2$
Pure endowment	$I(T > n)$	$v^n \cdot I(T > n)$	$A_{x:\overset{1}{\overline{n}\rceil}}$ or ${}_nE_x$	${}^2A_{x:\overset{1}{\overline{n}\rceil}} - \left(A_{x:\overset{1}{\overline{n}\rceil}}\right)^2$
Endowment	1	$v^{\min(T,n)}$	$\bar{A}_{x:\overline{n}\rceil}$	${}^2\bar{A}_{x:\overline{n}\rceil} - \left(\bar{A}_{x:\overline{n}\rceil}\right)^2$
Deferred	$I(T > n)$	$v^T \cdot I(T > n)$	${}_{n\vert}\bar{A}_x$	${}^{2}_{n\vert}\bar{A}_x - \left({}_{n\vert}\bar{A}_x\right)^2$

endowment

$$Z = \underbrace{\begin{cases} v^T, & T \le n \\ v^n, & T > n \end{cases}}$$

$$= v^{\min(T,n)}, \quad \text{simplifies}$$

$$= \underbrace{v^T I(T \le n) + v^n I(T > n)}, \quad \text{defines}$$

$$E\left[v^{\min(T,n)}\right] = \bar{A}_{x:\overline{n}|}$$

Benefit = 1

$Z = b_T v_T$

$b_T = 1 \Rightarrow E[Z] = E[v_T]$

$b_T = B \Rightarrow E[Z] = B \cdot \underbrace{E[v_T]}$ ✓

$Var(Z) = B^2 \, Var(Z @ \$1)$

---

constant force = .05

constant δ = .03

$\bar{A}_x = \frac{5}{8}$

$100 \bar{A}_x = 100 \times \frac{5}{8} < 100$

---

endowment = pure + term $\Rightarrow$ $\bar{A}_{x:\overline{n|}} = A_{x:\overline{n|}}^{\;\;1} + \bar{A}^{1}_{x:\overline{n|}}$

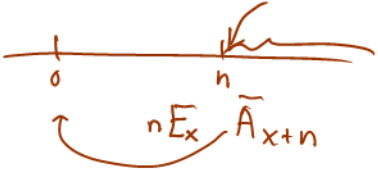
whole life = term + deferred $\Rightarrow$ $\bar{A}_x = \bar{A}^{1}_{x:\overline{n|}} + {}_{n|}\bar{A}_x$

deferred insurance

$\downarrow$

= discounted whole life insurance

$$= {}_{n|}\bar{A}_x = {}_nE_x \cdot \bar{A}_{x+n}$$



const force μ, δ

$${}_{n|}\bar{A}_x = \underbrace{\frac{\mu}{\mu+\delta}}_{\bar{A}_{x+n}} \cdot \underbrace{e^{-(\mu+\delta)n}}_{\text{discount factor}}$$

Pure endowment insurance

- For an n-year pure endowment insurance, we can also express the PV random variable as:

$$Z = v^n I(T_x > n),$$

where $I(E)$ is 1 if the event E is true, and 0 otherwise.

- The term $I(I(T_x > n))$ is a binary random variable with mean $\mathsf{E}[I(T_x > n)] = {}_np_x$ and $\mathsf{Var}[I(T_x > n)] = {}_np_x(1 - {}_np_x)$.

- APV for pure endowment:

$$A_{x:\overset{1}{\overline{n}|}} = {}_nE_x = v^n\, {}_np_x.$$

- Variance (show also using rule of moments):

$$\mathsf{Var}[Z] = v^{2n}\, {}_np_x \cdot {}_nq_x = {}^2A_{x:\overset{1}{\overline{n}|}} - \left(A_{x:\overset{1}{\overline{n}|}}\right)^2.$$

Endowment insurance

- An n-year endowment insurance is the sum of an n-year term and and n-year pure endowment:

$$Z = Z_1 + Z_2 = v^{\min(T,n)} = \overbrace{\underbrace{v^T \cdot I(T \leq n)}_{Z_1}}^{\text{term}} + \overbrace{\underbrace{v^n I(T_x > n)}_{Z_2}}^{\text{pure endowment}},$$

where $Z_1 = v^T \cdot I(T \leq n)$ is the term component and $Z_2 = v^n I(T_x > n)$ is the pure endowment component.

- Therefore, it is clear that:

where ${}_nE_x = v^n\,{}_np_x$

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + {}_nE_x = \cancel{v^n\,{}_np_x}.$$

- One can also use the variance of sums of random variables to get:

$$\mathsf{Var}[Z] = \mathsf{Var}[Z_1] + \mathsf{Var}[Z_2] + 2\mathsf{Cov}[Z_1, Z_2]$$

where one can show that $\mathsf{Cov}[Z_1, Z_2] = \mathsf{E}[Z_1]\mathsf{E}[Z_2]$ since $Z_1 \cdot Z_2 = 0$.

$$Z = \underset{\swarrow}{Z_1} + Z_2 \longrightarrow v^n I(T>n)$$

$$Z_1 = v^T I(T \le n)$$

P.S. can always use the rule of moment

$$Var(Z) = \underbrace{var(Z_1)}_{\text{variance of term insurance}} + \underbrace{var(Z_2)}_{\text{variance of pure endowment}} + 2\underbrace{Cov(Z_1, Z_2)}_{\left(\cancel{E[Z_1 Z_2]}^{0} - E[Z_1]\cdot E[Z_2]\right)}$$

$$Z_1 \cdot Z_2 = v^T \underbrace{I(T \le n)}_{\substack{1\\0}} \cdot v^n \underbrace{I(T>n)}_{\substack{0\\1}} = 0 \Rightarrow E[Z_1 Z_2] = 0$$

$$E[Z_1]\,E[Z_2] = \bar{A}^{\,1}_{x:\overline{n}|} \cdot A_{x:\overline{n}|}^{\;\;\;1}$$

Deferred insurance

- An n-year deferred insurance can be viewed as a discounted (with life) whole life insurance:

$$
{}_{n|}\bar{A}_x = {}_nE_x \cdot \bar{A}_{x+n}
$$

- The pure endowment insurance is used as a discounting with life contingent payments.

Constant force of mortality - all throughout life

Assume mortality is based on a constant force, say μ, and interest is also based on a constant force of interest, say δ.

- Find expressions for the APV for the following types of insurances:
  - whole life insurance;
  - n-year term life insurance;
  - n-year endowment insurance; and
  - n-year deferred life insurance.
- Check out the (corresponding) variances for each of these types of insurance.

[Details in class]

whole = term + deferred

deferred = whole - term

APVs under constant force of mortality

Assume constant force of mortality μ and constant force of interest δ.

Type	APV of 1 payable at moment (continuous)
Term	$\bar{A}^{\,1}_{x:\overline{n}\rceil} = \dfrac{\mu}{\mu+\delta}\left[1-e^{-(\mu+\delta)n}\right]$
Whole	$\bar{A}_x = \dfrac{\mu}{\mu+\delta}$ $n \to \infty$
Pure	$A_{x:\overline{n}\rceil}^{\ \ 1} = {}_nE_x = e^{-(\mu+\delta)n}$
Endowment	$\bar{A}_{x:\overline{n}\rceil} = \dfrac{\mu}{\mu+\delta}\left[1-e^{-(\mu+\delta)n}\right] + e^{-(\mu+\delta)n}$
Deferred	${}_{n\vert}\bar{A}_x = \dfrac{\mu}{\mu+\delta}e^{-(\mu+\delta)n}$ = whole - term

De Moivre's law

Uniform $(0, \omega)$

$\downarrow$

limiting age

$T_x \sim (0, \omega - x)$ $\quad f_T(t) = \frac{1}{\omega - x}, \; 0 \leq t \leq \omega - x$

Find expressions for the APV for the same types of insurances in the case where you have:

- De Moivre's law.

n-year term

whole life

pure endowment

n yr endowment

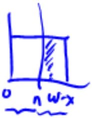
m year deferred

de Moivre's

term insurance: $\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n v^t \underbrace{f_T(t)}_{\frac{1}{\omega-x}}\, dt = \frac{1}{\omega-x} \underbrace{\int_0^n v^t\, dt}_{\text{Continuous annuity-certain } \bar{a}_{\overline{n}|}} = \frac{\bar{a}_{\overline{n}|}}{\omega-x}$

$n\to\infty \Rightarrow n \to \omega-x$

whole life: $\bar{A}_x = \frac{\bar{a}_{\overline{\omega-x}|}}{\omega-x}$



pure endowment $A_{x:\overline{n}|}^{\;\;1} = v^n\, {}_np_x = e^{-\delta n}\left(1-\frac{n}{\omega-x}\right)$

endowment (term + pure) $\bar{A}_{x:\overline{n}|} = \frac{\bar{a}_{\overline{n}|}}{\omega-x} + e^{-\delta n}\left(1-\frac{n}{\omega-x}\right)$

deferred m-year ${}_{m|}\bar{A}_x = {}_mE_x\, \bar{A}_{x+m}$

$= e^{-\delta m}\left(1-\frac{m}{\omega-x}\right)\cdot \frac{\bar{a}_{\overline{\omega-x-m}|}}{\omega-x-m}$

Illustrative example 1

For a whole life insurance of $1,000 on (x) with benefits payable at the moment of death, you are given:

$$\delta_t = \begin{cases} 0.04, & 0 < t \leq 10 \\ 0.05, & t > 10 \end{cases}$$

and

$$\mu_{x+t} = \begin{cases} 0.006, & 0 < t \leq 10 \\ 0.007, & t > 10 \end{cases}$$

Calculate the actuarial present value for this insurance.

whole life

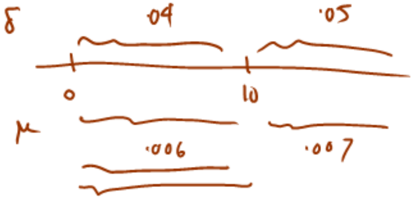
1000 multiply by 1000

= term + deferred

= term + discounted whole life

$$= \frac{\mu}{\mu+\delta}\left[1-e^{-(\mu+\delta)n}\right] + \underbrace{e^{-(\mu+\delta)n}}\ \frac{\mu'}{\mu'+\delta'}$$

$\delta' = .05 \quad \mu' = .007$



$\delta = .04 \quad n = 10$

$\mu = .006$

$$APV = \underbrace{\left[\frac{.04}{.046}\left(1-e^{-.046(10)}\right) + e^{-.046(10)} \cdot \frac{.05}{.057}\right] * 1000}_{86.02166}$$

86.02166

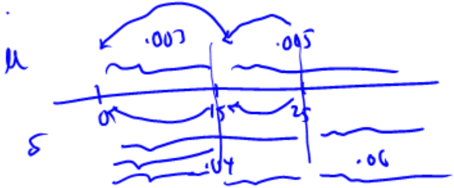
Suppose Benefit = 1000 —

whole life

$$\mu_{x+t} = \begin{cases} .003, & 0 < t \le 15 \\ .005, & t > 15 \end{cases} \qquad \delta_t = \begin{cases} .04, & 0 < t \le 25 \\ .06, & t > 25 \end{cases}$$

Calculate APV of this whole life.

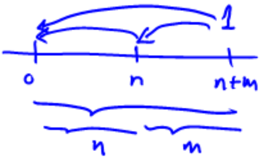
1000



$$APV(\text{benefit}) = 1000 * \Bigg[\underbrace{\frac{.003}{.003+.04}\left(1-e^{-(.043)(15)}\right)}_{\text{1st 15 years}} + \underbrace{e^{-.043(15)} \cdot \frac{.005}{.005+.04}\left(1-e^{-.045(10)}\right)}_{\text{next 10 years}}$$

$$\underline{\underline{80.02166}} = \qquad \underbrace{+\, e^{-.043(15)}\, e^{-.045(10)} \cdot \frac{.005}{.005+.06}}_{\text{thereafter}} \Bigg]$$

$${}_{n+m}E_x = \underbrace{{}_nE_x \;\; {}_mE_{x+n}}$$



v is multiplicative

p is multiplicative

$$v^{n+m} \underbrace{{}_{n+m}p_x} = v^n \cdot v^m \cdot {}_np_x \cdot {}_mp_{x+n}$$

$$= \underbrace{\left(v^n \, {}_np_x\right)}_{{}_nE_x} \cdot \underbrace{\left(v^m \, {}_mp_{x+n}\right)}_{{}_mE_{x+n}}$$

$$\underbrace{{}_{50}E_x = {}_{10}E_x \cdot {}_{20}E_{x+10} \cdot {}_{20}E_{x+30}}$$

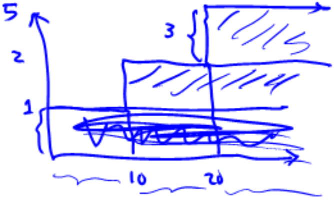
Benefits are varying

whole life to (x) payable at the moment of death where the benefits vary according to

$$b_T = \begin{cases} 1, & T \le 10 \\ 2, & 10 < T \le 20 \\ 5, & T > 20 \end{cases}$$

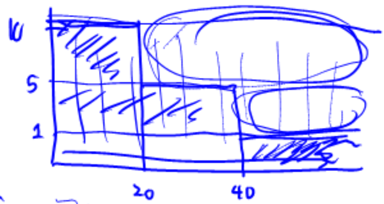
Express this (APV) in terms of standard symbols.

$$\boxed{APV = 1 \cdot \bar{A}_x + 1 \cdot {}_{10}E_x \, \bar{A}_{x+10} + 3 \cdot {}_{20}E_x \, \bar{A}_{x+20}}$$



$$b_T = \begin{cases} 10, & 0<T\le 20 \\ 5, & 20<T\le 40 \\ 1, & T>40 \end{cases}$$

(x)



$$APV = \underbrace{\bar{A}_x + 4\cdot \bar{A}^{1}_{x:\overline{40}|} + 5\cdot \bar{A}^{1}_{x:\overline{20}|}}_{\text{assuming term insurance are available}}$$

$$= 10\cdot \bar{A}_x - 5\,{}_{20}E_x\,\bar{A}_{x+20} - 4\,{}_{40}E_x\,\bar{A}_{x+40}$$

Consider whole life $B = 1$ PV rv $= Z = v^T$ $v = e^{-\delta}$

What is the probability that the PV is below some $\underbrace{\text{fixed number}}_{\alpha}$?

$\log v^T = T \cdot \underbrace{\log v}$

$\log v = \underbrace{\ln v}_{-\delta}$

$$Pr(Z \le \alpha) = Pr(v^T \le \alpha)$$

$$= Pr(T \underbrace{\log v}_{-\delta} \le \log \alpha)$$

$T \le$

$T >$

$$= \underbrace{Pr\left(T > \frac{\log \alpha}{-\delta}\right)}_{\text{positive number}}$$

1

$\alpha \le 1$

$\log \alpha \Rightarrow$ negative

$-\delta \Rightarrow$ negative

$Pr[T > a] = {}_a p_x$

$\frac{\log \alpha}{-\delta} p_x$

$Pr[Z \text{ something small}] \approx Pr[T \text{ is later}]$

$Pr[Z \text{ something large}] \approx Pr[T \text{ is earlier}]$

Equivalent probability calculations

We can also compute probabilities of Z as follows. Consider the present value random variable Z for a whole life issued to age x. For $0 < \alpha < 1$, the following is straightforward:

$$\begin{aligned} \underbrace{\mathsf{Pr}[Z \leq \alpha]} &= \mathsf{Pr}[e^{-\delta T_x} \leq \alpha = \mathsf{Pr}[-\delta T_x \leq \log(\alpha)] \\ &= \mathsf{Pr}[T_x > -(1/\delta)\log(\alpha)] = {}_u p_x, \end{aligned}$$

$\left(\frac{\log \alpha}{-\delta}\right)$

where

$$u = (1/\delta)\log(1/\alpha) = \log(1/\alpha)^{1/\delta}.$$

- Consider the case where $\alpha = 0.75$ and $\delta = 0.05$. Then $u = \log(1/0.75)^{1/0.05} = 5.753641$.
- Thus, the probability $\mathsf{Pr}[Z \leq 0.75]$ is equivalent to the probability that (x) will survive for another 5.753641 years.

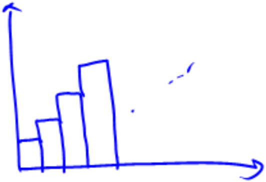
$\Pr[Z \leq .90] \Rightarrow$

> 6 years

Insurances with varying benefits

Type	b_T	Z	APV
✓ Increasing whole life	$\lfloor T+1 \rfloor$	$\lfloor T+1 \rfloor v^T$	$(I\bar{A})_x$
Whole life increasing m-thly	$\lfloor Tm+1 \rfloor / m$	$v^T \lfloor Tm+1 \rfloor / m$	$(I^{(m)}\bar{A})_x$
Constant increasing whole life	T	Tv^T	$(\bar{I}\bar{A})_x$
✓ Decreasing n-year term	$\begin{cases} n - \lfloor T \rfloor, & T \leq n \\ 0, & T > n \end{cases}$	$\begin{cases} (n - \lfloor T \rfloor) v^T, & T \leq n \\ 0, & T > n \end{cases}$	$(D\bar{A})^{1}_{x:\overline{n}\rceil}$

* These items will be discussed in class.



increasing whole life

1, 2, 3

$$Z = \lfloor T+1 \rfloor v^T$$

$T = 0.5$

$\lfloor \cdot \rfloor$ = greatest integer

$T+1 = 1.5$

$$APV = \int_0^\infty \lfloor t+1 \rfloor v^t \underbrace{f_T(t)}_{\to 0}\, dt = (I\bar{A})_x$$

$\lfloor T+1 \rfloor = 1$

increasing whole life

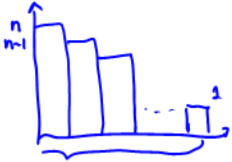
$\longrightarrow$ converges

n, n-1, n-2.

I = increasing

D = decreasing

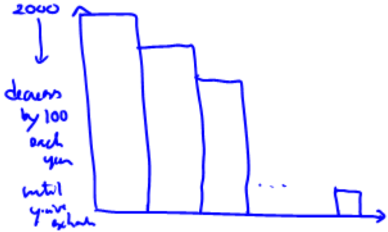
n-year decreasing term



$$Z = \begin{cases} n - \lfloor T \rfloor, & 0 < T \le n, \\ 0, & T > n \end{cases}$$

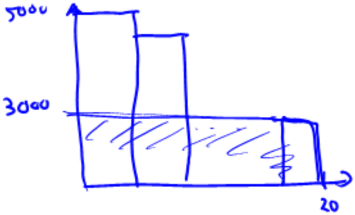
$$APV = (D\bar{A})^{1}_{x:\overline{n}|}$$

$\underbrace{(D\bar{A})^{1}_{x:\overline{n}|}}$ $\Rightarrow$ decreasing by 1 unit



$$APV = \underbrace{(D\bar{A})^{1}_{x:\overline{20}|}}_{2000,\ 1900,\ 1800,\ \ldots,\ 100} \cdot 100$$

5000 decreasing by 100 each year and stops at the end of 20-year



$$APV(\text{benefits}) = (D\bar{A})^{1}_{x:\overline{20}|} \cdot 100$$

$$+\ 3000 \cdot \bar{A}^{1}_{x:\overline{20}|}$$

Illustrative example 2

skip.

For a whole life insurance on (50) with death benefits payable at the moment of death, you are given:

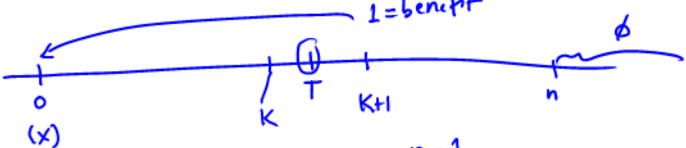
- Mortality follows De Moivre's law with $\omega = 110$.
- $b_t = 10000(1.10)^t$, for $t \geq 0$
- $\delta = 5\%$
- Z denotes the present value random variable for this insurance.

Calculate $\mathrm{E}[Z]$ and $\mathrm{Var}[Z]$.

Can you find an explicit expression for the distribution function of Z, i.e. $\Pr[Z \leq z]$?

continuous

discrete
(at end of year of death)



<u>n-year term</u>

$$PV_{r.v.} = \begin{cases} 1 \cdot v^{K+1}, & K = 0, 1, 2, \ldots, n-1 \\ \phi, & K \geq n \end{cases}$$

$$Z = v^{K+1}\, I(K<n)$$

$$APV = E[Z] = E\big[\underbrace{v^{K+1} I(K<n)}\big] = A^{1}_{x:\overline{n}|}$$

$$= \sum_{k=0}^{\infty} v^{k+1} \underbrace{I(k<n)} \cdot Pr\{K=k\} = \underbrace{\sum_{k=0}^{n-1} v^{k+1} Pr[K=k]}$$

$$Pr[K=k] = {}_{k|}q_x = {}_{k}p_x \cdot q_{x+k} = \frac{d_{x+k}}{l_x}$$

The APV of n-year term insurance, payable at the EOY of death, is

$$A^{\,1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \underbrace{{}_{k|}q_x}_{\frac{d_{x+k}}{l_x}} = \sum_{k=0}^{n-1} v^{k+1} \frac{d_{x+k}}{l_x}$$

no bar
$\Rightarrow$ discrete

$Z = $ PV r.v.

$$\mathrm{Var}(Z) = \underbrace{{}^2A^{\,1}_{x:\overline{n}|}}_{\substack{\text{evaluated @ } 2\delta \\ \text{evaluated @ } e^{2\delta}-1}} - \left(A^{\,1}_{x:\overline{n}|}\right)^2$$

rule of moments

$i \to \delta$

$i^* \to 2\delta$

$1+i^* = e^{2\delta}$

$i^* = e^{2\delta}-1$

$n \to \infty$ (discrete) whole life insurance

$Z = v^{K+1}$, no restriction K

$$\sum_{k=0}^{\infty} v^{k+1} \,{}_{k|}q_x = \sum_{k=0}^{\infty} v^{k+1} \frac{d_{x+k}}{l_x}$$

$E[v^{K+1}] = A_x$

$\mathrm{Var}(Z) = {}^2A_x - (A_x)^2$

$${}^2A_x = \sum_{k=0}^{\infty} e^{-2\delta(k+1)} \frac{d_{x+k}}{l_x}$$

mortality assumption

- Makeham (details in book)

- $\underline{\underline{i = .05}}$,

SULT

Standard Ultimate Life Table

e.g. APV of a whole life of 1 issued to

(i) 50 years old $\quad A_{50} = 0.18931$

(ii) $\underline{\underline{65 \text{ years}}}$ old $\quad A_{65} = 0.35477$

$$\mathrm{Var}(Z) = {}^2A_{50} - (A_{50})^2 = .05108 - (.18931)^2 > 0$$

5 years $\quad {}_5E_x$

10 years $\quad {}_{10}E_x$

20 years $\quad {}_{20}E_x$

Insurances payable at EOY of death

- For insurances payable at the end of the year (EOY) of death, the PV r.v. Z clearly depends on the curtate future lifetime K_x.
- It is $Z = b_{K+1} v_{K+1}$.
- To illustrate, consider an n-year term insurance which pays benefit at the end of year of death:

$$b_{K+1} = \begin{cases} 1, & K = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}, \; v_{K+1} = v^{K+1},$$

and therefore

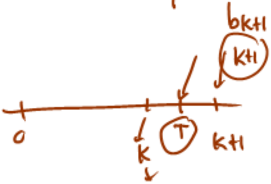
$$Z = \begin{cases} v^{K+1}, & K = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}.$$

$Z = b_T \cdot v_T$

Continuous (at the moment of death)

$Z = b_{K+1} \cdot v_{K+1}$

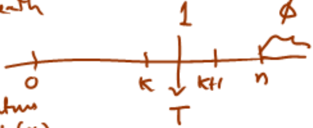
discrete (at the end of the year of death)



$v = \frac{1}{1+i}$ $\quad v_{K+1} = v^{K+1}$

$i = \text{constant}$

n-year term insurance payable at eoy of death



$b_{K+1} = 1$

K_x = curtate future lifetime of (x)

$Z = v^{K+1} I(K<n)$

$$\text{APV (benefits)} = E[Z] = \sum_{K=0}^{\infty} v^{K+1} \underbrace{I(K<n)}_{\substack{=1, \text{ if true} \\ =0}} P_r(K=k) = \underbrace{\sum_{K=0}^{n-1} v^{K+1} P_r[K=k]}$$

or EPV

$$APV(\text{n-yr term}) = \sum_{k=0}^{n-1} v^{k+1} \underbrace{\Pr[K=k]}_{\text{deferred probability}}$$

$${}_{k|}q_x = {}_kp_x \, q_{x+k}$$

$${}_kp_x = \frac{l_{x+k}}{l_x}$$

$$q_{x+k} = \frac{d_{x+k}}{l_{x+k}}$$

$$= \sum_{k=0}^{n-1} v^{k+1} \frac{\cancel{l_{x+k}}}{l_x} \cdot \frac{d_{x+k}}{\cancel{l_{x+k}}}$$

$$A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1} \frac{d_{x+k}}{l_x}$$

<u>3-year term</u>

x	l_x	
35	1000	5
36	995	10
37	985	20
38	965	

$B = 1$

3-year term

$$\underbrace{A^{1}_{35:\overline{3|}}}_{\Rightarrow B=1} = v \cdot \frac{5}{1000} + v^2 \frac{10}{1000} + v^3 \frac{20}{1000}$$

$\Rightarrow B = \underline{1}$

$i = 5\%$ let you calculate

$\lll 1$

$$Var[Z] = E[Z @ 2\delta] - (E[Z])^2$$

$$= \underbrace{{}^2A^1_{x:\overline{n}|}}_{2\delta} - (A^1_{x:\overline{n}|})^2$$

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \frac{d_{x+k}}{l_x}$$

$${}^2A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{2(k+1)} \frac{d_{x+k}}{l_x}$$

$\delta \to 2\delta$

$v = e^{-\delta}$

$v^* = e^{-2\delta} = v^2$

Back to our example:

$${}^2A^1_{35:\overline{3}|} = v^2 \frac{5}{1000} + v^4 \frac{10}{1000} + v^6 \frac{20}{1000}$$

check: $Var[Z] \geq 0$ non-negative

$n \to \infty$ whole life $Z = v^{K+1}$, $K = 0, 1, \ldots, \infty$ life table $\frac{d_{x+k}}{l_x}$

$$E[Z] = A_x = \sum_{k=0}^{\infty} v^{k+1} Pr[K=k]$$

$$Var[Z] = {}^2A_x - (A_x)^2$$

n-year endowment insurance

$$\underbrace{\text{endowment insurance}}_{\text{term insurance + pure endowment}}$$

$$Z = \underbrace{v^{K+1} I(K<n)} + \underbrace{v^{n} I(K \geq n)}$$

$$= v^{\min(K+1,\,n)}$$

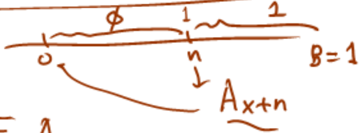
$$\longrightarrow {}_nE_x$$

$$E[Z] = A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$$

$$Var[Z] = {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2 \qquad \text{using } 2\delta \text{ rule}$$

---

deferred insurance



$$Z = v^{K+1} I(K \geq n)$$

$$E[Z] = {}_{n|}A_x = {}_nE_x\, A_{x+n}$$

$$Var[Z] = {}^2_{n|}A_x - \left({}_{n|}A_x\right)^2$$

$$\longrightarrow A_{x+n} \cdot {}_nE_x$$

$${}_{n|}A_x = \sum_{K=n}^{\infty} v^{K+1}\, {}_{K|}q_x$$

$K^* = K-n \Rightarrow$ replace K by K^*+n

$$ {}_{n|}A_x = \sum_{k=n}^{\infty} v^{k+1} \, {}_kp_x \, q_{x+k} \qquad k^* = k-n \Leftrightarrow k = k^*+n $$

$$ = \sum_{k^*=0}^{\infty} v^{k^*+1} \cdot \underbrace{v^n \, {}_np_x}_{{}_nE_x} \cdot {}_{k^*}p_{x+n} \, \underbrace{q_{x+k^*+n}}_{x+n+k^*} \qquad {}_{k^*+n}p_x = {}_np_x \cdot {}_{k^*}p_{x+n} $$

$$ = {}_nE_x \underbrace{\sum_{k^*=0}^{\infty} v^{k^*+1} \, {}_{k^*}p_{x+n} \, q_{x+n+k^*}}_{A_{x+n}} $$

- deferred = discounted whole life (starts later)
- endowment = term + pure endowment
- whole life = term + deferred

- continued

- APV of n-year term:

$$A^{1}_{x:\overline{n|}} = \mathsf{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x \cdot q_{x+k}$$

- Rule of moments also apply in discrete situations. For example,

$$\mathsf{Var}[Z] = {}^2A^{1}_{x:\overline{n|}} - \left(A^{1}_{x:\overline{n|}}\right)^2,$$

where

$${}^2A^{1}_{x:\overline{n|}} = \mathsf{E}\left[Z^2\right] = \sum_{k=0}^{n-1} e^{-2\delta(k+1)}\, {}_kp_x \cdot q_{x+k}.$$

Traditional insurances - discrete

Type	Benefit b_{K+1}	PV r.v. $Z = b_{K+1} v_{K+1}$	APV E$[Z]$	Variance Var$[Z]$
Term life	$I(K < n)$	$v^{K+1} \cdot I(K < n)$	$A^{\ 1}_{x:\overline{n}\rvert}$	${}^2A^{\ 1}_{x:\overline{n}\rvert} - \left(A^{\ 1}_{x:\overline{n}\rvert}\right)^2$
Whole life	1	v^{K+1}	A_x	${}^2A_x - (A_x)^2$
Endowment	1	$v^{\min(K+1,n)}$	$A_{x:\overline{n}\rvert}$	${}^2A_{x:\overline{n}\rvert} - \left(A_{x:\overline{n}\rvert}\right)^2$
Deferred	$I(K \geq n)$	$v^{K+1} \cdot I(K \geq n)$	${}_{n\rvert}A_x$	${}^{\ 2}_{n\rvert}A_x - \left({}_{n\rvert}A_x\right)^2$

$P_x \quad q_x \quad A_x$

SULT @ $i = 5\%$

Makeham's $\mu_x = A + Bc^x$, realistic,

$A = .00022$

$B = 2.7 \times 10^{-6}$

$c = 1.124$

$A_{x:\overline{10|}} \quad {}_{10}E_x$

term + pure

e.g. 10 year term insurance

$A^{1}_{x:\overline{10|}} = A_{x:\overline{10|}} - {}_{10}E_x$

$x = 40$

$A^{1}_{40:\overline{10|}} = \underbrace{A_{40:\overline{10|}}}_{.61494} - \underbrace{{}_{10}E_{40}}_{.60920} = .00574$ per dollar of benefit

If benefit is, say 100, $100 \times A^{1}_{40:\overline{10|}} = 0.574$

What if you need a term other than 10 or 20 years?

SULT ✓ Mortality follows the Standard Ultimate Life Table

✓ $i = 5\%$

Calculate the APV of a 30-year term insurance of 1 issued to (40).

$$\underbrace{A^{1}_{x:\overline{30|}} = A_{x:\overline{30|}} - {}_{30}E_x}$$

term $\underline{A^{1}_{40:\overline{30|}}} = A_{40} - \dfrac{{}_{30|}A_{40}}{{}_{30}E_{40}\, A_{70}}$

A_{40}: .12106

A_{70}: .42818

${}_{30}E_{40} = {}_{20}E_{40} \times {}_{10}E_{60}$

.36663 .57864

$= 0.03022299$

endowment $A_{40:\overline{30|}} = \underbrace{A^{1}_{40:\overline{30|}} + {}_{30}E_{40}}$

$= \underline{0.2423698}$ ✓

Mortality follows SULT @ $i=5\%$

$Z=$ PV of a whole life of 100 payable at eoy of death issued to (40)

Calculate $Var[Z]$.

$$E[Z] = \underbrace{100 \cdot A_{40}}_{.12106} = \underbrace{12.106}$$

$$E[Z^2] = 100^2 \cdot {}^2A_{40} = \underbrace{100^2 \cdot (.02347)}$$

$$Var[Z] = 100^2 [.02347 - (.12106)^2] = 88.14476$$

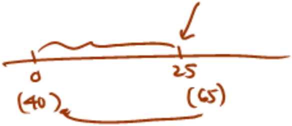
Calculate: 20-year term insurance of 100 to $(40) = 100 * A^{1}_{40:\overline{20|}} = \underline{1.463464}$

practice

Mortality follows SULT @ 5%

$Z =$ PV r.v. of a 25-year deferred of 100 to (40)
(payable at eoy of death)

Calculate $E[Z]$ and $Var[Z]$



$$E[Z] = 100 * {}_{25|}A_{40}$$

$$= 100 * {}_{25}E_{40} * A_{65}$$

$$= \underbrace{100 * \underbrace{{}_{20}E_{40}}_{.36663} * \underbrace{{}_{5}E_{60}}_{.76687} * \underbrace{A_{65}}_{.35477}}_{9.974626}$$

$^{2}A_{65} = .15420$

$E[Z^2] =$ same formula @ 2δ

$${}_nE_x = v^n \, {}_np_x$$

$${}_nE_x @ 2\delta = v^{2n} \, {}_np_x = \underbrace{\underbrace{v^n} \cdot {}_nE_x}$$

$$\underbrace{v^n} \cdot \underbrace{v^n \, {}_np_x}_{{}_nE_x}$$

$$v = \frac{1}{1.05}$$

$$E[Z^2] = 100^2 * {}_{25}E_{40} @ 2\delta * {}^2A_{65}$$

$$= 100^2 * v^{25} \cdot {}_{25}E_{40} * {}^2A_{65} \rightarrow .15420$$

$${}_{25}E_{40} = {}_{20}E_{40} * {}_5E_{60}$$

$${}_{20}E_{40} = .36663 \qquad {}_5E_{60} = .76687$$

$$v^{25} = \left(\frac{1}{1.05}\right)^{25}$$

$$= **$$

$$Var[Z] = ** - (9.974626)^2 = ***$$

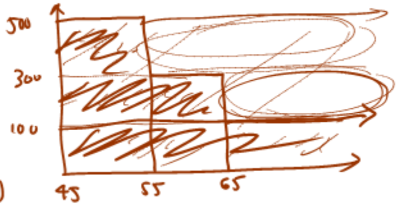
practice

varying payments
end of year of death
to (45)

500 in the first 10 years
300 in the following 10 years
100 thereafter

$\underbrace{\hphantom{500 \text{ in the first 10 years}}}_{\text{sums of whole life}}$

Calculate APV!



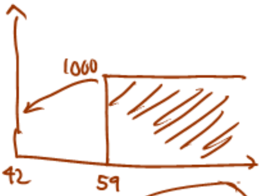
$$APV = 500 A_{45} - 200\, {}_{10}E_{45} A_{55} - 200\, {}_{20}E_{45} A_{65}$$

$$= \underbrace{500(.15161) - 200(.60655)(.23524) - 200(.35994)(.35477)}_{21.72885}$$

in terms of term insurance

$$= 100 A_{45} + 200 \underbrace{A^{1}_{45:\overline{20}|}}_{(A_{45:\overline{20}|} - {}_{20}E_{45})} + 200 \underbrace{A^{1}_{45:\overline{10}|}}_{(A_{45:\overline{10}|} - {}_{10}E_{45})}$$

17-year deferred to (42) of 1000



APV (benefit) = ??

SULT @ 5%

$$1000 * {}_{17|}A_{42} = 1000 * \underbrace{{}_{17}E_{42}}\, A_{59} \rightarrow 118.7005$$

$A_{59} \rightarrow .27852$

Method 1: $${}_{17}E_{42} = v^{17}\,{}_{17}p_{42} = \left(\frac{1}{1.05}\right)^{17} \frac{l_{59}}{l_{42}} \quad \frac{96929.6}{99229.8}$$

Method 2: $${}_{17}E_{42} = {}_{10}E_{42} * {}_{5}E_{52} * {}_{2}E_{57}$$

${}_{10}E_{42}$: .60832

${}_{5}E_{52}$: .77643

${}_{2}E_{57} = v^2\,{}_2p_{57} = \left(\frac{1}{1.05}\right)^2 \frac{l_{59}}{l_{57}}$ 96929.6 / 97435.2

Recursive formula / one year to the next –

$$\underbrace{A_x} = A^1_{x:\overline{1|}} + \underbrace{{}_1E_x} \cdot \underbrace{A_{x+1}}$$

$${}_nE_x = v^n \, {}_np_x$$

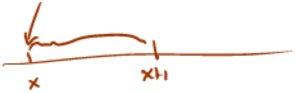
$$= \underbrace{v \cdot q_x} + v \cdot p_x \underbrace{A_{x+1}}$$

$$\boxed{A_x = v q_x + v p_x A_{x+1}}$$

$$\boxed{A_{x+1} = \frac{A_x - v q_x}{v p_x}}$$

commonly used /

VIF / intuitive



$$A_x = \underbrace{v q_x + v p_x A'_{x+1}}$$

$$^2A_x = v^2 q_x + v^2 p_x \, \underbrace{^2A_{x+1}}_{2\delta}$$

works!

$$A'_x \longleftrightarrow A'_{x+1}$$

x $\quad$ $x+1$

$v p_x = {}_1E_x$

$$\boxed{A_x = 1 \cdot v q_x + v p_x A_{x+1}}$$

v LF –

$v = e^{-\delta}$

$v^2 = e^{-2\delta}$

2nd moment $\leftarrow$ $$^2A_x = v^2 \cdot q_x + v^2 p_x \, {}^2A_{x+1}$$

<u>whole life</u>

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \cdot \underbrace{{}_{k|}q_x}_{{}_kp_x\, q_{x+k}}$$

$E[Z]$

$\downarrow$

v^{k+1}

$$= v q_x + \underbrace{\sum_{k=1}^{\infty} v^{k+1}\, {}_kp_x\, q_{x+k}}$$

$k^* = k-1 \rightarrow k = k^*+1$

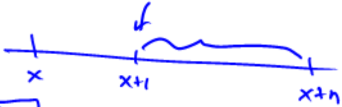
$$\sum_{k^*=0}^{\infty} v^{k^*+1+1}\, \underbrace{{}_{k^*+1}p_x}\, q_{x+1+k^*}$$

$$\underline{p_x} \cdot {}_{k^*}p_{x+1}\, q_{x+1+k^*}$$

$$= v q_x + v p_x \underbrace{\sum_{k^*=0}^{\infty} v^{k^*+1}\, {}_{k^*}p_{x+1}\, q_{x+1+k^*}}_{A_{x+1}}$$

$$A_x = v q_x + v p_x A_{x+1}$$

term insurance



$$A^1_{x:\overline{n}|} = v\,q_x + v\,p_x\,A^1_{x+1:\overline{n-1}|}$$

endowment

$$A_{x:\overline{n}|} = v\,q_x + v\,p_x\,A_{x+1:\overline{n-1}|}$$

$${}_nE_x = {}_1E_x \cdot {}_{n-1}E_{x+1}$$

~~continuous~~ $$A_x = A^1_{x:\overline{1}|} + {}_1E_x\,A_{x+1}$$

continuous $$\bar{A}_x = \bar{A}^1_{x:\overline{1}|} + {}_1E_x\,\bar{A}_{x+1}$$

$$\underbrace{\int_0^1 v^t\,{}_tp_x\,\mu_{x+t}\,dt}_{vs}$$

$$v \cdot q_x$$

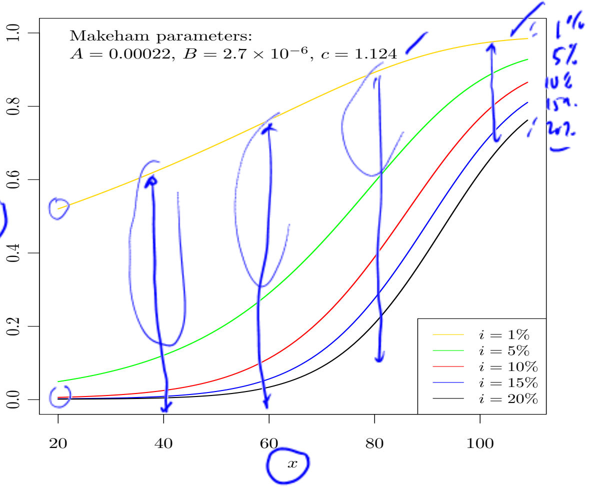
Recursive relationships

- The following will be derived/discussed in class:
  - whole life insurance: $A_x = vq_x + vp_x\, A_{x+1}$
  - term insurance: $A^{1}_{x:\overline{n|}} = vq_x + vp_x\, A^{\;\;1}_{x+1:\overline{n-1|}}$
  - endowment insurance: $A_{x:\overline{n|}} = vq_x + vp_x\, A_{x+1:\overline{n-1|}}$

develop recursion for a deferred -

A_x

1



fix mortality

Figure: Actuarial Present Value of a discrete whole life insurance for various interest rate assumptions fix intest

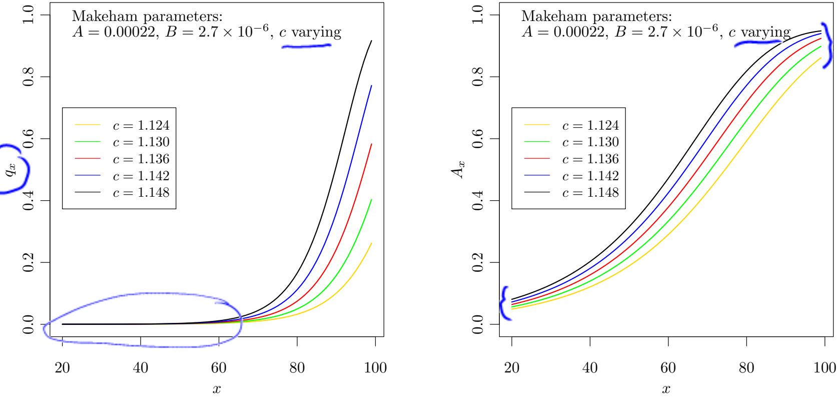


Figure: Actuarial Present Value of a discrete whole life insurance for various mortality rate assumptions with interest rate fixed at 5%

Illustrative example 3

$Var[Z] = {}^2A_{41} - (A_{41})^2$

For a whole life insurance of 1 on (41) with death benefit payable at the end of the year of death, let Z be the present value random variable for this insurance.

You are given:

- $i = 0.05$;
- $p_{40} = 0.9972$;
- $A_{41} - A_{40} = 0.00822$; and
- ${}^2A_{41} - {}^2A_{40} = 0.00433$.

Calculate Var$[Z]$.

$v = 1/1.05$

$p_{40} = .9972$

$q_{40} = .0028$

recursion

$A_{41} - A_{40} = .00822$

$A_{41} - (vq_{40} + vp_{40}A_{41}) = .00822$

$A_{41}(1 - vp_{40}) = \dfrac{.00822 + vq_{40}}{1 - vp_{40}}$

$A_{41} = .2169621$

$${}^2A_{41} - \underbrace{{}^2A_{40}}_{} = .00433 \qquad v^2 = 1/1.05^2$$

$$\left(v^2 q_{40} + v^2 p_{40}\, {}^2A_{40}\right) = .00433$$

$${}^2A_{41}\cancel{\left(1 - v^2 p_{40}\right)} = \frac{.00433 - v^2 q_{40}}{1 - v^2 p_{40}} \quad .0028 \quad .9972$$

$${}^2A_{41} = .0712616$$

$$Var[Z] = {}^2A_{41} - (A_{41})^2$$

$$= .0712616 - (.2169621)^2 \approx \underline{\underline{0.025}}$$

Other forms of insurance

- Varying benefit insurances
- Very similar to the continuous cases
- You are expected to read and understand these other forms of insurances.
- It is also useful to understand the various (possible) recursion relations resulting from these various forms.

Illustration of varying benefits

endowment

For a special life insurance issued to (45), you are given:

- Death benefits are payable at the end of the year of death. discrete
- The benefit amount is \$100,000 in the in the first 10 years of death, decreasing to \$50,000 after that until reaching age 65.
- An pure endowment benefit of \$100,000 is paid if the insured reaches age 65.
- There are no benefits to be paid past the age of 65.
- Mortality follows the `Standard Ultimate Life Table` at $i = 0.05$. SULT

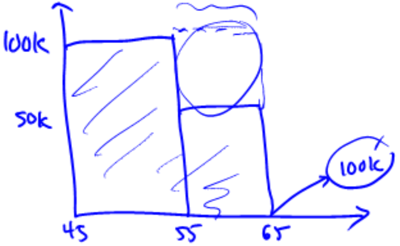
Calculate the actuarial present value (APV) for this insurance.

SULP

whole life insurance,

10 year endowment

20 year endowment



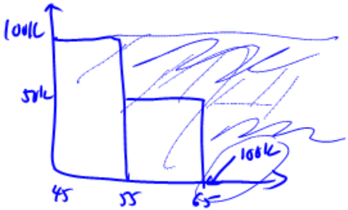
$$APV(\text{benefit}) = 100000 \, \overset{.38385}{A_{45:\overline{20}|}}$$

$$- 50000 \; \underset{.60655}{{}_{10}E_{45}} \; A^1_{55:\overline{10}|}$$

$$\left(A_{55:\overline{10}|} - {}_{10}E_{55}\right) \quad .35477$$

$$\left(\underset{.23524}{A_{55}} - \underset{.59344}{{}_{10}E_{55}} A_{65}\right)$$

$$= \underline{\underline{37{,}635.96}}$$



$$APV(\text{benefit}) = 100000\, A_{45} - 50000\, {}_{10}E_{45} A_{55} - 50000\, {}_{20}E_{45} A_{65} + 100000\, {}_{20}E_{45}$$

verify the same result!

Illustrative example 4

do this later!

For a whole life insurance issued to age 40, you are given:

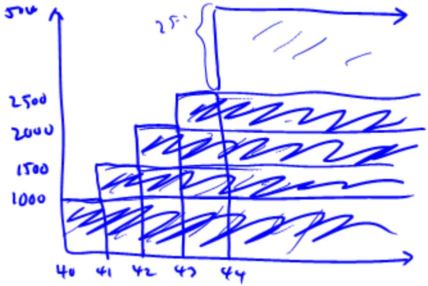
- Death benefits are payable at the moment of death.
- The benefit amount is \$1,000 in the first year of death, increasing by \$500 each year thereafter for the next 3 years, and then becomes level at \$5,000 thereafter.
- Mortality follows the `Standard Ultimate Life Table` at $i = 0.05$.
- Deaths are uniformly distributed over each year of age. UDD approximate

Calculate the APV for this insurance.

(40)

moment of death

SULT @ $i=5\%$



$$APV(\text{benefits}) = 1000\,\bar{A}_{40} + 500\,{}_1E_{40}\,\bar{A}_{41} + 500\,{}_2E_{40}\,\bar{A}_{42} + 500\,{}_3E_{40}\,\bar{A}_{43} + 2500\,{}_4E_{40}\,\bar{A}_{44}$$

$$= 500\cdot\frac{\overset{.05}{i}}{\underset{\log(1.05)}{\delta}}\Big[2A_{40} + {}_1E_{40}\,A_{41} + {}_2E_{40}\,A_{42} + {}_3E_{40}\,A_{43} + 5\,{}_4E_{40}\,A_{44}\Big]$$

$A_{40} = .12106$ $\quad A_{42} = .13249$ $\quad A_{44} = .14496$

$A_{41} = .12663$ $\quad A_{43} = .13859$

$${}_1E_{40} = v\,p_{40}$$

$$p_{40} = 1 - q_{40} = l_{41}/l_{40}$$

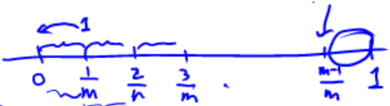
$${}_2E_{40} = v^2 p_{40} \cdot p_{41} = v^2\, l_{42}/l_{40}$$

$${}_3E_{40} = v^3 p_{40}\, p_{41}\, p_{42} = v^3\, l_{43}/l_{40}$$

$${}_4E_{40} = v^4 p_{40}\, p_{41}\, p_{42}\, p_{43} = v^4\, l_{44}/l_{40}$$

verify $\boxed{APV(\text{benefits}) = 613.4042}$

Just consider 1 year term insurance,



$$A^{(m)\,1}_{x:\overline{1|}} = v^{1/m}\,{}_{1/m}q_x + v^{1/m}\,{}_{1/m}p_x\, v^{1/m}\,{}_{1/m}q_{x+1/m} + v^{2/m}\,{}_{2/m}p_x\, v^{1/m}\,{}_{1/m}q_{x+2/m} + \cdots$$

monthly $m=12$
semiannual $m=2$
quarterly $m=4$

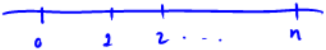
$$= \sum_{r=0}^{m-1} v^{(r+1)/m}\, \underbrace{{}_{\frac{r}{m}}p_x\ {}_{\frac{1}{m}}q_{x+\frac{r}{m}}}_{\frac{1}{m}\cdot q_x}$$

UDD assumption

$$= q_x \cdot \frac{1}{m} \underbrace{\sum_{r=0}^{m-1} v^{(r+1)/m}}_{v^{1/m} + v^{2/m} + \cdots + v^{m/m}} = \underbrace{q_x \cdot \frac{1}{m} \cdot v^{1/m}\, \frac{1-v^{m/m}}{1-v^{1/m}}}_{} = q_x\, v \cdot \frac{i}{i^{(m)}}$$

$$A^{(m)\,1}_{x:\overline{n}|} \approx \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}$$

adjustment



$$A^{(m)\,1}_{x:\overline{n}|} = \underbrace{A^{(m)\,1}_{x:\overline{1}|}} + {}_1E_x \underbrace{A^{(m)\,1}_{x+1:\overline{1}|}} + {}_2E_x \underbrace{A^{(m)\,1}_{x+2:\overline{1}|}} + \cdots$$

$$\approx \frac{i}{i^{(m)}} \Big[\underbrace{A^{1}_{x:\overline{1}|} + {}_1E_x A^{1}_{x+1:\overline{1}|} + {}_2E_x A^{1}_{x+2:\overline{1}|} + \cdots}_{A^{1}_{x:\overline{n}|}} \Big]$$

exact UDD

$n \to \infty$

$$\lim_{m\to\infty} i^{(m)} \to \delta$$

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$$

$m \to \infty \Rightarrow$ continuum

$$\bar{A}_x = \frac{i}{\delta} A_x$$

apply $\frac{i}{i^{(m)}}$ or $\frac{i}{\delta}$ only on insurances but not on pure endowment

$$A^{(m)\,1}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}$$

$$A^{(m)}_{x:\overline{n}|} = \underbrace{A^{(m)\,1}_{x:\overline{n}|}}_{\frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}} + A_{x:\overline{n}|}^{\;\;1}$$

$$\frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$$

endowment does not have a similar adjustment

$$\bar{A}_{x:\overline{n}|} = \underbrace{\bar{A}^{1}_{x:\overline{n}|}} + A_{x:\overline{n}|}^{\;\;1}$$

$$= \frac{i}{\delta} A_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$$

Insurances payable m-thly

- Consider the case where we have just one-year term and the benefit is payable at the end of the m-th of the year of death.
- We thus have

$$A^{(m)}_{\substack{1\\x:\overline{1}|}} = \sum_{r=0}^{m-1} v^{(r+1)/m} \cdot {}_{r/m}p_x \cdot {}_{1/m}q_{x+r/m}.$$

- We can show that under the UDD assumption, this leads us to:

$$A^{(m)}_{\substack{1\\x:\overline{1}|}} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{1}|}.$$

n-year

- In general, we can generalize this to:

$$A^{(m)}_{\substack{1\\x:\overline{n}|}} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}.$$

Other types of insurances with m-thly payments

- For other types, we can also similarly derive the following (with the UDD assumption):
  - whole life insurance: $A_x^{(m)} = \dfrac{i}{i^{(m)}} A_x$
  - deferred life insurance: ${}_{n|}A_x^{(m)} = \dfrac{i}{i^{(m)}} {}_{n|}A_x$
  - endowment insurance: $A_{x:\overline{n|}}^{(m)} = \dfrac{i}{i^{(m)}} A^{1}_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}}$

$$ {}_{n|}A_x^{(m)} = \underline{{}_nE_x}\, \underbrace{A_{x+n}^{(m)}}_{\frac{i}{i^{(m)}} A_{x+n}} $$

$$ = \frac{i}{i^{(m)}}\, {}_{n|}A_x $$

Relationships - continuous and discrete

$\frac{i}{i^{(m)}} \rightarrow \frac{i}{\delta}$

- For some forms of insurances, we can get explicit relationships under the UDD assumption:
  - whole life insurance: $\bar{A}_x = \frac{i}{\delta} A_x$
  - term insurance: $\bar{A}^1_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|}$
  - increasing term insurance: $(I\bar{A})^1_{x:\overline{n}|} = \frac{i}{\delta} (IA)^1_{x:\overline{n}|}$

$(IA^{(m)})^1_{x:\overline{n}|}$

$(\bar{I}\bar{A})^1_{x:\overline{n}|}$

$(IA)^1_{x:\overline{n}|}$ discrete

1, 2, 3, ..., n

payable at moment of death

A_x $\quad \bar{A}_x$ $\quad A_x^{(m)}$ $\qquad$ compare.

$$\frac{i}{\delta}A_x \qquad \frac{i}{i^{(m)}}A_x$$

$$\bar{A}_x > A_x^{(m)} > A_x$$

$A^{(6)}$ $\quad A^{(4)}$ $\quad A^{(12)}$

$$A_x^{(12)} > A_x^{(6)} > A_x^{(4)}$$

Illustrative example 5

issue age

first select

For a three-year term insurance of 1000 on $[50]$, you are given:

- Death benefits are payable at the end of the quarter of death.
- Mortality follows a select and ultimate life table with a two-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	9706	9687	9661	52
51	9680	9660	9630	53
52	9653	9629	9596	54

19 26 31

- Deaths are uniformly distributed over each year of age.
- $i = 5\%$

Calculate the APV for this insurance.

3-year term of 1000 to [50]

end of quarter

$m=4$

$$1000\, A^{(4)\,1}_{[50]:\overline{3}|} = 1000\, \frac{i}{i^{(4)}}\, \underbrace{A^{\,1}_{[50]:\overline{3}|}}$$

$$= 1000\, \frac{i}{i^{(4)}} \left[v \cdot \frac{d_{[50]}}{\ell_{[50]}} + v^2\, \frac{d_{[50]+1}}{\ell_{[50]}} + v^3\, \frac{d_{[50]+2}}{\ell_{[50]}} \right]$$

0 1 2 3

$i = .05$

$i^{(4)}$

$\left(1+\frac{i^{(4)}}{4}\right)^4 = 1+i$

$i^{(4)} = \underbrace{4\left[(1.05)^{1/4}-1\right]}$

$= \ldots$

$$\underbrace{= 1000\, \frac{i}{i^{(4)}} \left[v \cdot \frac{19}{9706} + v^2\, \frac{26}{9706} + v^3\, \frac{31}{9706} \right]}$$

$$= 7.183958$$

vs 1000

$v = \frac{1}{1.05}$

reasonable

select

quarter

life table

J = the number of completed periods in the year of death

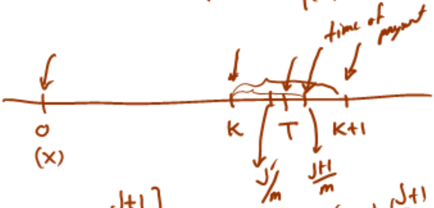
$J = \{\underbrace{0, 1, \dots, m-1}\}$

APV of this insurance payable m-thly

UDD in year of death

$\Pr(J=j) = \frac{1}{m}$

approximation $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$



$$A_x^{(m)} = E\left[v^{K+\frac{J+1}{m}}\right] \qquad v^{(K+1)+\left(\frac{J+1}{m}-1\right)}$$

J, K are independent

$$= E\left[v^{K+1} \cdot v^{\frac{J}{m}} \cdot v^{\frac{1}{m}-1}\right]$$

$$= \underbrace{v^{\frac{1}{m}-1}} \; \underbrace{E\left[v^{K+1}\right]}_{A_x} \; \underbrace{E\left[v^{J/m}\right]}_{\frac{1}{m}\left(\underbrace{v^{0/m}+v^{1/m}+\dots+v^{m-1/m}}_{\frac{1-v^{\frac{1}{m}\cdot m}}{1-v^{1/m}}}\right)}$$

$$A_x^{(m)} = v^{\frac{1}{m}\cancel{-1}} \cdot A_x \cdot \frac{1-v}{1-v^{1/m}}$$

$d = i/1+i$

$= i\cancel{v}$

$1-d = v$

$d = \frac{i}{1+i}$

$$= i \cdot A_x \underbrace{\left(\frac{v^{1/m}}{1-v^{1/m}}\right)}_{\frac{1}{\frac{1-v^{1/m}}{v^{1/m}}}}$$

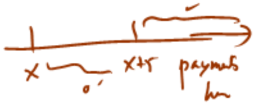
$i^{(m)}$

financial math

$$\boxed{A_x^{(m)} = \frac{i}{i^{(m)}} A_x}$$

Illustrative example 6

nice problem



Continuous

Each of 100 independent lives purchases a single premium 5-year deferred whole life insurance of 10 payable at the moment of death.

You are given:

- $\mu = 0.004$
- $\delta = 0.006$
- F is the aggregate amount the insurer receives from the 100 lives.
- The 95th percentile of the standard Normal distribution is 1.645.

$Z = Z_1 + Z_2 + \cdots + Z_{100} \sim$ Normal because CLT

$Z_i \sim$ PV rv

$E[Z_1 + \cdots + Z_{100}] = 100\, E[Z_i]$

$Var[Z_1 + \cdots + Z_{100}] = 100\, Var(Z_i)$

Using a Normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

$Z =$ total claims $\qquad F =$ collect fixed amount

$Pr[F \geq Z] = 0.95 \Rightarrow Pr[Z \leq F] = 0.95$

$\Rightarrow Pr\left[\underbrace{\frac{Z-E[Z]}{\sqrt{Var[Z]}}} \leq \underbrace{\frac{F-E[Z]}{\sqrt{Var(Z)}}}\right] = 0.95$

$N \sim N(0,1)$

standard normal

$Pr[N \leq \;\;] = 0.95$

≈ 1.645

${}_5E_x = v^5 {}_5p_x$

$= e^{-5(.006+.004)}$

$= e^{-.05}$

.95 th of $N(0,1)$

$= 1.645$

${}_nE_x = v^n {}_np_x$ ✓

${}_nE_x$ @ 2δ

$= v^n \cdot {}_nE_x$

$$\frac{F-E[Z]}{\sqrt{Var(Z)}} = 1.645$$

$$F = E[Z] + 1.645\sqrt{Var(Z)}$$

100*

$= \left(4e^{-.05} + 1.645\sqrt{10^2 e^{-.08}\frac{1}{4} - 10^2\left(\frac{2}{5}\right)^2 e^{-.10}}\right)$

100*

$= \left(4e^{-.05} + 1.645(10)e^{-.05}\sqrt{e^{.02}\left(\frac{1}{4}\right) - \left(\frac{4}{25}\right)}\right)$

$= 860$

$\frac{.004}{.004+2(.003)}$

$\frac{4}{16} = \frac{1}{4}$

$\frac{860}{100} = 8.6$

5yr deferred WL of 10

$E[Z_i] = 10\; {}_{5|}\bar{A}_x$

$= 10 \cdot {}_5E_x \bar{A}_{x+5}$

$= 10\left(\frac{2}{5}\right)e^{-.05}$ ✓

$\frac{\mu}{\mu+\delta} = \frac{.004}{.010}$

$= \frac{4}{10}$

$= \frac{2}{5}$

$E[Z_i^2] = 10^2 \cdot {}_{5|}{}^2\bar{A}_x = 10^2 v^5 {}_5E_x {}^2\bar{A}_x$

$\frac{\mu}{\mu+2\delta}$

$= 10^2 e^{-5(.006)} e^{-.05} \frac{1}{4}$

$e^{-.08}$

If you pay the APV to cover losses, what is the probability that insurer has enough funds to cover your losses?

$$Z = Z_1 + Z_2 + Z_3 + \dots + Z_m$$

$$F = \underbrace{E[Z_1] + E[Z_2] + \dots}_{E[Z]}$$

$$\Pr[F \geq Z] = \Pr[Z \leq F]$$

$$\approx \Pr\left[\underbrace{\frac{Z - E[Z]}{\sqrt{Var(Z)}}}_{N} \leq \frac{\overset{E[Z]}{F} - E[Z]}{\sqrt{Var(Z)}}\right]$$

$$\Pr[N \leq 0] = 1/2$$

Illustrative example 7

$v = \frac{1}{1.06}$

Suppose interest rate $i = 6\%$ and mortality is based on the following life table:

deaths: 60, 60, 60, 60, 60, 60, 60, 60, 220, 100

x	90	91	92	93	94	95	96	97	98	99	100
ℓ_x	800	740	680	620	560	500	440	380	320	100	0

Calculate the following:

(a) $A_{94} = \left(v \cdot \frac{60}{560} + v^2 \frac{60}{560} + v^3 \frac{60}{560} + v^4 \frac{60}{560}\right) + v^5 \frac{220}{560} + v^6 \frac{100}{560} = 0.797128$

(b) $A^{1}_{90:\overline{5}|} = v \frac{60}{800} + v^2 \frac{60}{800} + v^3 \frac{60}{800} + v^4 \frac{60}{800} + v^5 \frac{60}{800} = 0.3159273$

(c) ${}_{3|}A^{(4)}_{92}$, assuming UDD between integral ages $= \frac{i}{i^{(4)}}\left[v^4 \frac{60}{680} + v^5 \frac{60}{680} + v^6 \frac{60}{680} + v^7 \frac{220}{680} + v^8 \frac{100}{680}\right]$

$= 0.5166944$

(d) $A_{95:\overline{3}|} = A^{1}_{95:\overline{3}|} + A_{95:\overline{3}|}^{\ \ 1}$

$= \left(v \frac{60}{500} + v^2 \frac{60}{500} + v^3 \frac{60}{500}\right) + \left(v^3 \cdot \frac{320}{500}\right) = 0.8581178$

Illustrative example 8

5-year term

A five-year term insurance policy is issued to (45) with benefit amount of $10,000 payable at the end of the year of death. disct

Mortality is based on the following select and ultimate life table:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
45	5282	5105	4856	4600	48
46	4753	4524	4322	4109	49
47	4242	4111	3948	3750	50
48	3816	3628	3480	3233	51

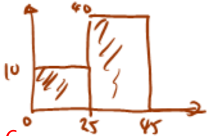
$i=5\%$

Calculate the APV for this insurance if $i = 5\%$.

Practice: $10000 \; A_{[45]:\overline{5}|}$

Illustrative example 9

continuous



Note: This is a modified question from SOA Spring 2016 exam.

A life insurance policy is issued to (35) with present value random variable:

$$Z = \begin{cases} 10v^{T_{35}}, & 0 < T_{35} \le 25 \\ 40v^{T_{35}}, & 25 < T_{35} \le 45 \\ 0, & T > 45 \end{cases}$$

varying payment
issued to (35)
with maturity 45 years

You are also given:

- Mortality follows the `Standard Ultimate Life Table`.
- Deaths are uniformly distributed over each year of age. UDD
- $i = 0.05$

Calculate the expected value and the standard deviation of Z.

$$APV = E[Z] = 10\bar{A}_{35} + 30\,{}_{25}E_{35}\,\bar{A}_{60} - 40\,{}_{45}E_{35}\,\bar{A}_{80}$$

.05

$$= 10\,\frac{i}{\delta}\left[A_{35} + 3\,{}_{20}E_{35}\,{}_{5}E_{55}\,A_{60} - 4\,{}_{20}E_{35}\,{}_{20}E_{55}\,{}_{5}E_{75}\,A_{80}\right]$$

log(1.05)

.09653 .37041 .77382 .29028 .37041 .32819 .69574 .59293

$$\boxed{1.49153}$$

$$E[Z^2] = 10^2 \cdot \frac{i@2\delta}{\delta@2\delta}\left[{}^2A_{35} + 3^2\,v^{25}\,{}_{20}E_{35}\,{}_{5}E_{55}\,{}^2A_{60} - 4^2\,v^{45}\,{}_{20}E_{35}\,{}_{20}E_{55}\,{}_{5}E_{75}\,{}^2A_{80}\right]$$

.01601 .10834 .38134

$i@2\delta$

$\downarrow$

$v \to v^2$

$\downarrow$

$i \to (1+i)^2 - 1$

$i \to i^*$

$v \to v^* = \frac{1}{1+i^*}$

$$\frac{1.05^2 - 1}{2\delta}$$

4.318038

$$Var(Z) = 4.318038 - (1.49153)^2 = 2.093376$$

$$\sqrt{Var(Z)} = \sqrt{2.093376} = 1.44685 \approx 1.45$$

Other terminologies and notations used

$Z = PV_{r.v.}$

Expression	Other terms/symbols used
$E[Z]$ Actuarial Present Value (APV)	Expected Present Value (EPV) ✓ Net Single Premium (NSP) ✓ single benefit premium ✓
basis ✓	assumptions ✓
interest rate (i) ✓	interest per year effective ✓ discount rate
benefit amount (b)	sum insured (S) ✓ death benefit ✓
Expected value of Z ✓	E(Z) ✓ $E[Z]$
Variance of Z	Var(Z) $V[Z]$ $Var[Z]$

d ✓