Survival Models

Lecture: Weeks 2-3

Chapter summary

- Survival models
  - Age-at-death random variable
  - Time-until-death random variables
  - Force of mortality (or hazard rate function)
  - Some parametric models
    - De Moivre's (Uniform), Exponential, Weibull, Makeham, Gompertz
    - Generalization of De Moivre's
  - Curtate future lifetime
- Chapter 2 (Dickson, Hardy and Waters = DHW)

Age-at-death random variable

random Variable

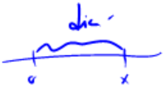
- X is the age-at-death random variable; continuous, non-negative
- X is interpreted as the lifetime of a newborn (individual from birth)
- Distribution of X is often described by its survival distribution function (SDF):

$$\underset{\text{newborn}}{S_0}(x) = \boxed{\text{Pr}[X > x]}$$

  - other term used: survival function

- Properties of the survival function:
  - $S_0(0) = 1$: probability a newborn survives 0 years is 1.
  - $S_0(\infty) = \lim_{x\to\infty} S_0(x) = 0$: all lives eventually die.
  - non-increasing function of x: not possible to have a higher probability of surviving for a longer period.

Cumulative distribution and density functions

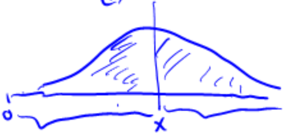


- Cumulative distribution function (CDF): $F_0(x) = \Pr[X \leq x]$ $= 1 - \Pr(X > x)$
  - nondecreasing; $F_0(0) = 0$; and $F_0(\infty) = 1$.
- Clearly we have: $F_0(x) = 1 - S_0(x)$ ✓

- Density function: $f_0(x) = \dfrac{dF_0(x)}{dx} = -\dfrac{dS_0(x)}{dx}$
  - non-negative: $f_0(x) \geq 0$ for any $x \geq 0$
  - in terms of CDF: $F_0(x) = \int_0^x f_0(z)dz$ ✓
  - in terms of SDF: $S_0(x) = \int_x^\infty f_0(z)dz$

$$\left.\begin{array}{l} \dfrac{d}{dx} F_0(x) \\ -\dfrac{d}{dx} S_0(x) \end{array}\right\} = f_0(x)$$

$$\int_0^\infty f_0(x) = 1$$



$f_0(x)$ $S_0(x)$ $F_0(x)$

Force of mortality

- The force of mortality for a newborn at age x:

$$\mu_x = \frac{f_0(x)}{1 - F_0(x)} = \frac{f_0(x)}{S_0(x)} = -\frac{1}{S_0(x)} \frac{dS_0(x)}{dx} = -\frac{d \log S_0(x)}{dx}$$

- Interpreted as the conditional instantaneous measure of death at x.
- For very small Δx, $\mu_x \Delta x$ can be interpreted as the probability that a newborn who has attained age x dies between x and $x + \Delta x$:

$$\mu_x \Delta x \approx \Pr[x < X \leq x + \Delta x | X > x]$$

- Other term used: hazard rate at age x.

force of mortality

$A \mid B$

$$\lim_{\Delta x \to 0} \frac{\Pr(x < X \leq x+\Delta x \mid X > x)}{\Delta x} = \mu_x$$

$\underbrace{\mu_x}\Delta x$

failure rate

$$\frac{\Pr(\overbrace{x < X \leq x+\Delta x, X > x})}{\Pr(X > x)} = \frac{\overbrace{\Pr(x < X \leq x+\Delta x)}}{\Delta x}$$

$S_0(x) = \Pr(X > x)$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \lim_{\Delta x \to 0} \frac{\Pr(X > x) - \Pr(X > x+\Delta x)}{\Delta x \Pr(X > x)}$$

$$\Pr(a < X \leq b)$$
$$= \Pr(X > a) - \Pr(X > b)$$
$$= \Pr(X \leq b) - \Pr(X \leq a)$$

$$= -\lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{S_0(x+\Delta x) - S_0(x)}{S_0(x)}$$

$$= -\frac{1}{S_0(x)} \boxed{\lim_{\Delta x \to 0} \frac{S_0(x+\Delta x) - S_0(x)}{\Delta x}}$$

$$\frac{d}{dx} S_0(x)$$

$$\mu_x = \underbrace{-\frac{1}{S_0(x)}\frac{d}{dx}S_0(x)}$$

$$= -\frac{d}{dx}\log S_0(x) \checkmark$$

$$\mu_x = \frac{-\frac{d}{dx}S_0(x)}{S_0(x)} = \frac{f_0(x)}{S_0(x)}$$

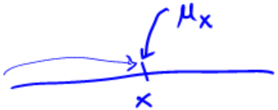
$$\mu_x = \frac{f_0(x)}{\int_x^{\infty} f_0(z)\,dz}$$

$$\mu_x = \frac{\frac{d}{dx}F_0(x)}{1-F_0(x)}$$

μ_x $\quad f_0(x)$ $\quad S_0(x)$ $\quad F_0(x)$

$\log = \ln$

$\log = \log_e$



Some properties of μ_x force of mortality

> 1

Some important properties of the force of mortality:

- non-negative: $\mu_x \geq 0$ for every $x > 0$
- divergence: $\int_0^\infty \mu_x dx = \infty$.
- in terms of SDF: $\boxed{S_0(x) = \exp\left(-\int_0^x \mu_z dz\right).}$ VIP
- in terms of PDF: $f_0(x) = \mu_x \exp\left(-\int_0^x \mu_z dz\right).$

$\mu_x \quad f_0(x) \quad S_0(x) \quad F_0(x)$

$$\mu_x = -\frac{d}{dx} \log S_0(x)$$

$$\underbrace{\int d \log S_0(x)}_{\log S_0(x)}$$

$$\mu_x \, dx = -d \log S_0(x)$$

$$\int_0^x \mu_z \, dz = \int_0^x -d \log S_0(z)$$

$$-\int_0^x \mu_z \, dz = \log S_0(z) \Big|_0^x$$

$$= \log S_0(x) - \underbrace{\log S_0(0)}_{1}$$

$$S_0(x) = e^{-\int_0^x \mu_z \, dz}$$

$$\mu_x = \frac{f_0(x)}{S_0(x)}$$

$$\mu_x S_0(x) = f_0(x)$$

$$f_0(x) = S_0(x) \mu_x = \mu_x S_0(x)$$

$$= \underbrace{\mu_x e^{-\int_0^x \mu_z \, dz}}$$

$\mu e^{-\mu x}$

$a e^{-ax}$

X = age at death (of a newborn)

$S_0(x) = Pr(X > x)$ prob that a newborn will survive to reach age x

Properties:

① $S_0(0) = 1$ alive at birth

② $S_0(\infty) = 0$ eventually die

③ $S_0(x)$ is non-increasing in X. For any ages $a > b$, $S_0(a) \le S_0(b)$.

Prove $\frac{d}{dx} S_0(x) \le 0$

$S_0(x)$

$f_0(x)$

$F_0(x)$

μ_x

$S_0(x) = e^{-\mu x}$, $x \geq 0$ μ = constant

$S_0(0) = 1$

$S_0(\infty) = e^{-\infty} \to 0$

$\frac{d}{dx} S_0(x) = -\mu e^{-\mu x} \le 0$

Exponential

$f_0(x) = \frac{-d}{dx} S_0(x) = \mu e^{-\mu x}$

$F_0(x) = 1 - e^{-\mu x}$

force of mortality $\mu_x = \frac{-d}{dx} \log S_0(x) = \frac{d}{dx}(\mu x)$

Constant force $\Leftarrow \mu$

$X \geq 0$ - nonnegative

Given

✓ $S_0(x)$

- $F_0(x)$

- $f_0(x)$

✓ μ_x

$f_0(x) = -\frac{d}{dx} S_0(x)$, $F_0(x) = 1 - S_0(x)$, $\mu_x = \frac{-\frac{d}{dx} S_0(x)}{S_0(x)}$ ✓✓ $= -\frac{d}{dx} \log S_0(x)$

$f_0(x) = \frac{d}{dx} F_0(x)$, $S_0(x) = 1 - F_0(x)$, $\mu_x = \frac{\frac{d}{dx} F_0(x)}{1 - F_0(x)}$

$F_0(x) = \int_0^x f_0(z)\,dz$, $S_0(x) = \int_x^\infty f_0(z)\,dz$, $\mu_x = \frac{f_0(x)}{\int_x^\infty f_0(z)\,dz}$ ✓

$S_0(x) = e^{-\int_0^x \mu_z\,dz}$, $F_0(x) = 1 - e^{-\int_0^x \mu_z\,dz}$, $f_0(x) = \mu_x\, e^{-\int_0^x \mu_z\,dz}$

Moments of age-at-death random variable

$E(x) = 1/\mu$

$m \quad S_0(x) = e^{-\mu x}$

$S_0(m) = e^{-m\mu} = .5$

- The mean of X is called the complete expectation of life at birth:

$-m = -\frac{1}{\mu}\log 2$

mean'

birth $\mathring{e}_0$

$$\mathring{e}_0 = \mathsf{E}[X] = \int_0^\infty x f_0(x)\,dx = \int_0^\infty S_0(x)\,dx.$$

$m = \frac{1}{\mu}\log 2$

- The RHS of the equation can be derived using integration by parts.
- Variance:

$$\mathsf{Var}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 = \mathsf{E}\left[X^2\right] - (\mathring{e}_0)^2.$$

- The median age-at-death m is the solution to

median'

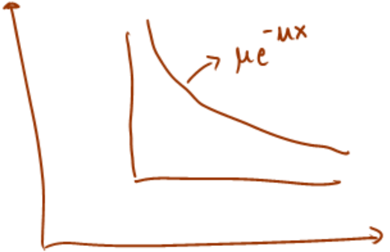
$P(X \leq m) = 1/2$

$P(X > m) = 1/2$

$$S_0(m) = F_0(m) = \frac{1}{2}.$$

Constant μ

memoryless



$$Pr(X>x) = e^{-\mu x} = S_0(x)$$

$X > x+z$

$$Pr(X > z+x \mid X > z) = \frac{Pr(X > x+z,\ X > z)}{Pr(X>z)} = \frac{Pr(X > x+z)}{Pr(X>z)}$$

$$= \frac{e^{-\mu(x+\cancel{z})}}{\cancel{e^{-\mu z}}} = e^{-\mu x}$$

memoryless property

Some special parametric laws of mortality

Law/distribution	μ_x	$S_0(x)$	Restrictions
De Moivre (uniform)	$1/(\omega - x)$	$1 - (x/\omega)$	$0 \leq x < \omega$
Constant force (exponential)	μ	$\exp(-\mu x)$	$x \geq 0, \mu > 0$
Gompertz	Bc^x	$\exp\left[-\frac{B}{\log c}(c^x - 1)\right]$	$x \geq 0, B > 0, c > 1$
Makeham	$A + Bc^x$	$\exp\left[-Ax - \frac{B}{\log c}(c^x - 1)\right]$	$x \geq 0, B > 0, c > 1,$ $A \geq -B$
Weibull	kx^n	$\exp\left(-\frac{k}{n+1}x^{n+1}\right)$	$x \geq 0, k > 0, n > 1$

Uniform $X \sim U(0, \omega)$ $\longrightarrow$ limiting age

$$f_0(x) = \frac{1}{\omega}$$

$$F_0(x) = \int_0^x \frac{1}{\omega}\, dz$$

$$= x/\omega$$

$$S_0(x) = 1 - x/\omega$$

$$\mu_x = \frac{-\frac{d}{dx} S_0(x)}{S_0(x)}$$

$$= \frac{1/\omega}{1 - x/\omega}$$

$$= \frac{1/\omega}{(\omega - x)/\omega}$$

$$= \frac{1}{\omega - x}$$

De Moivre's law of mortality

* Mortality follows de Moivre's law.

uniform

Makeham's

(realistic)

$$\mu_x = \underbrace{A}_{\text{constant term independent of } x} + \underbrace{Bc^x}_{\text{geometrically increasing term of } x}$$

A, B, c parameters

$A \geq -B$

$B > 0$

$c > 1$

accidents

$$\int c^z dz = \underline{\int e^{z \cdot \log c} dz}$$

$$= \frac{1}{\log c} c^z + K$$

$$S_0(x) = e^{-\int_0^x (A + Bc^z)\, dz}$$

$$= e^{-Ax} e^{-B\int_0^x c^z dz}$$

$$\underline{S_0(x) = e^{-Ax} e^{-\frac{B}{\log c}(c^x - 1)}}$$

$F_0(x)$

$f_0(x) = \underline{\mu_x}\, \underline{S_0(x)}$

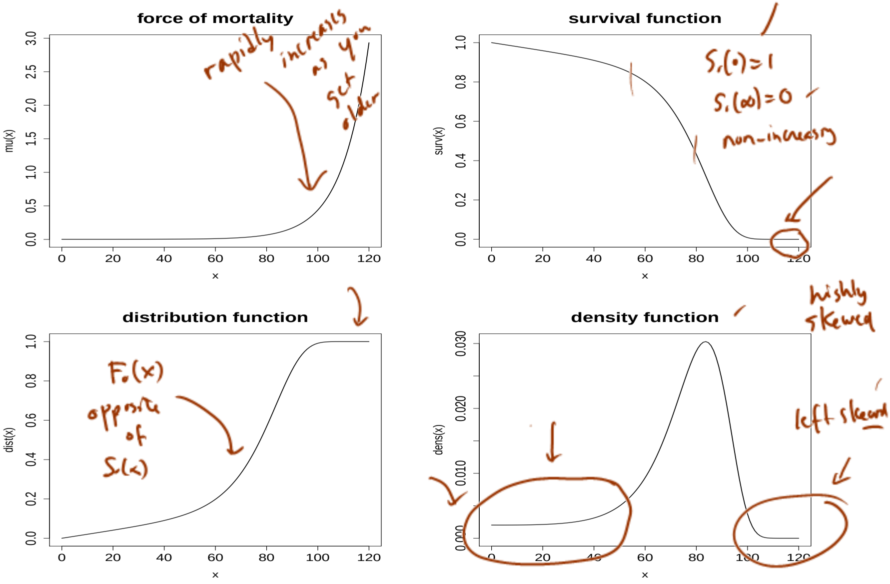


Figure: Makeham's law: $\boxed{A = 0.002,\ B = 10^{-4.5},\ c = 1.10}$

Illustrative example 1

Suppose X has survival function defined by

$$\boxed{S_0(x) = \frac{1}{10}(100-x)^{1/2}}, \quad \text{for} \quad 0 \leq x \leq 100.$$

1. Explain why this is a legitimate survival function.
2. Find the corresponding expression for the density of X.
3. Find the corresponding expression for the force of mortality at x.
4. Compute the probability that a newborn with survival function defined above will die between the ages 65 and 75.

Solution to be discussed in lecture.

$$S_0(x) = \frac{1}{10}(100-x)^{1/2}, \quad x \le 100$$

① $S_0(0) = 1$

$S_0(\infty) = S_0(100) = 0$

$\frac{d}{dx}S_0(x) = \frac{1}{10}\,\frac{1}{2}(100-x)^{-1/2}(-1) \le 0$ non-increasing

positive

$$\boxed{\frac{1}{\omega - x}}$$

② $f_0(x) = -\frac{d}{dx}S_0(x) = \frac{1}{20}\frac{1}{(100-x)^{1/2}}$

③ $\mu_x = \dfrac{-\frac{d}{dx}S_0(x)}{S_0(x)} = \dfrac{\frac{1}{\cancel{20}_{\,2}}\frac{1}{(100-x)^{1/2}}}{\frac{1}{\cancel{10}}(100-x)^{1/2}} = \frac{1}{2}\,\frac{1}{(100-x)}$

④ $Pr(65 < X \le 75) = Pr(X > 65) - Pr(X > 75)$

$= S_0(65) - S_0(75) = \frac{1}{10}\sqrt{35} - \frac{1}{10}(5) = \underline{\underline{.09161}}$

Practice problem - SOA MLC Spring 2016 Question #2

Generalized de Moivre's

$X \quad T_x$

$\downarrow$

T_0

You are given the survival function:

$$\underline{S_0(x)} = \underbrace{\left(1 - \frac{x}{60}\right)^{1/3}}, \quad \text{for} \quad 0 \le x \le 60.$$

Calculate $1000\mu_{35}$.

$$\mu_x = \frac{-\frac{d}{dx}S_0(x)}{S_0(x)} = -\frac{d}{dx}\log S_0(x)$$

$$= -\frac{d}{dx}\left[\frac{1}{3}\log\left(1-\frac{x}{60}\right)\right]$$

$$= -\frac{1}{3}\cdot\frac{1}{1-\frac{x}{60}}\cdot\left(-\frac{1}{60}\right) = \frac{1}{3}\,\frac{\frac{1}{60}}{\frac{60-x}{60}} = \frac{1}{3(60-x)}$$

$$1000\cdot\frac{1}{3(60-35)} = \underline{\underline{13.333}}$$

2.2 Future lifetime random variable

- For a person now age x, its future lifetime is $T_x = X - x$. For a newborn, $x = 0$, so that we have $T_0 = X$.
- Life-age-x is denoted by (x).
- SDF: It refers to the probability that (x) will survive for another t years.

$$S_x(t) = \mathrm{Pr}[T_0 > x + t | T_0 > x] = \frac{S_0(x+t)}{S_0(x)} = {}_tp_x = 1 - {}_tq_x$$

- CDF: It refers to the probability that (x) will die within t years.

$$F_x(t) = \mathrm{Pr}[T_0 \leq x + t | T_0 > x] = \frac{S_0(x) - S_0(x+t)}{S_0(x)} = {}_tq_x$$

T_x = future lifetime of $\underset{\text{a person now age } x}{(x)}$

T_{45}

$T_x \to t$

$$T_x = X - x$$

f, F, S, μ

$\underline{\underline{S_0(x)}}$

$t \geq 0$

$$S_x(t) = Pr(T_x > t)$$
$$= Pr(X - x > t \mid X > x)$$
$$= \frac{Pr(X > x+t,\ X > x)}{Pr(X > x)} = \frac{Pr(X > x+t)}{Pr(X > x)}$$
$$= \frac{S_0(x+t)}{S_0(x)}$$

$x = 0 \to$ birth

$$S_0(t) = \frac{S_0(t)}{\cancel{S_0(0)}}$$

$$\boxed{S_x(t) = \frac{S_0(x+t)}{S_0(x)}} \quad \text{vlF} = Pr(T_x > t)$$

$$F_x(t) = 1 - S_x(t)$$

$$f_x(t) = \frac{d}{dt} F_x(t)$$

$$S_x(0) = 1$$

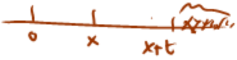
$$S_x(\infty) = \frac{S_0(\infty)}{S_0(x)} \rightarrow 0$$

$$\frac{d}{dt} S_x(t) = \frac{1}{S_0(x)} \frac{d}{dt} S_0(x+t) \leq 0 \text{ non-increasing}$$

small

$$Pr(T_x > t) = {}_t p_x$$

survival



$${}_t q_x$$

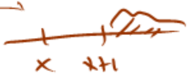
mortality

$_tp_x$ = probability that (x) will survive another t years

$_tq_x$ = probability that (x) will die before t years

Special cases: $t=1$

$_1p_x = p_x \longrightarrow t=1$



$_1q_x = q_x \longrightarrow t=1$

$T_x \rightarrow$ $S_x(t) = {}_tp_x = \dfrac{S_0(x+t)}{S_0(x)}$ ✓

x is known

$1-F_0(x)$ $\quad$ $1-F_0(x+t)$

$F_x(t) = {}_tq_x = 1 - \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{S_0(x) - S_0(x+t)}{S_0(x)} = \dfrac{F_0(x+t) - F_0(x)}{1 - F_0(x)}$

$f_x(t) = \dfrac{-d}{dt} S_x(t)$

$= \dfrac{1}{S_0(x)} \dfrac{-d}{dt} S_0(x+t) = \dfrac{f_0(x+t)}{S_0(x)}$

$\mu_x(t) = \dfrac{-\frac{d}{dt} S_x(t)}{S_x(t)} = \dfrac{f_0(x+t)/\cancel{S_0(x)}}{S_0(x+t)/\cancel{S_0(x)}} = \dfrac{f_0(x+t)}{S_0(x+t)} = \mu_{x+t}$

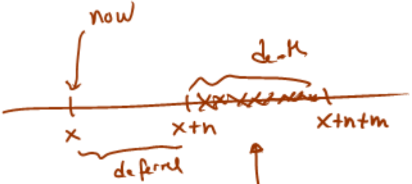
$\mu_x = \dfrac{f_0(x)}{S_0(x)}$

If X is exponential, μ is constant. $\mu_x = \mu$

T_x is exponential, μ is constant. $\mu_{x+t} = \mu$

$_tp_x$ $\quad$ $_tq_x$

deferred probability



$$_{n|m}q_x = \Pr(n < T_x \leq n+m)$$

Survival model $S_0(x) = \frac{1}{x+1}, \quad x \geq 0$

show: valid survival function

$${}_{10}p_{10} = S_{10}(10) = \frac{S_0(20)}{S_0(10)} = \frac{1/21}{1/11} = 11/21$$

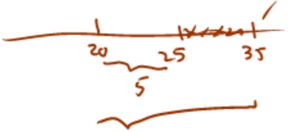
$$S_x(t) = {}_tp_x = \frac{S_0(x+t)}{S_0(x)}$$

$${}_{10}q_{10} = 10/21$$

$${}_{5|10}q_{20} = {}_5p_{20} - {}_{15}p_{20} = \frac{1/26}{1/21} - \frac{1/36}{1/21}$$

$$= 21\left(\frac{10}{26(36)}\right)$$

$$= ?$$



$$S_0(x) = \frac{1}{x+1}$$

$$\mu_x = \frac{-\frac{d}{dx}S_0(x)}{S_0(x)} = \frac{(x+1)^{-2}}{(x+1)^{-1}} = \frac{1}{x+1}$$

$$\mu_{25} = \frac{1}{26}$$

$S_0(25)$

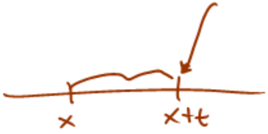
density of T_x $\qquad f_x(t) = \dfrac{f_0(x+t)}{S_0(x)}$

$$f_x(t) = \frac{f_0(x+t)}{S_0(x)} \cdot \frac{S_0(x+t)}{S_0(x+t)}$$

$$= \underbrace{\frac{f_0(x+t)}{S_0(x+t)}}_{\mu_{x+t}} \cdot \underbrace{\frac{S_0(x+t)}{S_0(x)}}_{S_x(t) = {}_t p_x}$$

$$\boxed{f_x(t) = {}_t p_x \cdot \mu_{x+t}}$$

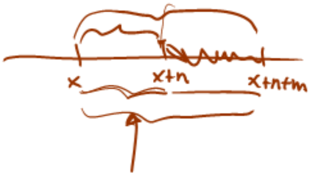
VIF



$${}_{n|m}q_x = {}_np_x - {}_{n+m}p_x$$

$$= {}_{n+m}q_x - {}_nq_x$$

$$= \underline{{}_np_x} \cdot \underline{{}_mq_{x+n}}$$



$$Pr(n < T_x \le n+m)$$

$$= Pr(T_x \le n+m) - Pr(T \le n)$$

$$= Pr(T_x > n) - Pr(T_x > n+m)$$

- continued

${}_tp_x \, \mu_{x+t}$

- Density:
$$f_x(t) = \frac{dF_x(t)}{dt} = -\frac{dS_x(t)}{dt} = \frac{f_0(x+t)}{S_0(x)}.$$
- Remark: If $t = 1$, simply use p_x and q_x.
- p_x refers to the probability that (x) survives for another year.
- $q_x = 1 - p_x$, on the other hand, refers to the probability that (x) dies within one year.

${}_tp_x$ ${}_tq_x$

Conditions to be valid

$S_x(t)$ valid

To reiterate, these are the conditions for a survival function to be considered valid:

- ✓ $S_x(0) = 1$: probability a person age x survives 0 years is 1.
- ✓ $S_x(\infty) = \lim_{t\to\infty} S_x(0) = 0$: all lives, regardless of age, eventually die.
- ✓ The survival function $S_x(t)$ for a life (x) must be a <u>non-increasing</u> function of t.

  ✓

  $$\frac{d}{dt} S_x(t) \le 0$$

2.3 Force of mortality of T_x

- In deriving the force of mortality, we can use the basic definition:

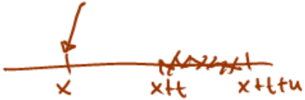
$$\mu_x(t) = \frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)}{S_0(x)} \cdot \frac{S_0(x)}{S_0(x+t)}$$
$$= \frac{f_0(x+t)}{S_0(x+t)} = \underline{\underline{\mu_{x+t}}}.$$

- This is easy to see because the condition of survival to age $x+t$ supercedes the condition of survival to age x.
- This results implies the following very useful formula for evaluating the density of T_x:

$$\boxed{f_x(t) = {}_tp_x \times \mu_{x+t}}$$

density of Tx

Special probability symbol



- The probability that (x) will survive for t years and die within the next u years is denoted by ${}_{t|u}q_x$. This is equivalent to the probability that (x) will die between the ages of $x+t$ and $x+t+u$.
- This can be computed in several ways:

survive t years but die the following u yrs

$$\begin{aligned}
{}_{t|u}q_x &= \boxed{\Pr[t < T_x \le t+u]} \\
&= \Pr[T_x \le t+u] - \Pr[T_x < t] \\
&= {}_{t+u}q_x - {}_tq_x \\
&= {}_tp_x - {}_{t+u}p_x \\
&= {}_tp_x \times {}_uq_{x+t}.
\end{aligned}$$

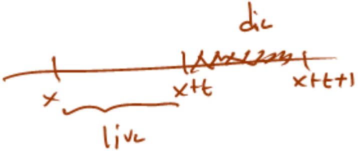
- If $u = 1$, prefix is deleted and simply use ${}_{t|}q_x$.

UCONN

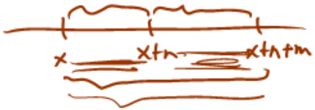
$_{t|u}q_x$ deferred prob

$u = 1$

$_{t|1}q_x = {}_{t|}q_x$ = prob that (x) will die between $x+t$ and $x+t+1$



Results: p's are multiplicative



$$_{n+m}p_x = {}_np_x \; {}_mp_{x+n} \quad ✓$$

$$= {}_mp_x \; {}_np_{x+m} \quad ✓$$

$$_tp_x = p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+t-1}$$

t years = t terms

q's are not multiplicative $\quad {}_{n+m}q_x \neq {}_nq_x \cdot {}_mq_{x+m}$

$$_{n+m}q_x = {}_nq_x + {}_np_x \; {}_mq_{x+n}$$

TRUE Prove ✓

$$1 - {}_{n+m}p_x$$

$$= 1 - {}_np_x \; {}_mp_{x+n}$$

Other useful formulas

- It is easy to see that

$$F_x(t) = \int_0^t f_x(s)ds$$

which in actuarial notation can be written as

$${}_tq_x = \int_0^t {}_sp_x \ \mu_{x+s} ds$$

- See Figure 2.3 for a very nice interpretation.
- We can generalize this to

$${}_{t|u}q_x = \int_t^{t+u} {}_sp_x \ \mu_{x+s} ds$$

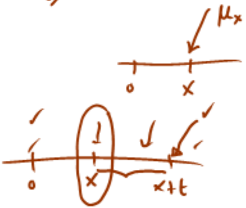
Sept 4

X age at death r.v. $\quad F_0(x),\ f_0(x),\ S_0(x),\ \mu_x$

$S_0(x) = 1-F_0(x) = Pr(X>x)$

T_x future lifetime for (x) $= X-x \mid X>x$

(x): life age x



$$f_x(t) = \frac{f_0(x+t)}{S_0(x)} = {}_tp_x\,\mu_{x+t}$$

$$F_x(t) = 1 - S_x(t) = Pr(T_x \le t) = {}_tq_x$$

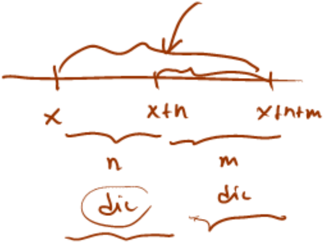
$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = Pr(T_x > t) = Pr(T_x \geq t) = {}_tp_x$$

$$\mu_{x+t} = \mu_x(t)$$

$${}_tp_x = e^{-\int_0^t \mu_{x+z}\,dz} = e^{-\int_x^{x+t} \mu_z\,dz}$$

multiplicative property of p

$${}_{n+m}p_x = {}_np_x \cdot {}_mp_{x+n}$$



q is not multiplicative

$${}_{n+m}q_x = 1 - {}_{n+m}p_x$$

$$= {}_nq_x + {}_np_x \cdot {}_mq_{x+n}$$

$$\left.{}_tp_x\right|_{t=1} = {}_1p_x = \underline{\underline{p_x}}$$

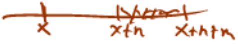
$$\left.{}_tq_x\right|_{t=1} = {}_1q_x = \underline{\underline{q_x}}$$

$$\underbrace{p_x} \cdot \underbrace{p_{x+1}} \cdot \underbrace{p_{x+2}} \cdots p_{x+n+m-1}$$

T_x (continuous) future lifetime

Curtate future lifetime (discrete)

${}_{n|m}q_x$ ✓



(discrete) ✓

2.6 Curtate future lifetime

count only the integral years

- Curtate future lifetime of (x) is the number of future years completed by (x) prior to death.
- $K_x = \lfloor T_x \rfloor$, the greatest integer of T_x.

$\lfloor 26.3 \rfloor = 26$
$\lfloor 26.99 \rfloor = 26$

- Its probability mass function is

$$\begin{aligned} \mathsf{Pr}[K_x = k] &= \mathsf{Pr}[k \le T_x < k+1] = \mathsf{Pr}[k < T_x \le k+1] \\ &= S_x(k) - S_x(k+1) = {}_{k+1}q_x - {}_kq_x = {}_{k|}q_x, \end{aligned}$$

for $k = 0, 1, 2, \ldots$

- Its distribution function is

$$\mathsf{Pr}[K_x \le k] = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x.$$

x, T_x, K_x

discrete random variable

K_x = curtate future lifetime of (x)

$\rightarrow 0, 1, 2, \ldots \infty$

x
K is integer

probability mass function

$$\Pr[K_x = k] = \Pr[\underbrace{\lfloor T_x \rfloor} = k]$$

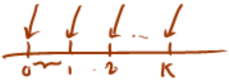
$$= {}_{k|1}q_x = {}_{k|}q_x$$

$$= {}_k p_x \cdot q_{x+k}$$

$$= {}_k p_x - {}_{k+1}p_x$$

$$= {}_{k+1}q_x - {}_k q_x$$

$$= \Pr[k < T_x \le k+1]$$



cumulative distribution function

$$\underbrace{\Pr[K_x \le k]} = {}_{k+1}q_x$$

$$\Pr[K_x > k] = {}_{k+1}p_x$$

$$\Pr[K_x \le k] = \sum_{j=0}^{k} \underbrace{\Pr[K_x = j]} \longrightarrow {}_{j|}q_x = {}_{j+1}q_x - {}_{j}q_x$$

$$= \sum_{j=0}^{k} \left({}_{j+1}q_x - {}_{j}q_x \right)$$

$$= \left(\cancel{{}_{1}q_x} + \cancel{{}_{2}q_x} + \dots + {}_{k+1}q_x \right) - \left({}_{0}q_x + \cancel{{}_{1}q_x} + \dots + \cancel{{}_{k}q_x} \right)$$

$$= {}_{k+1}q_x - \cancel{{}_{0}q_x} \to 0$$

$$= {}_{k+1}q_x$$

constant f-n of mortality $\quad \underline{\underline{\mu = .05}}$

$T_x \sim$ Exponential $\quad X \sim$ Exponential

$K_x \sim$

$_t p_x = e^{-\mu t} \quad _t q_x = 1 - e^{-\mu t}$

$\mu_x = \mu$ ✓

$\mu_{x+t} = \mu$ ✓

pmf: $\Pr[K_x = k] = {}_{k|}q_x = \underbrace{_k p_x}_{e^{-\mu k}} \cdot \underbrace{q_{x+k}}_{(1-e^{-\mu})}$ $\quad \mu = .05 \Rightarrow (1 - e^{-.05}) \cdot e^{-.05k}$

$\Pr[K_x \le k] = {}_{k+1}q_x = 1 - e^{-\mu(k+1)}$ $\quad \mu = .05 \Rightarrow 1 - e^{-.05(k+1)}$

$\underbrace{(1-e^{-.05})e^{-.05k}}_{\text{special distribution}}$ ✓, $k = 0, 1, 2, \ldots \infty$

$\underline{\text{Geometric}}$

$\underline{c\, e^{-\mu k}}$

$\sum_{k=0}^{\infty} \underline{(1-e^{-.05})}\, e^{-.05k} = 1$

$\cancel{(1-e^{-.05})} \underbrace{\sum_{k=0}^{\infty} e^{-.05k}}_{\frac{1}{\cancel{1-e^{-.05}}}} = 1$ ✓

2.5/2.6 Expectation of life

average lifetime

- The expected value of T_x is called the complete expectation of life:

$$\mathring{e}_x = \mathsf{E}[T_x] = \int_0^\infty t f_x(t)dt = \int_0^\infty t\, {}_tp_x \mu_{x+t} dt = \boxed{\int_0^\infty {}_tp_x dt.}$$

- The expected value of K_x is called the curtate expectation of life:

$$e_x = \mathsf{E}[K_x] = \sum_{k=0}^\infty k \cdot \Pr[K_x = k] = \sum_{k=0}^\infty k \cdot {}_{k|}q_x = \boxed{\sum_{k=1}^\infty {}_kp_x.}$$

- Proof can be derived using discrete counterpart of integration by parts (summation by parts). Alternative proof will be provided in class.
- Variances of future lifetime can be similarly defined.

Recall: If X is continuous, $x \geq 0$ $\quad E[X] = \int_0^\infty (1 - F_0(x))\,dx$

$$= \int_0^\infty S_0(x)\,dx \checkmark$$

$$E[T_x] = \int_0^\infty t \cdot \underbrace{f_x(t)}_{{}_tp_x\,\mu_{x+t}}\,dt = \mathring{e}_x = \int_0^\infty S_x(t)\,dt \qquad \underline{E[g(T_x)]}$$

$$\boxed{\mathring{e}_x = \int_0^\infty {}_tp_x\,dt} \quad \text{VIF}$$

$$E[K_x] = \boxed{e_x = \sum_{k=1}^\infty {}_kp_x}$$

$$e_x = \sum_{k=0}^{\infty} k \cdot \underbrace{{}_{k|}q_x}_{({}_kp_x - {}_{k+1}p_x)}$$

$${}_{k|}q_x = {}_kp_x \, q_{x+k} ,$$

$$= {}_kp_x - {}_{k+1}p_x ,$$

$$= {}_{k+1}q_x - {}_kq_x ,$$

$$= \underbrace{\sum_{k=0}^{\infty} k \cdot {}_kp_x}_{(\cancel{0} + \underbrace{p_x} + \underbrace{2 \cdot {}_2p_x} + \underbrace{3 \, {}_3p_x} + \cdots)} - \underbrace{\sum_{k=0}^{\infty} k \cdot {}_{k+1}p_x}_{(\underbrace{{}_2p_x} + 2 \cdot {}_3p_x + \cdots)}$$

$$= \underbrace{p_x + {}_2p_x + {}_3p_x + \cdots}_{\sum_{k=1}^{\infty} {}_kp_x}$$

Illustrative example 2

Let X be the age-at-death random variable with

$$\mu_x = \frac{1}{2(100 - x)}, \quad \text{for } 0 \le x < 100.$$

newborn

1. Give an expression for the survival function of X.
2. Find $f_{36}(t)$, the density function of future lifetime of (36).
3. Compute ${}_{20}p_{36}$, the probability that life (36) will survive to reach age 56.
4. Compute $\mathring{e}_{36}$, the average future lifetime of (36).

$e_{36} = ?$

① $S_0(x) = e^{-\int_0^x \mu_z \, dz}$

$$= e^{-\int_0^x \frac{1}{2(100-z)} dz}$$

$$= e^{-\frac{1}{2}\left(-\log(100-z)\Big|_0^x\right)}$$

$$= e^{-\frac{1}{2}\left(-\log(100-x) + \log(100)\right)}$$

$$= e^{-\frac{1}{2}\left(\log\frac{100}{100-x}\right)} = \left(\frac{100}{100-x}\right)^{-1/2} = \left(\frac{100-x}{100}\right)^{1/2}$$

$0 \le x < 100$

$S_0(0) \to 1$

$S_0(100) \to 0$

$\mu_x = \frac{1}{2(100-x)}, \quad 0 \le x < 100$

$\int \frac{1}{100-z} dz$

$-\log(100-z)$

$\log = \ln$

(2) $S_0(x) = \left(\frac{100-x}{100}\right)^{1/2}$ Generalized de Moivre ✓

$f_0(x) = -\frac{d}{dx} S_0(x) = \cancel{-}\frac{1}{10}\left(\frac{1}{2}\right)(100-x)^{-1/2}\cancel{(-1)} = \underline{\frac{1}{20}(100-x)^{-1/2}}$ ✓

agree ✓

$\underbrace{100^{-1/2}}_{10^{-1}}(100-x)^{1/2} = \frac{1}{10}(100-x)^{1/2}$ ✓

$f_{36}(t) = \text{density of } T_{36} = \frac{f_0(36+t)}{S_0(36)} = \frac{\frac{1}{\cancel{20}_2}(100-36-t)^{-1/2}}{\frac{1}{\cancel{10}}(100-36)^{1/2}}$

$= \frac{1}{16}(64-t)^{-1/2}, \quad \underline{0 \le t < 64}$ ✓

(3) ${}_{20}p_{36}$ = Prob that (36) will live for another 20 years $= \frac{S_0(36+20)}{S_0(36)} = \frac{\cancel{\frac{1}{10}}(44)^{1/2}}{\cancel{\frac{1}{10}}(64)^{1/2}}$

$= \frac{1}{8}44^{1/2} = .8291562$

$S_0(x) = \frac{1}{10}(100-x)^{1/2}$

④ $\underline{\mathring{e}_{36}} = E[T_{36}] = \int_0^\infty \underbrace{S_{36}(t)}\, dt$

$$= \int_0^{64} \underbrace{\frac{S_0(36+t)}{S_0(36)}}_{\frac{\cancel{\frac{1}{10}}(64-t)^{1/2}}{\cancel{\frac{1}{10}}\, 64^{1/2}}} dt$$

$e_{36} = ?$

$\Downarrow$

no closed form

$\underline{\text{to do !!!}}$

$$= \frac{1}{8}\int_0^{64} (64-t)^{1/2}\, dt$$

$$= \frac{1}{8}\left[-\frac{(64-t)^{3/2}}{3/2} \Big|_0^{64} \right]$$

$$= \frac{1}{8}\cdot\frac{1}{3/2}\, 64^{3/2} = \frac{1}{\cancel{8}}\,\frac{2}{3}\, 64 \cdot \cancel{64^{1/2}}^{\,8} = \frac{128}{3}$$

$$= \underline{42.66667}$$

Class Test 1

topic all of Chap. 2

bring , own calculator

, 1 formulasheet

On Monday

classes

Illustrative example 3

$\mathring{e}_x = E[T_x]$

$E[X]$

$E[x] = \frac{\omega}{2} = 30 \Rightarrow \omega = 60$

$\int_0^\omega (1-\frac{x}{\omega})\,dx = \frac{\omega}{2}$

Suppose you are given that:

- $\mathring{e}_0 = 30$; and
- $S_0(x) = 1 - \dfrac{x}{\omega}$, for $0 \leq x \leq \omega$.

de Moivre's

$T_{15} \sim$ uniform$(0, 45)$

Evaluate $\mathring{e}_{15}$.

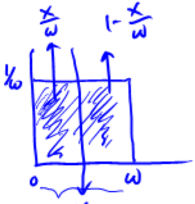
$\Pr[T_{15} > t] = \dfrac{\Pr[X > 15+t]}{\Pr[X > 15]}$

Solution to be discussed in lecture.

$E[T_{15}] = \frac{45}{2} = 22.5$

Review de Moivre

Mortality follows de Moivre



preserved for T_x

$X \sim \text{Uniform}(0, \omega)$ → parameter

$f_0(x) = \frac{1}{\omega}, \quad 0 < x \leq \omega$

$E[X] = \frac{\omega}{2}$

$S_0(x) = 1 - \frac{x}{\omega}$

$F_0(x) = \frac{x}{\omega}$

$\mu_x = \frac{f_0(x)}{S_0(x)} = \frac{\frac{1}{\omega}}{1-\frac{x}{\omega}} = \frac{1}{\omega - x}$

$E[X] = \int_0^{\omega} x \cdot \frac{1}{\omega}\, dx = \frac{\omega}{2}$

$T_x \sim \text{Uniform}(0, \omega - x)$

$f_x(t) = \frac{1}{\omega - x}, \quad 0 < t \leq \omega - x$

$S_x(t) = 1 - \frac{t}{\omega - x}$

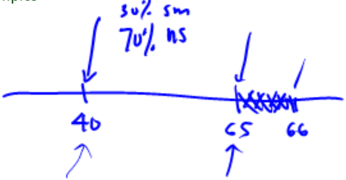
$E[T_x] = \frac{\omega - x}{2}$

$F_x(t) = \frac{t}{\omega - x}$

$\mu_{x+t} = \frac{1}{\omega - x - t}$

Illustrative example 4

homogeneous, heterogeneous,



For a group of lives aged 40 consisting of 30% smokers (sm) and the rest, non-smokers (ns), you are given:

- For non-smokers, $\mu_x^{\text{ns}} = 0.05$, for $x \geq 40$ $\quad {}_tp_{40}^{ns} = e^{-.05t}, \; t \geq 0$
- For smokers, $\mu_x^{\text{sm}} = 0.10$, for $x \geq 40$ $\quad {}_tp_{40}^{sm} = e^{-.10t}, \; t \geq 0$

Calculate q_{65} for a life randomly selected from those who reach age 65.

$$p_{65} = \Pr[T_{65} > 1] = \underbrace{\Pr[T_{65} > 1 \mid ns]}_{{}_tp_{65}^{ns}} \Pr[ns] + \underbrace{\Pr[T_{65} > 1 \mid sm]}_{{}_tp_{65}^{sm}} \Pr[sm]$$

law of total probability

law of total probability

$$P_{cs} = \underline{P_{cs}^{ns}} \cdot Pr(ns) + \underline{P_{cs}^{sm}}\, Pr(sm)$$

$$Pr(ns) \to \cancel{70\%} \quad .87 \qquad Pr(sm) \to \cancel{30\%} \quad .11$$

$$P_{cs}^{ns} \to e^{-.05} \qquad P_{cs}^{sm} \to e^{-.10}$$

$$= \underbrace{e^{-.05} \cdots e^{-.10}}_{1 - P_{cs} \;=\; .05384399}$$

40 → 30% sm, 70% ns

65 → proportion with change

$T_{40} \sim Exp$

$T_{65} \sim Exp$

$\mu^{ns} = .05$

$\mu^{sm} = .10$

L = # of lives age 40

$$Pr[ns@65] = \frac{L \cdot 70\%\, e^{-.05(25)}}{L \cdot 70\%\, e^{-.05(25)} + L \cdot 30\%\, e^{-.10(25)}} \approx .8906403$$

$$L * 70\% * \underbrace{{}_{25}P_{40}^{ns}}_{e^{-.05(25)}}$$

$$L * 30\% * \underbrace{{}_{25}P_{40}^{sm}}_{e^{-.10(25)}}$$

$$Pr[sm@65] = 1 - .8906403 = .1093597$$

constant force $\mu_x = \mu =$ constant
independent of x

$X \sim$ exponential

$f_0(x) = \mu e^{-\mu x}, \quad x > 0$

$S_0(x) = e^{-\mu x}$

$F_0(x) = 1 - e^{-\mu x}$

$\mu_x = \mu$

memoryless

$T_x \sim$ exponential

$\mu_{x+t} = \mu$

$f_x(t) = \mu e^{-\mu t}$

$S_x(t) = e^{-\mu t}$

$F_x(t) = 1 - e^{-\mu t}$

$\underbrace{\Pr[T_x > t]}_{{}_t p_x}$

$\Pr[T_x \le t] = {}_t q_x$

Temporary (partial) expectation of life

$$\underbrace{E[T_x]}_{\mathring{e}_x} = \int_0^\infty t\,p_x\,dt$$

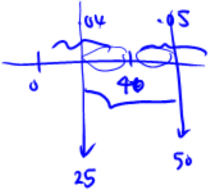
We can also define temporary (or partial) expectation of life:

$$\mathring{e}_{x:\overline{n}|} = \mathrm{E}\left[\min(T_x, n)\right] = \mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,dt \longrightarrow \leq n$$

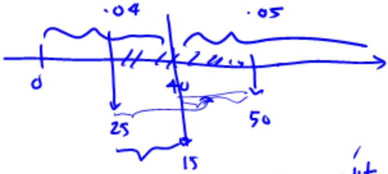
This can be interpreted as the average future lifetime of (x) within the next n years.

Suppose you are given:

$$\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x \geq 40 \end{cases}$$



Calculate $\mathring{e}_{25:\overline{25}|}$



$${}_tp_x = e^{-\mu t}$$

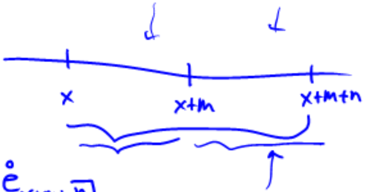
$$\mathring{e}_{25:\overline{25|}} = \int_0^{25} {}_tp_{25}\,dt$$

$$= \int_0^{15} {}_tp_{25}\,dt + \int_{15}^{25} {}_tp_{25}\,dt$$

$$\int_0^{15} e^{-.04t}\,dt \qquad \int_0^{10} {}_{15}p_{25} \cdot {}_tp_{40}\,dt$$

$$= \frac{1-e^{-.04(15)}}{.04} + \underbrace{{}_{15}p_{25}}_{e^{-.04(15)}} \underbrace{\int_0^{10} e^{-.05t}\,dt}_{\frac{1-e^{-.05(10)}}{.05}} = 15.59852 < 25$$

average future life of a person (25) for the next 25 years



$$\mathring{e}_{x:\overline{m+n}|} = \underbrace{\mathring{e}_{x:\overline{m}|} + {}_mp_x \cdot \mathring{e}_{x+m:\overline{n}|}}_{\text{VIF}}$$

${}_mp_x \; {}_{t-x}p_{x+m}$

$$\int_0^{m+n} {}_tp_x\,dt = \underbrace{\int_0^m {}_tp_x\,dt}_{\mathring{e}_{x:\overline{m}|}} + \underbrace{\int_m^{m+n} {}_tp_x\,dt}_{{}_mp_x \int_0^n {}_tp_{x+m}\,dt}$$

$${}_mp_x \underbrace{\int_0^n {}_tp_{x+m}\,dt}_{\mathring{e}_{x+m:\overline{n}|}}$$

$$\boxed{\mathring{e}_{x:\overline{m+n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x \cdot \mathring{e}_{x+m:\overline{n}|}}$$

analogy: ${}_{m+n}q_x = {}_mq_x + {}_mp_x \cdot {}_nq_{x+m}$

Given: $\underbrace{\overset{\circ}{e}_{20:\overline{4}|}} = 3.7$ $\quad \overset{\circ}{e}_{20:\overline{10}|} = 8.2$ $\quad \underbrace{\overset{\circ}{e}_{24:\overline{6}|}} = 5.4$

Calculate prob that a life (20) will **not** survive to reach another 4 years

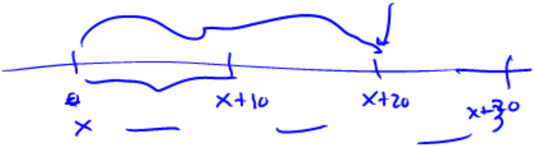
$${}_4q_{20}$$

$$\underset{8.2}{\overset{\circ}{e}_{20:\overline{10}|}} = \underset{3.7}{\overset{\circ}{e}_{20:\overline{4}|}} + {}_4p_{20}\ \underset{5.4}{\overset{\circ}{e}_{24:\overline{6}|}}$$

Important

$${}_4p_{20} = \frac{8.2 - 3.7}{5.4} = \frac{5}{6}$$

$${}_4q_{20} = \frac{1}{6}$$



$$\ddot{e}_{x:\overline{30}|} = \ddot{e}_{x:\overline{10}|} + {}_{10}p_x \, \ddot{e}_{x+10:\overline{10}|} + \underbrace{{}_{20}p_x}_{{}_{10}p_x \cdot {}_{10}p_{x+10}} \, \ddot{e}_{x+20:\overline{10}|}$$

STOP here

Generalized De Moivre's law

$S_0(x) = 1 - \frac{x}{\omega}, \ 0 < x \leq \omega$

GDM(ω, α)

The SDF of the so-called Generalized De Moivre's Law is expressed as

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \text{ for } 0 \leq x \leq \omega.$$

$\alpha = 1$

$\Rightarrow$ de Moivre's

Derive the following for this special type of law of mortality:

1. force of mortality $\mu_x = \frac{\alpha}{\omega - x}$
2. survival function associated with T_x $\quad S_x(t) = \left(1 - \frac{t}{\omega - x}\right)^{\alpha}, \ 0 < t \leq \omega - x$
3. expectation of future lifetime of x $\quad E[T_x] = ? \ \frac{\omega - x}{\alpha + 1}$
4. can you find explicit expression for the variance of T_x? $\quad Var[T_x]$

$\alpha = 1, \ \frac{\omega - x}{2}$

GDM ω, α $\quad S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}, \quad 0 < x \le \omega \longrightarrow$ max. age

$$\mu_x = -\frac{d}{dx} \log S_0(x)$$

$$= -\frac{d}{dx}\, \alpha \cdot \log\left(1 - \frac{x}{\omega}\right) = \cancel{-}\alpha \cdot \frac{1}{1 - \frac{x}{\omega}} \cdot \left(\cancel{-}\frac{1}{\omega}\right) = \alpha \cdot \frac{\frac{1}{\cancel{\omega}}}{\frac{\omega - x}{\cancel{\omega}}} = \frac{\alpha}{\omega - x}$$

$$F_0(x) = 1 - \left(1 - \frac{x}{\omega}\right)^{\alpha}$$

$X \sim$

$T_x \sim$

$$f_0(x) = \frac{d}{dx} F_0(x) = \frac{\alpha}{\omega}\left(1 - \frac{x}{\omega}\right)^{\alpha - 1}$$

T_x is also de Moivre's

GDM $\omega - x, \alpha$

$$S_x(t) = \Pr[T_x > t] = \frac{\Pr[X > x + t]}{\Pr[X > x]} = \frac{S_0(x+t)}{S_0(x)}$$

$$\mu_{x+t} = \frac{\alpha}{\omega - x - t}$$

$$\left(1 - \frac{t}{\omega - x}\right)^{\alpha} = \cdots = \frac{\left(1 - \frac{x+t}{\omega}\right)^{\alpha}}{\left(1 - \frac{x}{\omega}\right)^{\alpha}}$$

$$f_x(t) = \frac{\alpha}{\omega - x}\left(1 - \frac{t}{\omega - x}\right)^{\alpha - 1}$$

$$E[x] = \int_0^{\omega} S_0(x)\,dx = \underbrace{\int_0^{\omega} \left(1-\frac{x}{\omega}\right)^{\alpha} dx}$$

$$u = 1-\frac{x}{\omega}$$

$$du = -\frac{1}{\omega}dx \Rightarrow dx = -\omega\, du$$

$$\int_a^b = -\int_b^a$$

$$= \int_1^0 u^{\alpha}\cdot(-\omega)\,du$$

$$= \omega\int_0^1 u^{\alpha}\,du = \omega\,\frac{u^{\alpha+1}}{\alpha+1}\Big|_0^1 = \frac{\omega}{\alpha+1}$$

$$Var[x] = E(x^2) - E[x]^2$$

$$E[T_x] = \frac{\omega - x}{\alpha+1}$$

Illustrative example

- We will do **Example 2.6** in class.

Example 2.3

i) $S_0(0) = 1$ alive at birth

ii) $S_0(\infty) = 0$ everybody dies eventually

iii) non-increasing $\frac{d}{dx} S_0(x) \le 0 \rightarrow$ decreasing

if <0,

$c = e^{\log c}$

$$\int_0^t Bc^{x+s}\,ds = Bc^x \underbrace{\int_0^t c^s\,ds} = Bc^x \left(\frac{1}{\log c}(c^t - 1)\right)$$

Gompertz

Let $\mu_x = Bc^x$, for $x > 0$, where B and c are constants such that $0 < B < 1$ and $c > 1$.

$$Pr[T_x > t] = e^{-\int_0^t \mu_{x+s}\,ds}$$

$c > 0$

Derive an expression for $S_x(t)$.

derive $S_0(x)$

$0 < B < 1$

$$\frac{S_0(x+t)}{S_0(x)} = S_x(t) = Pr[T_x > t] = {}_tp_x$$

$c > 1$

$$ {}_tp_x = S_x(t) = e^{-\int_0^t \mu_{x+s}\,ds} = e^{-\frac{Bc^x}{\log c}(c^t - 1)}$$

Gompertz $\mu_x = Bc^x \Rightarrow {}_tp_x = e^{-\frac{Bc^x}{\log c}(c^t-1)}$

Makeham $\mu_x = \underset{\substack{\uparrow \\ \text{due to} \\ \text{accident}}}{A} + Bc^x$

$${}_tp_x = e^{-At}\, e^{-\frac{Bc^x}{\log c}(c^t-1)}$$

$$\mathring{e}_{x:\overline{n|}} = \int_0^n {}_tp_x\, dt$$

$x = 35$

$A = .002 \quad B = 10^{-4.5} \quad c = 1.10$

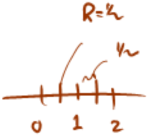
Calculate $\mathring{e}_{35:\overline{2|}}$

$$= \int_0^2 {}_tp_{35}\, dt = \int_0^1 {}_tp_{35}\, dt + \int_1^2 {}_tp_{35}\, dt$$

trapezoidal rule for 2 years

$$\underbrace{\tfrac{1}{2}(1 + .9970719) + \tfrac{1}{2}(.9970719 + .9940579)}_{= 1.994102}$$

t	${}_tp_{35}$
0	1
1	.9970719
2	.9940579



$$\mathring{e}_{35:\overline{2|}} = \underbrace{\int_0^1 {}_tp_{35}\,dt}_{\text{mid } 1/2} + \underbrace{\int_1^2 {}_tp_{35}\,dt}_{\text{mid } 1\frac{1}{2}}$$

$$= \underbrace{\frac{1/2}{3}\left[1 + 4\,{}_{1/2}p_{35} + {}_1p_{35}\right] + \frac{1/2}{3}\left[{}_1p_{35} + 4\cdot{}_{1\frac{1}{2}}p_{35} + {}_2p_{35}\right]}$$

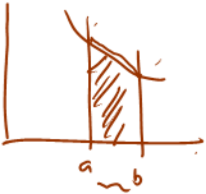
$$= \underline{\underline{1.994116}} \checkmark$$

t	${}_tp_{35}$
$1/2$	.998546
$1\frac{1}{2}$	.9955768

Exact Value = $\mathring{e}_{35:\overline{2|}}$ = $\underline{\underline{1.994116}}$ ✓

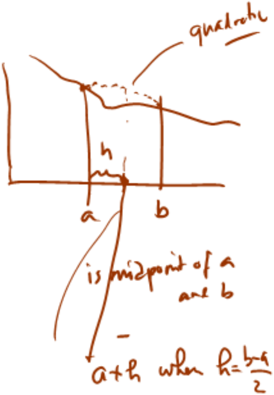
① Trapezoidal Rule

$$\int_a^b f(x)\,dx = \underbrace{\tfrac{1}{2}(b-a)\left[f(a)+f(b)\right]}_{\text{area of a trapezoid}}$$

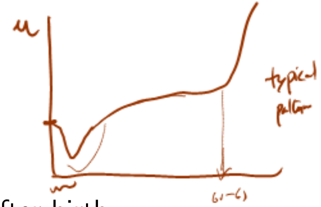


② Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a)+4f(a+h)+f(b)\right]$$



Typical mortality pattern observed



- High (infant) mortality rate in the first year after birth.
- Average lifetime (nowadays) range between 70-80 - varies from country to country.
- Fewer lives/deaths observed after age 110 - supercentenarian is the term used to refer to someone who has reached age 110 or more.
- The highest recorded age at death (I believe) is 122.
- Different male/female mortality pattern - females are believed to live longer.

- bear pain
- lifestyle

122

People are living longer

underwriting / selection - preferred

Substandard mortality

- A substandard risk is generally referred to someone classified by the insurance company as having a higher chance of dying because of:
  - some physical condition
  - family or personal medical history
  - risky occupation
  - dangerous habits or lifestyle (e.g. skydiving)
- Mortality functions are superscripted with s to denote substandard: q_x^s and μ_x^s.
- For example, substandard mortality may be obtained from a standard table using:
  1. adding a constant to force of mortality: $\mu_x^s = \mu_x + c$
  2. multiplying a fixed constant to probability: $q_x^s = \min(kq_x, 1)$
- The opposite of a substandard risk is preferred risk where someone is classified to have better chance of survival.

$\mu_x^s = \mu_x + c \quad c>0$ worse mortality addition

$${}_tp_x^s = e^{-\int_0^t (\mu_{x+s}+c)\,ds}$$

$c>0$

$$= \underbrace{e^{-\int_0^t \mu_{x+s}\,ds}}_{{}_tp_x} e^{-ct} < {}_tp_x \quad \text{or} \quad {}_tq_x^s > {}_tq_x$$

---

multiplication

$q_x^s = k \cdot q_x \quad \underline{\underline{k>1}}, \quad \underline{k \cdot q_x \le 1}$

$${}_tp_x^s = p_x^s \cdot p_{x+1}^s \cdots p_{x+t-1}^s$$

$$= (1-k\cdot q_x)(1-k\cdot q_{x+1}) \cdots (1-k\cdot q_{x+t-1})$$

$$< p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+t-1} = {}_tp_x$$

Practice problem - SOA MLC Fall 2000 Question #4

$T_{25} \sim \text{Uniform}(0,75)$

Mortality for Audra, age 25, follows De Moivre's law with $\omega = 100$. If she takes up hot air ballooning for the coming year, her assumed mortality will be adjusted so that for the coming year only, she will have a constant force of mortality of $\underline{\underline{0.1}}$.

Calculate the decrease in the 11-year temporary complete life expectancy for Audra if she takes up hot air ballooning.

without air ballooning

$$\mathring{e}_{25:\overline{11}|} = \int_0^{11} {}_tp_{25}\,dt = \int_0^{11}\left(1-\frac{t}{75}\right)dt = 11 - \frac{1}{75(2)}11^2 = 10.19333$$

with air ballooning

$$\mathring{e}^{s}_{25:\overline{11}|} = \int_0^1 {}_tp^{s}_{25}\,dt + p^{s}_{25}\cdot\mathring{e}_{26:\overline{10}|}$$

$$= \int_0^1 e^{-.1t}\,dt + e^{-.1}\int_0^{10}\left(1-\frac{t}{74}\right)dt = \text{next page}$$

$$\mathring{e}_x = \mathring{e}_{x:\overline{m}|} + \underline{\underline{{}_m p_x}}\, \mathring{e}_{x+m}$$

0 m

$${}_t p_x = e^{-\mu t}$$ ✓

m = 1

$$\boxed{\mathring{e}^s = \mathring{e}^s_{x:\overline{1}|} + p^s_x\, \mathring{e}_{x+1}}$$

$$\underbrace{\int_0^1 e^{-.1t}\,dt} + e^{-.1}\, \underbrace{\mathring{e}_{26:\overline{10}|}}_{\int_0^{10} {}_t p_{26}\,dt}$$

$$\frac{1-e^{-.1}}{.1} + e^{-.1} \underbrace{\int_0^{10}\left(1-\frac{t}{74}\right)dt}_{10 - \frac{1}{74(2)}10^2} = 9.388623$$

deduct from first

Difference is $10.19333 - 9.388623 = \underline{0.80471}$ ✓

Illustrative example 5

next chapter

You are given:

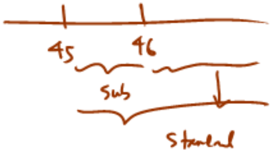
- Mortality for standard lives follows the Standard Ultimate Life Table (SULT).
- The force of mortality for standard lives age $45+t$ is represented as μ^{SULT}_{45+t}.
- The force of mortality for substandard lives age $45+t$, μ^{sub}_{45+t}, is defined by

standard

$$\mu^{s}_{45+t} = \mu^{sub}_{45+t} = \begin{cases} \mu^{\text{SULT}}_{45+t} + 0.05, & \text{for } 0 \le t < 1 \\ \mu^{\text{SULT}}_{45+t}, & \text{for } t \ge 1 \end{cases} = \begin{cases} \mu_{45+t} + .05, & 0 \le t < 1 \\ \mu_{45+t}, & t \ge 1 \end{cases}$$

Calculate the probability that a substandard 45-year-old will die within the next two years.

assume $q_{45} = 0.01$ $\quad q_{46} = 0.02$



$_2q_{45}^{sub} = \ ?$

$\mu' \longrightarrow p$

$$_2p_{45}^{sub} = {}_2p_{45}^{s} = e^{-\int_0^2 \mu^s_{45+t}\,dt}$$

$$= e^{-\left[\int_0^1 (\mu_{45+t}+.05)\,dt + \int_1^2 \mu_{45+t}\,dt\right]}$$

$$= e^{-\left[\underbrace{\int_0^1 \mu_{45+t}\,dt} + .05 + \underbrace{\int_1^2 \mu_{45+t}\,dt}\right]}$$

$$= e^{-.05}\, \underbrace{e^{-\int_0^2 \mu_{45+t}\,dt}}_{_2p_{45}} \longrightarrow \overset{.99}{p_{45}}\ \overset{.98}{p_{46}}$$

$$= \underbrace{e^{-.05}(.99)(.98)}_{< \underbrace{.99(.98)}_{\text{standard}}}$$

standard: prob that (40) will live for 10 years is 0.90

substandard: $\mu^{s}_{40+t} = \mu_{40+t} + a$

prob that (40) will live for 10 years is 0.85
↓
substandard

$_{10}p_{40} = .90$

$_{10}p^{s}_{40} = .85$

$.85 = .90\, e^{-10a}$

Solve for a

$$\boxed{\frac{-\log\left(\frac{.85}{.90}\right)}{10} = a}$$

Calculate a.

$$\underline{{}_{10}p^{s}_{40}} = e^{-\int_0^{10} \mu^{s}_{40+t}\, dt}$$

$$= e^{-\int_0^{10} (\underline{\mu_{40+t}} + \underline{a})\, dt}$$

$$= \underbrace{e^{-\int_0^{10} \mu_{40+t}\, dt}}_{\underline{{}_{10}p_{40}}} \; \underbrace{e^{-\int_0^{10} a\, dt}}_{e^{-10a}}$$

Practice problem - SOA LTAM Spring 2019 Question #3

Gompertz $\mu_x = Bc^x$

Makeham $\mu_x = A + Bc^x$

- de Moivre's
- Generalized de Moivre's
- exponential or constant force

You are given:

- A life table uses a Makeham's mortality model with parameters $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $c = 1.124$
- ${}_{10}p_{50} = 0.9803$

Calculate $\dfrac{d}{dt}\,{}_tq_{50}$ at $t = 10$.

$$\frac{d}{dt}\Big(\underbrace{\Pr[T_{50} \le t]}_{{}_tq_{50}}\Big) = f_{50}(t) = {}_tp_{50}\,\mu_{50+t}$$

at $t=10$, $\dfrac{d}{dt}\,{}_tq_{50} = {}_{10}p_{50}\,\mu_{60}$

$$= .9803\left(A + Bc^{60}\right) = .00315806$$

plug A, B, C

Final remark - other contexts

- The notion of a lifetime or survival learned in this chapter can be applied in several other contexts:
  - engineering: lifetime of a machine, lifetime of a lightbulb
  - medical statistics: time-until-death from diagnosis of a disease, survival after surgery
  - finance: time-until-default of credit payment in a bond, time-until-bankruptcy of a company
  - space probe: probability radios installed in space continue to transmit
  - biology: lifetime of an organism
  - other actuarial context: disability, sickness/illness, retirement, unemployment

Other symbols and notations used

Expression	Other symbols used
probability function	$P(\cdot)$ $\quad$ $\mathsf{Pr}(\cdot)$
survival function of newborn	$S_X(x)$ $\quad$ $S(x)$ $\quad$ $s(x)$
future lifetime of x	$T(x)$ $\quad$ T
curtate future lifetime of x	$K(x)$ $\quad$ K
survival function of x	$S_{T_x}(t)$ $\quad$ $S_T(t)$
force of mortality of T_x	$\mu_{T_x}(t)$ $\quad$ $\mu_x(t)$

Exercise 2.1

$$F_0(t) = 1-\left(1-\frac{t}{105}\right)^{1/5}, \quad 0 \le t < 105$$

$$F_0(x) = 1-\left(1-\frac{x}{105}\right)^{1/5}, \quad 0 \le x < 105 \to t$$

(b) prob that (30) survives to at least age 70

$$\underline{\underline{{}_{40}p_{30}}} = Pr[T_{30} > 40] = S_{30}(40) = \frac{S_0(70)}{S_0(30)}$$

$$= \frac{1-F_0(70)}{1-F_0(30)} = \left(\frac{7}{15}\right)^{1/5} = \underline{\underline{.8586}}$$

$$\underbrace{S_x(t) = \frac{S_0(x+t)}{S_0(x)}}$$

$$S_0(x) = e^{-\int_0^x \mu_z \, dz}$$

$$\underline{\underline{S_x(t)}} = e^{-\int_0^t \mu_{x+s} \, ds}$$

(f) Calculate $\overset{\circ}{e}_{50} = \int_0^\infty \underline{{}_tp_{50}} \, dt$

$${}_tp_{50} = \frac{S_0(50+t)}{S_0(50)} = \frac{1-F_0(50+t)}{1-F_0(50)}$$

$$\vdots$$

$$\left(1-\frac{t}{55}\right)^{1/5}, \quad \underline{0 < t \le 55}$$

$$\mathring{e}_{50} = \int_0^{55} {}_tp_{50}\,dt = \int_0^{55} \left(1-\tfrac{t}{55}\right)^{1/5} dt = \underbrace{\int_1^{0} u^{1/5}\cdot(-55\,du)}_{45.8333}$$

$u = 1 - t/55$

$du = -\tfrac{1}{55}\,dt$

Suppose mortality of (25) is given by

$$S_{25}(t) = \left(1-\tfrac{t}{50}\right)^{2/3}, \quad 0 < t \le 50$$

$$S_{25}(t) = \frac{S_0(25+t)}{S_0(25)}$$

Calculate ${}_{10}p_{50}$ or ${}_{10|20}q_{50}$

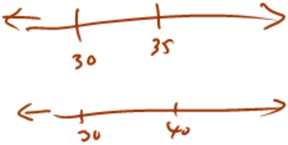
$${}_{10}p_{50} = \frac{S_0(50+10)/S_0(25)}{S_0(50)/S_0(25)} = \frac{S_0(25+35)/S_0(25)}{S_0(25+25)/S_0(25)} = \frac{S_{25}(35)}{S_{25}(25)} = \frac{15^{2/3}}{25^{2/3}} = \left(\frac{3}{5}\right)^{2/3}$$

Study Q #3 Fall 2018 Test 1

Q#11 $\mathring{e}_{30} = 51.50$ $\mathring{e}_{35} = 46.68$ $\mathring{e}_{40} = 41.91$ Study

$\mathring{e}_{30:\overline{5|}} = 4.988$ $\mathring{e}_{30:\overline{10|}} = 9.963$

Calculate ${}_5p_{35}$



$$\underbrace{\mathring{e}_{30}}_{51.50} = \underbrace{\mathring{e}_{30:\overline{5|}}}_{4.988} + {}_5p_{30}\,\underbrace{\mathring{e}_{35}}_{46.68} \Rightarrow {}_5p_{30} = 0.996401$$

$$\underbrace{\mathring{e}_{30}}_{51.50} = \underbrace{\mathring{e}_{30:\overline{10|}}}_{9.963} + {}_{10}p_{30}\,\underbrace{\mathring{e}_{40}}_{41.91} \Rightarrow {}_{10}p_{30} = 0.9911$$

$${}_{10}p_{30} = \underbrace{{}_5p_{30}}_{.996401} \cdot {}_5p_{35} = .9946798$$

$${}_{10}p_{50} = \frac{S_{10}(50)}{S_{10}(40)}$$

$$= \frac{S_{25}(35)}{S_{25}(25)}$$