mortality

Life Tables and Selection

Lecture: Weeks 4-5

Chapter summary

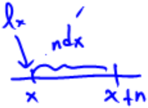
integral ages

15
16
17
⋮

- What is a life table?
  - also called a mortality table
  - tabulation of basic mortality functions
  - deriving probabilities/expectations from a life table
- Relationships to survival functions
- Assumptions for fractional (non-integral) ages
- Select and ultimate tables
  - national life tables
  - valuation or pricing tables
- Chapter 3, DHW

What is the life table?

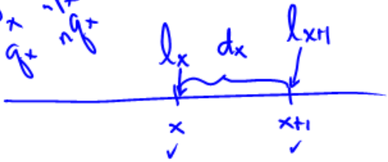
- A tabular presentation of the mortality evolution of a cohort group of lives.
- Begin with ℓ_0 number of lives (e.g. 100,000) - called the radix of the life table.
- (Expected) number of lives who are age x: $\ell_x = \ell_0 \cdot S_0(x) = \ell_0 \cdot {}_xp_0$
- (Expected) number of deaths between ages x and $x+1$: $d_x = \ell_x - \ell_{x+1}$.
- (Expected) number of deaths between ages x and $x+n$: ${}_nd_x = \ell_x - \ell_{x+n}$.



- Conditional on survival to age x, the probability of dying within n years is: ${}_nq_x = {}_nd_x/\ell_x = (\ell_x - \ell_{x+n})/\ell_x$. $= 1 - \frac{\ell_{x+n}}{\ell_x} = 1 - {}_np_x$
- Conditional on survival to age x, the probability of living to reach age $x+n$ is: ${}_np_x = 1 - {}_nq_x = \ell_{x+n}/\ell_x$.

p_x ${}_np_x$

q_x ${}_nq_x$



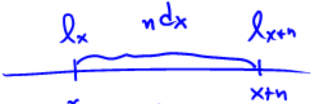
l_0 = radix = fixed

eg 100,000

l_x = (expected) number of lives who reach age x (who are alive at beginning of age x)

d_x = number of deaths between ages x & $x+1$

$= l_x - l_{x+1}$



${}_nd_x$ = number of deaths between age x & $x+n$

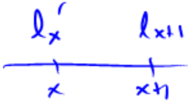
$= l_x - l_{x+n}$

$n=1$

${}_1d_x = d_x$

$$l_x = \underline{l_0} \cdot \underbrace{Pr(X>x)}_{\underline{S_0(x)}}$$

l_x is non-increasing in x (decreasing)



$$
\begin{aligned}
l_x &\geq l_{x+1} \\
&\geq l_{x+2} \\
&\geq l_{x+3} \\
&\geq \vdots \\
&\geq l_{x+n}
\end{aligned}
$$

Example of a life table

$q_x = \dfrac{d_x}{l_x}$

$q_x + p_x = 1$

d_0

0 1

x	ℓ_x	d_x	q_x	p_x	$\mathring{e}_x$
0	100,000	680	0.006799	0.993201	77.84
1	99,320	48	0.000483	0.999517	77.37
2	99,272	29	0.000297	0.999703	76.41
3	99,243	22	0.000224	0.999776	75.43
⋮	⋮	⋮	⋮	⋮	⋮
50	93,735	413	0.004404	0.995596	30.87
51	93,323	443	0.004750	0.995250	30.01
52	92,879	475	0.005113	0.994887	29.15
53	92,404	507	0.005488	0.994512	28.30
⋮	⋮	⋮	⋮	⋮	⋮
97	5,926	1,370	0.231201	0.768799	3.15
98	4,556	1,133	0.248600	0.751400	2.95
99	3,423	913	0.266786	0.733214	2.76

5 age year

2 age year

complete expectation of life

or

average lifetime

$50 + 30.87 = 80.37$

raw task

$${}_2p_{50} = \frac{l_{52}}{l_{50}} = \frac{92879}{93735} = p_{50} \cdot p_{51}$$

lag: 5 or maybe more

old

Source: U.S. Life Table for the total population, 2004, Center for Disease Control and Prevention (CDC)

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Radix of the life table

- The radix of the life table does not have to start at age 0, e.g. start with age x_0, so that the table starts with radix ℓ_{x_0}.
- The limiting age of the table is usually denoted by ω, in which case the table gives entries for only a period of $\omega - x_0$.
- All the formulas still work, e.g. conditional on survival to age x, the probability of surviving to reach age $x+n$ is:

$$ {}_np_x = 1 - {}_nq_x = \frac{\ell_{x+n}}{\ell_x}. $$

- Note that among ℓ_x independent lives who have reached age x, the number of survivors $\mathcal{L}_n$ within n years is a Binomial random variable with parameters ℓ_x and ${}_np_x$ so that

$$ \mathsf{E}(\mathcal{L}_n) = \ell_x \cdot {}_np_x. $$

l_x = expected

l_x $\mathcal{L}_n$

x $\quad$ $x+n$

1 or 0

survive $\quad$ die

$\mathcal{L}_n$ = number of survivors out of l_x lives at age x

with probability ${}_np_x$

$\mathcal{L}_n \sim$ ~~Bernoulli~~ Binomial random variable $\sim \text{bin}(n, p)$

$n = l_x$

$p = {}_np_x$

$n \cdot p \cdot (1-p)$

$$E[\mathcal{L}_n] = l_x \cdot \underbrace{{}_np_x}_{\frac{l_{x+n}}{l_x}} = l_{x+n}$$

$$\mathrm{Var}[\mathcal{L}_n] = l_x \cdot {}_np_x \cdot (1 - {}_np_x) = l_{x+n}\left(\frac{l_x - l_{x+n}}{l_x}\right)$$

Revised example 3.1

x	l_x	d_x
30	10000	34.78
31	9965.22	38.10
32	...	...
⋮		
39	9434.08	80.11

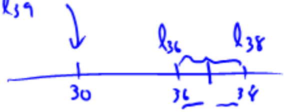
Using Table 3.1, page 43 of DHW, calculate the following:

- the probability that (30) will survive another 5 years
- the probability that (39) will survive to reach age 40
- the probability that (30) will die within 10 years
- the probability that (30) dies between ages 36 and 38

$$= {}_5p_{30} = \frac{l_{35}}{l_{30}} = \frac{9789.29}{10000}$$

$$p_{39} = \frac{l_{40}}{l_{39}} = \frac{9434.08 - 80.11}{9434.08} \quad \begin{matrix} \leftarrow l_{40} \\ \leftarrow l_{39} \end{matrix}$$

$${}_{10}q_{30} = 1 - {}_{10}p_{30} = 1 - \frac{l_{40}}{l_{30}}$$



$${}_{6|2}q_{30} = \frac{l_{36} - l_{38}}{l_{30}}$$

$${}_{6|2}q_{30} = \frac{l_{36} - l_{38}}{l_{30}}$$

$$= \frac{d_{36} + d_{37}}{l_{30}}$$

$${}_nd_x = l_x - l_{x+n}$$

$$= d_x + d_{x+1} + \dots + \underbrace{d_{x+n-1}}$$

$${}_np_x = \frac{l_{x+n}}{l_x}$$

$${}_nq_x = \frac{{}_nd_x}{l_x}$$

$${}_6p_{30} \cdot \underbrace{{}_2q_{36}}_{(1 - {}_2p_{36})}$$

$$\frac{\cancel{l_{36}}}{l_{30}}\left(1 - \underbrace{\frac{l_{38}}{l_{36}}}\right) = \frac{l_{36} - l_{38}}{l_{30}}$$

$$\frac{l_{36} - l_{38}}{\cancel{l_{36}}}$$

$$d_{36} + d_{37}$$

$$l_{36} - \cancel{l_{37}} + \cancel{l_{37}} - l_{38}$$

Illustrative example 1

$$\mathring{e}_x' = \int_0^\infty {}_tp_x \, dt$$

24983 - 24541 = 442

24541 - 24175 = 366

$\frac{\ell_{41}}{\ell_{40}} = \frac{24541}{24983} = .982308$

$1 - .982308 = \ldots$

Complete the following life table:

x	ℓ_x	d_x	p_x	q_x
40	24,983	442	·	·
41	24,541	366	·	·
42	24,175	259	·	·
43	23,880	224	·	·
44	23,656	159	·	·
45	23,495	—	—	—

not possible to calculate

.985086 .014914

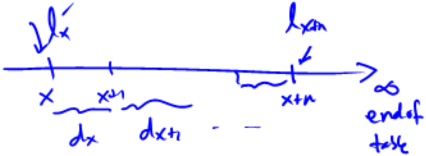
Additional useful formulas

From a life table, the following formulas can also easily be verified (or use your intuition):

- $\ell_x = \sum_{k=0}^{\infty} d_{x+k}$: the number of survivors at age x should be equal to the number of deaths in each year of age for all the following years.
- ${}_nd_x = \ell_x - \ell_{x+n} = \sum_{k=0}^{n-1} d_{x+k}$: the number of deaths within n years should be equal to the number of deaths in each year of age for the next n years.
- Finally, the probability that (x) survives the next n years but dies the following m years after that can be derived using:

$$ {}_{n|m}q_x = {}_np_x - {}_{n+m}p_x = \frac{{}_md_{x+n}}{\ell_x} = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}. $$

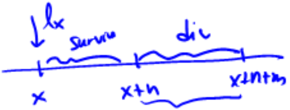
$$l_x = d_x + d_{x+1} + \cdots = \sum_{k=0}^{\infty} d_{x+k}$$



$${}_n d_x = l_x - l_{x+n}$$

$$= \underbrace{d_x + d_{x+1} + \cdots + d_{x+n-1}}_{\sum_{k=0}^{n-1} d_{x+k}}$$

n years



$${}_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x} = {}_n p_x - {}_{n+m} p_x = {}_{n+m} q_x - {}_n q_x$$

$$= \frac{d_{x+n} + d_{x+n+1} + \cdots + d_{x+n+m-1}}{l_x}$$

relates life table to survival models

$$S_0(x) = {}_x p_0 = \frac{l_{x+0}}{l_0} = \frac{l_x}{l_0},$$

l_0 = radix of life table from birth.

$$\mu_x = -\frac{d}{dx} \log S_0(x) = -\frac{1}{S_0(x)} \frac{d}{dx} S_0(x)$$

$$= -\frac{1}{l_x / \cancel{l_0}} \frac{d}{dx}\left(\frac{l_x}{l_0}\right) = \underbrace{-\frac{1}{l_x} \frac{d}{dx} l_x}_{} = \mu_x$$

$$\underbrace{-\frac{d}{dx} \log l_x}_{} = \underline{\underline{\mu_x}},$$

$$S_0(x) = {}_x p_0 = \frac{l_x}{l_0}$$

$$e^{-\int_0^x \mu_z \, dz} \longrightarrow l_x = l_0 \cdot e^{-\int_0^x \mu_z \, dz}$$

$$\boxed{\frac{d}{dx} l_x = -l_x \mu_x}.$$

$$\mu_{x+t} = -\frac{d}{dt} \log {}_tp_x$$

$$= -\frac{1}{{}_tp_x} \frac{d}{dt} {}_tp_x \quad \left(\frac{l_{x+t}}{l_x}\right)$$

$$\left(\frac{l_{x+t}}{\cancel{l_x}}\right)$$

$$= -\frac{1}{l_{x+t}} \frac{d}{dt} l_{x+t}$$

$$= -\frac{d}{dt} \log l_{x+t} = \mu_{x+t}$$

$S_x(t) = {}_tp_x$

$$\mu_{x+t} = -\frac{d}{dt} \log S_x(t)$$

$$S_x(t) = Pr[T_x > t]$$

$$= \frac{S_0(x+t)}{S_0(x)}$$

$$= \frac{{}_{x+t}p_0}{{}_xp_0}$$

$$= \frac{l_{x+t}/\cancel{l_0}}{l_x/\cancel{l_0}}$$

$$= l_{x+t}/l_x$$

$$= {}_tp_x$$

$S_x(t)$ = prob that (x) will survive the next t years

4:46 → 4:51

The force of mortality

- It is easy to show that the force of mortality can be expressed in terms of life table function as:

$$\mu_x = -\frac{1}{\ell_x} \cdot \frac{d\ell_x}{dx}.$$

- Thus, in effect, we can also write

$$\ell_x = \ell_0 \cdot \exp\left(- \int_0^x \mu_z dz \right).$$

- With a simple change of variable, it is easy to see also that

$$\mu_{x+t} = -\frac{1}{\ell_{x+t}} \cdot \frac{d\ell_{x+t}}{dt} = -\frac{1}{{}_tp_x} \cdot \frac{d\,{}_tp_x}{dt}.$$

- It follows immediately that:

${}_tp_x = S_x(t) = Pr[T_x > t]$

$$\frac{d}{dt}\,{}_tp_x = -\,{}_tp_x \mu_{x+t}.$$

$= \frac{d}{dt} Pr[T_x > t]$

Curtate expectation of life

Curtate lifetime is a discrete random variable

- Recall the expected value of K_x is called the curtate expectation of life. It can be expressed now as

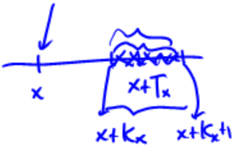
$$\mathsf{E}[K_x] = e_x = \sum_{k=1}^{\infty} {}_k p_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x}.$$

- The n-year temporary curtate expectation of life is

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x},$$

which gives the average number of completed years lived over the interval $(x, x+n]$ for a life (x).

life table $\Rightarrow$ probability distribution of curtate lifetime K_x



$$Pr[K_x = k] = {}_{k|}q_x$$

= prob of dying between k and k+1

$$= {}_kp_x \cdot q_{x+k} = \boxed{\frac{d_{x+k}}{l_x}}$$

$$E[K_x] = \underbrace{\sum_{k=0}^{\infty} k \cdot Pr[K_x = k]}$$

$$Var[K_x] = E[K_x^2] - e_x^2$$

$$\sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} \frac{l_{x+k}}{l_x} = e_x$$ = curtate expectation of life

$$e_{x:\overline{n}|} = E[\min(K_x, n)] = \sum_{k=1}^{n} {}_kp_x$$

$$\mathrm{Var}(K_x) = E[K_x^2] - e_x^2$$

$$\sum_{k=0}^{\infty} \underline{\underline{k^2}} \cdot \underbrace{\Pr[K_x = k]}_{\frac{d_{x+k}}{l_x}}$$

end of
ω = life table

Illustrative example 2

Suppose you are given the following extract from a life table:

x	ℓ_x
94	16,208
95	10,902
96	7,212
97	4,637
98	2,893
99	1,747
100	0

1. Calculate e_{95}. $= \sum_{k=1}^{\infty} {}_kp_{95} = \frac{l_{96}+l_{97}+\cdots+l_{99}}{l_{95}} = \underbrace{\frac{7212+4637+2893+1747}{10902}}_{= 1.512475}$
2. Calculate the variance of K_{95}, the curtate future lifetime of (95).
3. Calculate $e_{95:\overline{3}|}$. $= \frac{l_{96}+l_{97}+l_{98}}{l_{95}} = \frac{7212+4637+2893}{10902} = 1.352229$

$Var(K_{95}) = ?$

$e_{95} = 1.512475$ ✓

$\sum_{k=1}^{\infty} k^2 \cdot \underbrace{Pr[K_{95}=k]}_{d_{95+k}/l_{95}}$

$Pr[K_x = k] = \dfrac{d_{x+k}}{l_x}$

x	K	$Pr[K_{95}=k]$	k^2	$k^2 \cdot Pr[K_{95}=k]$	$k \cdot Pr[K_{95}=k]$
95	0	$\frac{2575}{10902}$	0	$2575/10902$	⋮
⋮	1	$\frac{1744}{10902}$	1	$4(1744/10902)$	
	2	⋮	4	⋮	
	3		9		
	4		16		
					1.512475

$\sum k^2 Pr[K_{95}=k] = \underline{4.386076}$

$Var(K_{95}) = \underline{E[K_{95}^2]} - e_{95}^2 = \underbrace{\sum_{k=0}^{\infty} k^2 Pr[K_{95}=k]}_{} - e_{95}^2$.

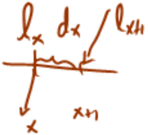
$4.386076 - (1.512475)^2 = \underline{2.098496}$

$\mathring{e}_x \geq e_x$

continuous expected lifetime

discrete (curtate) expected lifetime

fractional age



age last birthday

exact age 35.2

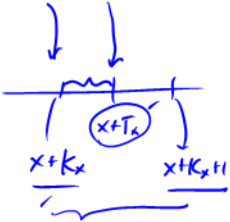
$$\boxed{\mathring{e}_x \approx e_x + 1/2}$$

approximately

K_x

$\downarrow$

T_x



$l_{1.5}$ ${}_{.2}p_{1.8}$

x	l_x	d_x	p_x	q_x	$\mathring{e}_x$
0					
1					
2					

Illustrative example 3

$\frac{\ell_x}{10{,}000}$ $\frac{d_x}{125}$ $\Rightarrow$ $10{,}000 - 125 = \underbrace{9875}_{\ell_{x+1}}$

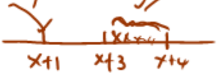
For a life (x), you are given $\ell_x = 10,000$ and the following extract from a life table:

ℓ_x	age	k	d_{x+k}
10,000	x	0	125
9875	$x+1$	1	250
9625	$x+2$	2	350
9275	$x+3$	3	500
8725	$x+4$	4	750

$_{2|}q_{x+1} = \frac{d_{x+3}}{\ell_{x+1}} = \frac{500}{9875} = .05063291$

$\frac{\ell_{x+3} - \ell_{x+4}}{\ell_{x+1}}$

Calculate $_{2|}q_{x+1}$ and interpret this probability.



prob that $(x+1)$ will survive 2 years and then die the following year

will die between ages $x+3$ and $x+4$

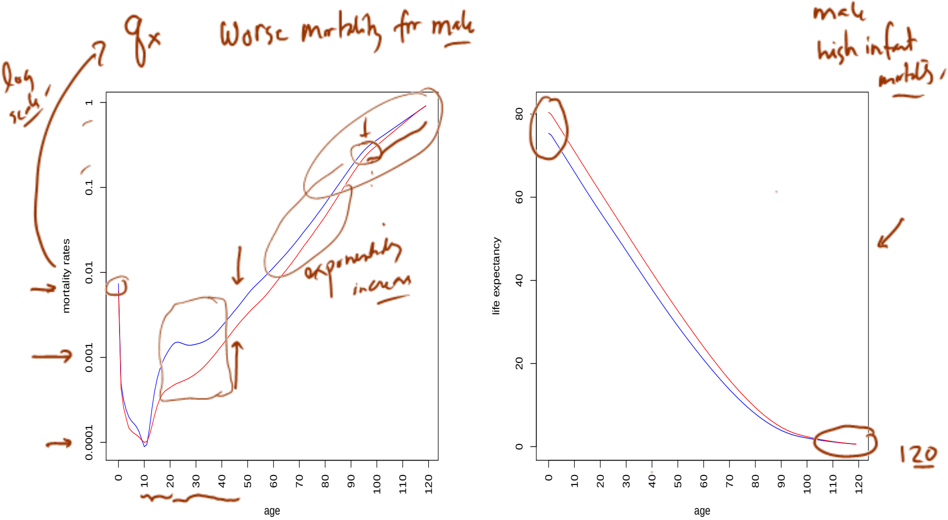
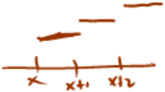


Figure: Source: Life Tables, 2007 from the Social Security Administration - male (blue), female (red)

Fractional age assumptions

- When adopting a life table (which may contain only integer ages), some assumptions are needed about the distribution between the integers.
- The two most common assumptions (or interpolations) used are (where $0 \leq t \leq 1$):



1. linear interpolation (also called UDD assumption):

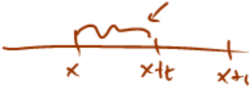
$$\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$$

2. exponential interpolation (equivalent to constant force assumption):

$$\log \ell_{x+t} = (1-t)\log \ell_x + t \log \ell_{x+1}$$

ln

$$ {}_tq_x = \frac{d_{x+t}}{l_x} = \underbrace{\frac{t\cdot d_x}{l_x}}_{q_x} = t\cdot q_x $$

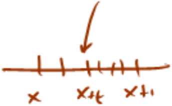


UDD: $\boxed{{}_tq_x = t\cdot q_x}$ VIF

$$ {}_tp_x = 1 - t\cdot q_x $$

$$ \mu_{x+t} = \frac{-1}{{}_tp_x}\,\frac{d}{dt}\,{}_tp_x = \frac{-1}{1-t\cdot q_x}\cdot(-q_x) = \frac{q_x}{1-t\cdot q_x} $$

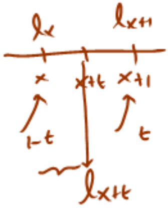
$$ {}_tp_x\cdot\mu_{x+t} = -\frac{d}{dt}\,{}_tp_x = -(-q_x) = q_x \quad \text{(independent of } t) $$



linear interpolation of ℓ $\quad 0 \le t \le 1$

x	ℓ_x
$x+t$	$\leftarrow \ell_{x+t}$
$x+1$	ℓ_{x+1}

$$\boxed{\ell_{x+t} = t\cdot \ell_{x+1} + (1-t)\,\ell_x}$$



$$d_{x+t} = \ell_x - \ell_{x+t}$$

$$= \cancel{\ell_x} - t\cdot \ell_{x+1} + (t-\cancel{1})\,\ell_x$$

$$= t\cdot(\underbrace{\ell_x - \ell_{x+1}}_{d_x}) = t\cdot d_x$$

x	ℓ_x	d_x
0	-	
1	-	
2	-	

uniformly distributing deaths between x to $x+1$

$$d_{x+1/2} = \frac{1}{2}\cdot d_x$$

UDD = uniform distribution of death

exponential interpolation

$$\log l_{x+t} = (1-t)\log l_x + t\log l_{x+1}$$

$$= \log l_x^{1-t} + \log l_{x+1}^{t}$$

$$= \log\left(l_x^{1-t}\, l_{x+1}^{t}\right)$$

$\log A + \log B = \log AB$

$$\boxed{l_{x+t} = l_x^{1-t}\, l_{x+1}^{t}}$$ VIF exponential

$l_x = l_x^{1-t}\, l_x^{t}$

$${}_tp_x = \frac{l_{x+t}}{l_x} = \frac{l_x^{1-t}\, l_{x+1}^{t}}{l_x} = \frac{\cancel{l_x^{1-t}}}{\cancel{l_x^{1-t}}}\,\frac{l_{x+1}^{t}}{l_x^{t}} = \Big(\underbrace{\frac{l_{x+1}}{l_x}}_{p_x}\Big)^{t}$$

$$\boxed{{}_tp_x = (p_x)^t}$$ VIF $0 \le t \le 1$

$${}_tq_x = 1-(p_x)^t = 1-(1-q_x)^t$$

$$\mu_{x+t} = \frac{-1}{{}_tp_x} \frac{d}{dt} {}_tp_x$$

$${}_tp_x = (p_x)^t = \underbrace{e^{t \log p_x}}$$

$$\frac{d}{dt} {}_tp_x = \underbrace{(p_x)^t \cdot \log p_x}$$

$$= \frac{-1}{\cancel{(p_x)^t}} \cancel{(p_x)^t} \log p_x$$

$$= \underbrace{-\log p_x} \leftarrow \text{independent of } t$$

$\mu_{x+1/2}$ ✓

$\mu_{x+1/3}$ ✓

$\mu_{x+3/4}$ ✓

$$x, x+1 \Rightarrow \underbrace{\mu = -\log p_x}$$

$$p_x = e^{-\mu}$$ constant frc

$${}_tp_x \, \mu_{x+t} = -\frac{d}{dt} {}_tp_x = -(p_x)^t \log p_x = (e^{-\mu})^t \cdot \mu = \mu e^{-\mu t}$$

between $0 \leq t \leq 1$

Some results on the fractional age assumptions

Function	Linear (UDD)	Exponential (constant force)
$_tq_x$	$t \cdot q_x$	$1-(1-q_x)^t$
μ_{x+t}	$\dfrac{q_x}{1-t\cdot q_x}$	$\boxed{\mu = -\log p_x}$
$_tp_x\mu_{x+t}$	q_x	$\mu e^{-\mu t}$

$_tp_x = (p_x)^t$

$\mu = -\log p_x$

$p_x = e^{-\mu}$

Here we have $0 \le t \le 1$.

Illustrative example 4

You are given the following extract from a life table:

x	ℓ_x
55	85,916
56	84,772
57	83,507
58	82,114

Estimate ${}_{1.4}p_{55}$ and ${}_{0.5|1.6}q_{55}$ under each of the following assumptions for non-integral ages:

(a) UDD; and

(b) constant force.

Interpret these probabilities.

UDD

$${}_{1.4}p_{55} = p_{55} \cdot \underbrace{{}_{.4}p_{56}}_{(1 - {}_{.4}q_{56})}$$

$$(1 - {}_{.4}q_{56}) = (1 - .4 * q_{56})$$

$q_{56} \rightarrow 1 - p_{56}$

$p_{55} \rightarrow \dfrac{l_{56}}{l_{55}}$

$l_{56} \rightarrow$ 84,772

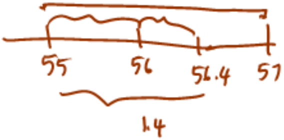
$l_{55} \rightarrow$ 85,916

$$= \frac{l_{56}}{l_{55}}\left(1 - .4 * \left(1 - \frac{l_{57}}{l_{56}}\right)\right)$$

$l_{57} \rightarrow$ 83,567

$l_{56} \rightarrow$ 84,772

$$= .9807952$$



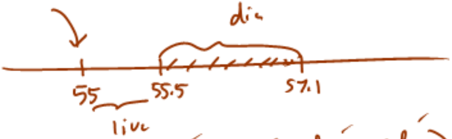
$${}_{1.4}p_{55} = \frac{l_{56.4}}{l_{55}} = \frac{.6\, l_{56} + .4\, l_{57}}{l_{55}} = \frac{.6(84772) + .4(83567)}{85916}$$

more direct

$$= .9807952$$

$_{0.5|1.6}q_{55}$

since l_x's are given $\Rightarrow$ directly with l_x



$$= \frac{l_{55.5} - l_{57.1}}{l_{55}} = \frac{(.5\,l_{55} + .5\,l_{56}) - (.9\,l_{57} + .1\,l_{58})}{l_{55}}$$

plus values

$$= \underline{\underline{.0230027}}$$

constant force

$$ {}_{.4}p_{55} = \frac{l_{56.4}}{l_{55}} = \frac{l_{56}^{.6}\, l_{57}^{.4}}{l_{55}} = \underbrace{\frac{84772^{.6}\; 83507^{.4}}{85916}} $$

$$ = .9807686 $$

l_x are given

$$ {}_t p_x = (p_x)^t $$

$$ l_{x+t} = l_x^{1-t}\, l_{x+1}^{t} $$

$$ {}_{0.5|1.6}q_{55} = \frac{l_{55.5} - l_{57.1}}{l_{55}} = \frac{l_{55}^{.5}\, l_{56}^{.5} - l_{57}^{.9}\, l_{58}^{.1}}{l_{55}} $$

plus values

$$ = \underline{.0229929} $$

Illustrative example 5

Assume the Uniform Distribution of Death (UDD) assumption holds between integer ages. You are given:

$$_{0.5}p_{65} = 0.95$$

$$_{0.3}p_{66} = 0.92$$

Calculate the probability that (65) will survive the next two years.

$.5q_{65} = 1 - .5p_{65} = 1 - .95 = .05 \Rightarrow .5 * q_{65} = .05$

$q_{65} = \frac{.05}{.5} = .025$

$.3q_{66} = 1 - .92 = .08$

$q_{66} = \frac{.08}{.3}$

65 66 67

$_2p_{65} = p_{65} \cdot p_{66}$

$= (1 - .025)\left(1 - \frac{.08}{.3}\right)$

$= .66$

Fractional part of the year lived

- Denote by R_x the fractional part of a year lived in the year of death. Then we have

$T_x - K_x = R_x$

$$\boxed{T_x = K_x + R_x}$$

where T_x is the time-until-death and K_x is the curtate future lifetime of (x).

- We can describe the joint probability distribution of (K_x, R_x) as

$$\text{Pr}\left[(K_x = k) \cap (R_x \leq s)\right] = \text{Pr}[k < T_x \leq k + s] = {}_k p_x \cdot {}_s q_{x+k},$$

for $k = 0, 1, \ldots$ and for $0 < s < 1$.

- The UDD assumption is equivalent to the assumption that the fractional part R_x occurs uniformly during the year, i.e. $R_x \sim \text{U}(0, 1)$.
  - It can be demonstrated that K_x and R_x are independent in this case.

$$\underbrace{\Pr[K_x = k \text{ and } R_x \le s]}$$

$$= \underline{{}_kp_x \cdot {}_sq_{x+k}} \;//$$

UDD

$$T_x = K_x + R_x$$

$$R_x \sim \text{Uniform}(0,1)$$

$$\underbrace{E[T_x]} = \underbrace{E[K_x]} + \underbrace{E[R_x]}$$

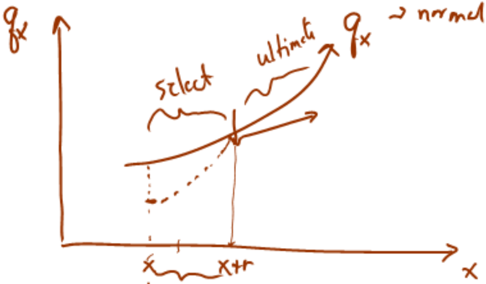
$$\boxed{\mathring{e}_x = e_x + 1/2}$$

$R_x \sim U(0,1)$

assumed

$$\boxed{\mathring{e}_x \approx e_x + 1/2}$$

Select and ultimate table



(x)

$[x]$ age x but select

buy policy

r = select period

$$q_{[x]} \le q_x$$

$$q_{[x]+1} \le q_{x+1}$$

$$\vdots$$

$$q_{[x]+j} = q_{x+j}, \quad j \geq r$$

r is small

$r = 2, 3, \ldots$

selection process

underwriting

↓

amount is big -

history

family

profession

health

medical -

Select and ultimate tables

${}_tq_{[x]}$ ${}_tp_{[x]}$ $\ell_{[x]+t}$

- Group of lives underwritten for insurance coverage usually has different mortality than the general population (some test required before insurance is offered).
- Mortality then becomes a function of age $[x]$ at selection (e.g. policy issue, onset of disability) and duration t since selection.
- For select tables, notation such as ${}_tq_{[x]}$, ${}_tp_{[x]}$, and $\ell_{[x]+t}$, are then used.
- However, impact of selection diminishes after some time - the select period (denoted by r).
- In effect, we have

$$q_{[x]+j} = q_{x+j}, \quad \text{for } j \geq r.$$

$q_{[x]} \leq q_x$

$q_{[x]+1} \leq q_{x+1}$

$\vdots$

Example of a select and ultimate table

$q_{[32]+1} = .377/1000$

$q_{33} = .459/1000$

$\ell_{[x]+1} \neq \ell_{[x+1]}$

$[x]$	$1000q_{[x]}$	$1000q_{[x]+1}$	$1000q_{x+2}$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
30	0.222	0.330	0.422	9,907	9,905	9,901	32
31	0.234	0.352	0.459	9,903	9,901	9,897	33
32	0.250	0.377	0.500	9,899	9,896	9,893	34
33	0.269	0.407	0.545	9,894	9,892	9,888	35
34	0.291	0.441	0.596	9,889	9,887	9,882	36

$r=2$ select period

$q_{[31]+2} = .459/1000 = q_{33}$

- From this table, try to compute probabilities such as:
  - (a) ${}_2p_{[30]}$;
  - (b) ${}_5p_{[30]}$;
  - (c) ${}_{1|}q_{[31]}$; and
  - (d) ${}_3q_{[31]+1}$.

$${}_2p_{[30]} = \frac{\ell_{[30]+2}}{\ell_{[30]}} = \frac{\ell_{32}}{\ell_{[30]}} = \frac{9901}{9907}$$

$${}_5p_{[30]} = \frac{\ell_{[30]+5}}{\ell_{[30]}} = \frac{\ell_{35}}{\ell_{[30]}} = \frac{9888}{9907}$$

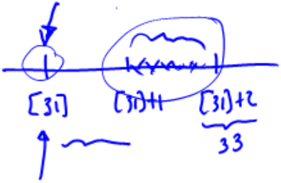
UCONN

$$_{1|}q_{[31]} = p_{[31]} * q_{[31]+1}$$

$$= (1 - q_{[31]}) * q_{[31]+1}$$

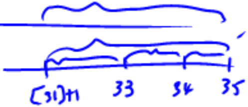
$$= \left(1 - \frac{.234}{1000}\right) * \frac{.352}{1000}$$

$$= \frac{\#\text{ of death}}{\#\text{ alive}} = \frac{9901 - 9897}{9903}$$



$$_3q_{[31]+1} = 1 - {}_3p_{[31]+1}$$

$$= 1 - (1 - \underbrace{q_{[31]+1}}_{\frac{.352}{1000}})(1 - \underbrace{q_{33}}_{\frac{.459}{1000}})(1 - \underbrace{q_{34}}_{\frac{.510}{1000}})$$



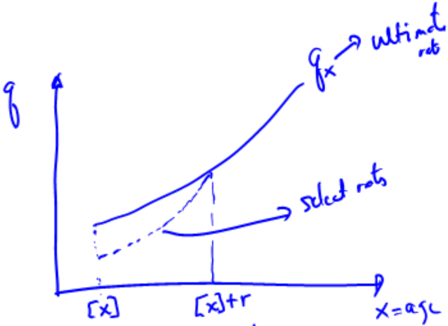
$$= \frac{l_{[31]+1} - l_{35}}{l_{[31]+1}}$$

$$= 1 - l_{35}/l_{[31]+1} = 1 - \frac{9888}{9901}$$

Select and ultimate table -

underwriting $\Rightarrow$ selection

determine how healthy you are to be insured -



r = select period

$$q_{[x]} \leq q_x$$

select

$$q_{[x]+1} \leq q_{x+1}$$

$$q_{[x]+r} = q_{x+r}$$

$$q_{[x]+j} = q_{x+j}, \text{ for } j \geq r$$

Illustrative example 6

A select and ultimate table with a three-year select period begins at selection age x.

You are given the following information:

- $\ell_{x+6} = 90,000$
- $q_{[x]} = \frac{1}{6}$ $\Rightarrow$ $p_{[x]} = \frac{5}{6} = \frac{\ell_{[x]+1}}{\ell_{[x]}} \Rightarrow \ell_{[x]} = \frac{6}{5}\ell_{[x]+1} = \frac{6}{5}\frac{10}{9}\ell_{[x+1]}$
- ${}_5p_{[x+1]} = \frac{4}{5}$ $\Rightarrow$ $\frac{\ell_{x+6}}{\ell_{[x+1]}} = \frac{4}{5} \Rightarrow \ell_{[x+1]} = \frac{5}{4}\ell_{x+6}$
- ${}_3p_{[x]+1} = \frac{9}{10} \cdot {}_3p_{[x+1]}$. $\Rightarrow \frac{\ell_{[x]+1+3}}{\ell_{[x]+1}} = \frac{\ell_{x+4}}{\ell_{[x]+1}} = \frac{9}{10}\frac{\ell_{[x+1]+3}}{\ell_{[x+1]}} \Rightarrow \ell_{[x]+1} = \frac{10}{9}\ell_{[x+1]}$

$\frac{6}{5}\frac{10}{9}\frac{5}{4}\ell_{x+6}$

90,000

$\frac{5}{3}\, 90000 = 150,000$

Evaluate $\ell_{[x]}$. = ?

Illustrative example 7

linear ⇒ UDD

exponential ⇒ constant force

constant force $\ell_{x+t} = \ell_x^{1-t} \, \ell_{x+1}^{t}$

You are given the following extract from a select and ultimate life table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	29,616	29,418	29,132	62
61	29,131	28,920	28,615	63
62	28,601	28,378	28,053	64

$[60]+.8$ $\quad$ $[60]+1.5$

Calculate $1000\,{}_{0.7}q_{[60]+0.8}$, assuming a constant force of mortality at fractional ages.

$$= 1000 \times \left(1 - \frac{\ell_{[60]+1.5}}{\ell_{[60]+.8}}\right) = \frac{\left(\ell_{[60]+1}^{.5}\, \ell_{[60]+2}^{.5}\right)}{\left(\ell_{[60]}^{.2}\, \ell_{[60]+1}^{.8}\right)} = 1000 \times \left(1 - \frac{29418^{.5}\, 29132^{.5}}{29616^{.2}\, 29418^{.8}}\right)$$

$= 6.207014$

UCONN

Illustrative example 8

$${}_t p_{[65]} = \frac{\ell_{[65]+t}}{\ell_{[65]}}$$

$$\int_0^2 = \int_0^{1/2} + \int_{1/2}^{1} + \int_1^{3/2} + \int_{3/2}^{1}$$

You are given the following extract from a select and ultimate life table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
65	80,625	79,954	78,839	67
66	79,137	78,402	77,252	68
67	77.575	76,770	75,578	69

Approximate $\mathring{e}_{[65]:\overline{2}|}$ using the trapezium (trapezoidal) rule with $h = 0.5$ and assuming UDD for fractional ages.

$$\mathring{e}_{[65]:\overline{2}|} = \int_0^2 {}_t p_{[65]}\, dt = \frac{1}{80625} \underbrace{\int_0^2 \ell_{[65]+t}\, dt}$$

$$\frac{1}{80625}\left[\underbrace{\int_0^{1/2} l_{[65]+t}\,dt} + \underbrace{\int_{1/2}^{1} l_{[65]+t}\,dt} + \ldots\ldots\right]$$

$$\int_a^b f(t)\,dt = \frac{1}{2}\{f(a)+f(b)\}$$

$$\frac{1}{80625}\left[\quad \frac{1}{2}\left[l_{[65]} + \underbrace{l_{[65]+1/2}}_{\frac{1}{2}(l_{[65]} + l_{[65]+1})}\right] + \frac{1}{2}\left[l_{[65]+1/2} + l_{[65]+1}\right] + \ldots\right.$$

$\vdots$

$= \;??$

Illustrative example 9

For a select-and-ultimate mortality table with a 3-year select period, you are given:

select rate

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(i) hint

(ii) Becky was a newly selected life on 01/01/2012.

(iii) Becky's age on 01/01/2012 is 61.

(iv) Q is the probability on 01/01/2012 that Becky will be dead by 01/01/2017.

Calculate Q.

die

1/1/2012 — 1/1/2017

[61] — 66

$Q = {}_5q_{[61]}$

$= 1 - p_{[61]} \cdot p_{[61]+1} \cdot p_{[61]+2} \cdot p_{64} \cdot p_{65}$

$= 1 - .9(.88)(.86)(.84)(.83)$

$= .5251231$

$_tq_x = t \cdot q_x$

Illustrative example 10 - modified SOA MLC Spring 2012

$e_{x:\overline{1}|} =$

$e_{50.5} = e_{50.5:\overline{1}|} + p_{50.5} \cdot e_{51.5}$

$e_{51.5} \to 22.4$

$e_{50.5:\overline{1}|} \to p_{50.5}$

$= p_{50.5}\left(\underbrace{1+22.4}_{23.4}\right)$

CF UDD

50 50.5 51 51.5 52

Suppose you are given:

- $p_{50} = 0.98$
- $p_{51} = 0.96$
- $e_{51.5} = 22.4$
- The force of mortality is constant between ages 50 and 51.
- Deaths are uniformly distributed between ages 51 and 52.

$= \underbrace{{}_{.5}p_{50.5}}_{CF} \cdot \underbrace{{}_{.5}p_{51}}_{UDD} \times 23.4$

$(.98)^{.5}$

${}_{.5}p_{51} \to (1 - .5\, q_{51})$, $q_{51} = .04$

$= 22.70152$

Calculate $e_{50.5}$.

UDD: $_tq_x = t \cdot q_x$

CF: $_tp_x = (p_x)^t$

$p_{50} = {}_{.5}p_{50} \cdot {}_{.5}p_{50.5}$

$p_{50} = (p_{50})^{.5} \cdot {}_{.5}p_{50.5} \Rightarrow {}_{.5}p_{50.5} = (p_{50})^{.5}$

$= (.98)^{.5}$

recursive formulas for e

$x \longrightarrow x+1$

$$\underbrace{e_x}_{50.5} = \underbrace{e_{x:\overline{1|}}}_{50.5} + p_x \cdot \underbrace{e_{x+1}}_{51.5}$$

Illustrative example 11 - modified SOA MLC Spring 2012

In a 2-year select and ultimate mortality table, you are given:

- $q_{[x]+1} = 0.96\, q_{x+1}$ → for any x
- $\ell_{65} = 82,358$
- $\ell_{66} = 81,284$

Calculate $\ell_{[64]+1}$.

for $x = 64 \Rightarrow \underbrace{p_{[64]+1}}_{81,284} = 1 - .96\,(1 - p_{65})$

$\ell_{66} = \dfrac{\ell_{[64]+2}}{\ell_{[64]+1}} = 1 - .96\left(1 - \dfrac{\ell_{66}}{\ell_{65}}\right)$ 81,284 82,358

$\Longrightarrow \ell_{[64]+1} = 82,315$

Survival

$S_0(0) = 1$

$S_0(\infty) = 0$

$\frac{d}{dx} S_0(x) = \quad . \quad . \quad < 0$

$_3p_x = p_x \cdot p_{x+1} \, p_{x+2}$

$_3p_x = \qquad =$

Illustrative example 12 - SOA MLC Fall 2014 MC#20

For a mortality table with a select period of two years, you are given:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
50	0.0050	0.0063	0.0080	52
51	0.0060	0.0073	0.0090	53
52	0.0070	0.0083	0.0100	54
53	0.0080	0.0093	0.0110	55

(ii) The force of mortality is constant between integral ages.

Calculate $1000\,{}_{2.5}q_{[50]+0.4}$.

Practice on this

$= 1000(.01642) = \underline{16.42}$

Mortality projection factors

later on.

Read Section 3.11

Only other symbol used in the MLC exam

l_x

ℓ_x

Expression	SOA adopts the symbol
number of lives	l_x

ℓ_x l_x