L_0 = loss at issue
random variable

P

Premium Calculation

Lecture: Weeks 10-12

Preliminaries

An insurance policy (life insurance or life annuity) is funded by contract premiums:

- once (single premium) made usually at time of policy issue, or
- a series of payments (usually contingent on survival of policyholder) with first payment made at policy issue
- to cover for the benefits, expenses associated with initiating/maintaining contract, profit margins, and deviations due to adverse experience.

Net premiums (or sometimes called benefit premiums)

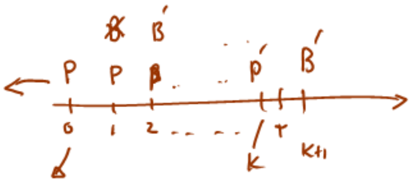
- considers only the benefits provided
- nothing allocated to pay for expenses, profit or contingency margins

Gross premiums (or sometimes called expense-loaded premiums)

- covers the benefits and includes expenses, profits, and contingency margins (discrete) whole life insurance of $B=1$, issued to (x)

Ⓖ, fully discrete whole life,

Ⓡ, P = net, (yearly)



n = net

$$L_0^n = \underline{\text{Benefit} - \text{Premium}}$$

$$= \underbrace{PVFB_0}_{} - PVFP_0 \rightarrow P\ddot{a}_{\overline{K+1}|}$$

$$= Bv^{K+1} -$$

$$\boxed{P=?}$$

$$E[L_0^n] = \underbrace{E[PVFB_0] - \underbrace{E[PVFP_0]}_{}}_{} = 0 \Rightarrow \underbrace{E[L_0^n]=0 \text{ and solve for } P}_{},$$

actuarial equivalence,

equivalence principle.

Set P: $E[PVFP_0] = E[PVFB_0]$

$$E[L_0] = 0 \Rightarrow E[P\ddot{a}_{\overline{K+1|}}] = \underbrace{E[v^{K+1}]}_{A_x}$$

$$P\ddot{a}_x = A_x \Rightarrow \boxed{P = \frac{A_x}{\ddot{a}_x}}$$

$\ddot{a}_x$

$$E[\ddot{a}_{\overline{K+1|}}] = \underbrace{E[1+v+\dots+v^{K+1}]}_{E[K+1] \text{ if } i=0}$$

$i=0 \Rightarrow v=1$

B is other than 1

$$\underbrace{E[PVFP_0]}_{P\ddot{a}_x} = E[PVFB_0]$$

$$P\ddot{a}_x = BA_x$$

$$P = B\frac{A_x}{\ddot{a}_x}$$

$$\overbrace{\underbrace{E[PVFP_0] = E[PVFB_0]}_{APV(FP_0) = APV(FB_0)}}^{0}$$

$$L_0 = PVFB_0 - PVFP_0$$

$$\swarrow$$

$$v^{K+1} - P\ddot{a}_{\overline{K+1|}}$$

$$Var(L_0) = Var\Big(\underbrace{v^{K+1} - P\ddot{a}_{\overline{K+1|}}}_{v^{K+1} - P\left(\frac{1-v^{K+1}}{d}\right)}\Big)$$

$$= Var\left(\underbrace{\left(1+\frac{P}{d}\right)}_{}v^{K+1} - \frac{P}{d}\right)$$

$$= \underbrace{\left(1+\frac{P}{d}\right)^2}_{} \underbrace{Var\left(v^{K+1}\right)}_{\left[{}^2A_x - (A_x)^2\right]}$$

fully discrete WL of $B=1$

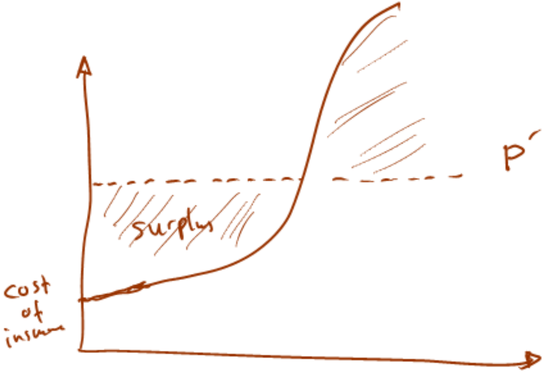
$$P = A_x / \ddot{a}_x \Leftrightarrow E[L_0] = 0$$

$$\underbrace{Cov\left(v^{K+1}, \ddot{a}_{\overline{K+1|}}\right) \neq 0}_{}$$

If $B \neq 1$,

$$= \left(B + \frac{P}{d}\right)^2 \left[{}^2A_x - (A_x)^2\right]$$

$$\boxed{P = B\frac{A_x}{\ddot{a}_x}}$$



p
surplus
cost of insurance

B - benefit fully discrete WL

$$L_0 = \underbrace{Bv^{K+1} - P\ddot{a}_{\overline{K+1|}}}$$

$$E[L_0] = 0$$

$$P = B\frac{A_x}{\ddot{a}_x}$$

$$B = 1$$

$$P = P_x' = \frac{A_x}{\ddot{a}_x}$$

$\downarrow$ capital P ✓

VIF ✓

$$Var(L_0) = \underbrace{B^2\left(1 + \frac{P_x'}{d}\right)^2\left[{}^2A_x - (A_x)^2\right]}$$

in terms of insurance

$$P_x = \frac{A_x}{\ddot{a}_x}$$

$$A_x = 1 - d\ddot{a}_x \; ✓$$

$$P_x = \frac{1}{\ddot{a}_x} - d \;"$$

$$P_x = \frac{A_x}{\frac{1-A_x}{d}} = \frac{dA_x'}{1-A_x} \; ✓$$

$$= B^2\underbrace{\left(1 + \frac{\frac{\cancel{d}A_x}{1-A_x}}{\cancel{d}}\right)}^2\left[{}^2A_x - (A_x)^2\right]$$

$$\vdots$$

$$= B^2\underbrace{\frac{1}{(1-A_x)^2}\left[{}^2A_x - (A_x)^2\right]}$$

$$\frac{{}^2A_x - (A_x)^2}{(1-A_x)^2} \; ✓$$

VIF ✓

$$\underbrace{\text{Probability of a positive loss at issue}}$$

$$\underbrace{\Pr[L_0 > 0]} = \Pr[\text{die early}] -$$

$$\Pr(v^{K+1} >> a)$$

$$\underbrace{= \Pr[K << \text{early}]}$$

$$\Pr[\ddot{a}_{\overline{K+1}|} >> a]$$

$$= \Pr[K >> \text{late}]$$

$$\boxed{B=1}$$

$$L_0 > 0 \Rightarrow v^{K+1} - P\ddot{a}_{\overline{K+1}|} > 0$$

$$\ddot{a}_{\overline{K+1}|} = \left(\frac{1-v^{K+1}}{d}\right)$$

$$\Rightarrow \underbrace{\left(1+\frac{P}{d}\right)v^{K+1} - \frac{P}{d} > 0}$$

$$\Rightarrow v^{K+1} > \frac{P/d}{1+P/d}$$

$$\Rightarrow (K+1)\underbrace{\log v}_{-\delta} > \log\left(\frac{P/d}{1+P/d}\right)$$

$$\boxed{K \text{ is discrete}}$$

$$\Pr(L_0>0) = \Pr\left[K < \underbrace{-\frac{1}{\delta}\log\left(\frac{P/d}{1+P/d}\right) - 1}_{a}\right] = \Pr(K<a]$$

Chapter summary

- Contract premiums
    - net premiums
    - gross (expense-loaded) premiums
- Present value of future loss random variable L_0^n L_0^g
- Premium principles
    - the equivalence principle (or actuarial equivalence principle)
    - portfolio percentile premiums
- Return of premium policies
- Chapter 6 of Dickson, et al.

Net random future loss

- An insurance contract is an agreement between two parties:
    - the insurer agrees to pay for insurance benefits;
    - in exchange for insurance premiums to be paid by the insured.
- Denote by PVFB_0 the present value, at time of issue, of future benefits to be paid by the insurer.
- Denote by PVFP_0 the present value, at time of issue, of future premiums to be paid by the insured.
- The insurer's net random future loss is defined by

$$L_0^n = \mathsf{PVFB}_0 - \mathsf{PVFP}_0.$$

- Note: this is also called the present value of future loss random variable (in the book), and if no confusion, we may simply write this as L_0.

The principle of equivalence

- The net premium, generically denoted by P, may be determined according to the principle of equivalence by setting

$$\boxed{\mathsf{E}\left[L_0^n\right] = 0.}$$

- The expected value of the insurer's net random future loss is zero.
- This is then equivalent to setting $\mathsf{E}\left[\text{PVFB}_0\right] = \mathsf{E}\left[\text{PVFP}_0\right]$. In other words, at issue, we have

$$\boxed{\text{APV(Future Premiums)} = \text{APV(Future Benefits)}}.$$

An illustration

Consider an n-year endowment policy which pays B dollars at the end of the year of death or at maturity, issued to a life with exact age x. Net premium of P is paid at the beginning of each year throughout the policy term.

- If we denote the curtate future lifetime of (x) by $K = K_x$, then the net random future loss can be expressed as

$$L_0^n = Bv^{\min(K+1,n)} - P\ddot{a}_{\overline{\min(K+1,n)}|}.$$

- The expected value of the net random future loss is

$$\begin{aligned} \mathsf{E}\left[L_0^n\right] &= B\mathsf{E}\left[v^{\min(K+1,n)}\right] - P\mathsf{E}\left[\ddot{a}_{\overline{\min(K+1,n)}|}\right] \\ &= BA_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}. \end{aligned}$$

n-year endowment of B = benefit - issued to (x) -

annual premiums at b.o.y.

$$\text{loss-at-issue} = L_0^n = L_0 = PVFB_0 - PVFP_0$$

$$= B\, v^{\min(K+1,n)} - P\, \ddot{a}_{\overline{\min(K+1,n)}|}$$

Equivalence principle $\underbrace{APV(FP_0)}$ = $APV(PVFB_0)$

$$P\, \underbrace{E\left[\ddot{a}_{\overline{\min(K+1,n)}|}\right]}_{\ddot{a}_{x:\overline{n}|}} = B\, \underbrace{E\left[v^{\min(K+1,n)}\right]}_{\underbrace{A_{x:\overline{n}|}}_{\text{term + pure}}}$$

$$\boxed{P = \frac{B\, A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}}$$

$$\mathrm{Var}(L_0) = \underbrace{B^2\left(1+\frac{P}{d}\right)^2\left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]}$$

$$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$$

$$= B^2 \cdot \frac{{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2}{\left(1-A_{x:\overline{n}|}\right)^2}$$

An illustration - continued

- By the principle of equivalence, $\mathsf{E}\left[L_0^n\right] = 0$, we then have

$$P = B\frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}. \checkmark$$

$$\ddot{a}_{\overline{\min(K+1,n)}|} = \frac{1-v^{\min\ldots}}{d}$$

- Rewriting the net random future loss as

$$L_0^n = \left(B + \frac{P}{d}\right)v^{\min(K+1,n)} - \frac{P}{d}, \checkmark$$

we can find expression for the variance:

$$\mathsf{Var}\left[L_0^n\right] = \left(B + \frac{P}{d}\right)^2\left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right].$$

- One can also show that this simplifies to

$$\mathsf{Var}\left[L_0^n\right] = B^2\frac{{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2}{\left(1 - A_{x:\overline{n}|}\right)^2}.$$

Some general principles

Note the following general principles when calculating premiums:

- For (discrete) premiums, the first premium is usually assumed to be made immediately at issue.
- Insurance benefit may have expiration or maturity:
  - in which case, it is implied that there are no premiums to be paid beyond expiration or maturity.
  - however, it is possible that premiums are to be paid for lesser period than expiration or maturity. In this case, it will be explicitly stated.

Fully discrete annual premiums - whole life insurance

Consider the case of a fully discrete whole life insurance where benefit of \$1 is paid at the end of the year of death with level annual premiums.

The net annual premium is denoted by P_x so that the net random future loss is

$$L_0 = v^{K+1} - P_x \, \ddot{a}_{\overline{K+1}|}, \quad \text{for } K = 0, 1, 2, \ldots$$

By the principle of equivalence, we have

$$P_x = \frac{\mathsf{E}\left[v^{K+1}\right]}{\mathsf{E}\left[\ddot{a}_{\overline{K+1}|}\right]} = \frac{A_x}{\ddot{a}_x}.$$

The variance of the net random future loss is

$$\mathsf{Var}[L_0] = \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2} = \frac{{}^2A_x - (A_x)^2}{(1 - A_x)^2}.$$

Other expressions

You can express the net annual premiums:

- in terms of annuity functions

$$P_x = \frac{1 - d\ddot{a}_x}{\ddot{a}_x} = \frac{1}{\ddot{a}_x} - d \qquad \checkmark$$

- in terms of insurance functions

$$P_x = \frac{A_x}{\big(1 - A_x\big)/d} = \frac{dA_x}{1 - A_x} \qquad \checkmark$$

Whole life insurance with *h* premium payments

SKIP

Consider the same situation where now this time there are only h premium payments.

- The net random future loss in this case can be expressed as

$$L_0 = v^{K+1} - P \times \begin{cases} \ddot{a}_{\overline{K+1}|}, & \text{for } K = 0, 1, \ldots, h-1 \\ \ddot{a}_{\overline{h}|}, & \text{for } K = h, h+1, \ldots \end{cases}$$

- Applying the principle of equivalence, we have

$$P = \frac{A_x}{\ddot{a}_{x:\overline{h}|}}.$$

Illustrative example 1

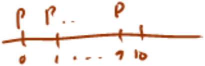
Consider a special endowment policy issued to (45). You are given: discrete

- Benefit of \$10,000 is paid at the end of the year of death, if death occurs before 20 years.
- Benefit of \$20,000 is paid at the end of 20 years if the insured is then alive. pure endowment
- Level annual premiums P are paid at the beginning of each year for 10 years and nothing thereafter.
- Mortality follows the `Survival Ultimate Life Table` with $i = 0.05$. SULT $i = 5\%$

Calculate P according to the equivalence principle.

$P = 921.10$

$E[L_0]=0 \rightarrow \underbrace{APV(F.P.)} = \underbrace{APV(F.B.)}_{\text{term + pure}}$ 10,000 20,000



$$P\,\ddot{a}_{45:\overline{10}|} = 10{,}000\,A^{1}_{45:\overline{20}|} + 20{,}000\,{}_{20}E_{45}$$

$\ddot{a}_{45} = 17.8162$

${}_{10}E_{45} = .60615$

$\ddot{a}_{55} = 16.0599$

$\ddot{a}_{45:\overline{10}|} = \ddot{a}_{45} - {}_{10}E_{45}\,\ddot{a}_{55}$

Solve for

$A_{45} = .15161$

${}_{20}E_{45} = .35994$

$A_{65} = .35477$

$A^{1}_{45:\overline{20}|} = A_{45} - {}_{20}E_{45}\,A_{65}$

$$P = \underbrace{\frac{10{,}000\,A^{1}_{45:\overline{20}|} + 20{,}000\,{}_{20}E_{45}}{\ddot{a}_{45:\overline{10}|}}}_{921.0995}$$

Benefits varying, Premiums vary

20-year term payable at the end of year of death:

issue age 45

SULT

i=5%

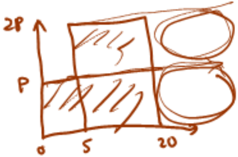
→ 10 in the first 5 years

→ 25 the following 15 years

premiums payable at the b.o.y.

→ P in the first 5 years

→ 2P after that . . .



Calculate P.

$$E[L_0]=0 \Rightarrow \underbrace{APV(FP_0)} = APV(FB_0)$$

$$P\ddot{a}_{45} + P\,{}_5E_{45}\ddot{a}_{50} - 2P\,{}_{20}E_{45}\ddot{a}_{65} = 10A_{45} + 15\,{}_5E_{45}A_{50} - 25\,{}_{20}E_{45}A_{65}$$

$$P = \frac{10A_{45} + 15\,{}_5E_{45}A_{50} - 25\,{}_{20}E_{45}A_{65}}{\ddot{a}_{45} + {}_5E_{45}\ddot{a}_{50} - 2\,{}_{20}E_{45}\ddot{a}_{65}},$$

(try to evaluate using table!)

Chap 3 Life Tables - (1)

Chap 4 Insurance 5

Chap 5 Annuities 5

Chap 6 Premiums (some) (1)

12 problems

Premiums

equivalence principle

$E[L_0] = 0$

$Var(L_0) =$

fully discrete WL

fully discrete endowment

$$P = B\frac{A_x}{\ddot{a}_x} = B\left(\frac{1}{\ddot{a}_x} - d\right)$$

$$= B\left(\frac{dA_x}{1-A_x}\right)$$

$B=1$, $P_x = \dfrac{A_x}{\ddot{a}_x}$

End of Midterm 3

Chap 3 Life Table select. ①

4 LI 5

5 LA 5

6 - Prem - ① -

calculator functions

SULT

- APV, variance.
- recursive formula.
- WL, endowment, term, pure endowment
- discounts, mthly
- approximate < UDD Woolhouse

2 formula,

SULT -

Illustrative Life Table

equivalence principle $\quad L_0^n$ loss-at-issue (net)

$$E[L_0^n] = 0$$

$$\swarrow$$

$$E[PVFB_0 - PVFP_0] = 0$$

$$E[PVFB_0] = \underbrace{E[PVFP_0]}$$

$$APV(FP_0) = APV(FB_0)$$

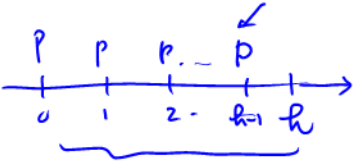
(discrete) whole life ins (x)

$$L_0 = \underbrace{PVFB_0}_{v^{K+1}} - PVFP_0$$

$$v^{K+1} - P\ddot{a}_{\overline{K+1|}} \implies P\ddot{a}_x = A_x$$

$$P = \frac{A_x}{\ddot{a}_x}$$

Whole life with fewer premiums



$$L_0 = PVFB_0 - PVFP_0$$

$$v^{K+1} - P\ddot{a}_{\overline{\min(K+1,h)}|}$$

$$E[L_0] = E\left[v^{K+1} - P\ddot{a}_{\overline{\min(K+1,h)}|}\right] = 0$$

$$\begin{cases}\ddot{a}_{\overline{K+1}|}, & K<h\\ \ddot{a}_{\overline{h}|}, & K\geq h\end{cases}$$

$$A_x - P\ddot{a}_{x:\overline{h}|} = 0$$

$$P = \frac{A_x}{\ddot{a}_{x:\overline{h}|}}$$

p.11

SOA-type question

$APV(FP_0) = APV(FB_0)$

$d = 1 - v$

2 years $v = 1 - d$

$= .95$

Two actuaries use the same mortality table to price a fully discrete two-year endowment insurance of 1,000 on (x). You are given:

- Kevin calculates non-level benefit premiums of 608 for the first year, and 350 for the second year.
- Kira calculates level annual benefit premiums of π.
- $d = 0.05$

Calculate π.

Kevin: $608 + 350\, p_x \cdot v = 1000\, A_{x:\overline{2}|}$

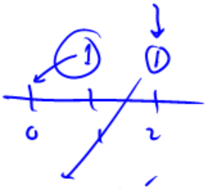
Kira: $\underbrace{\pi + \pi\, p_x \cdot v}_{} = 1000\, A_{x:\overline{2}|}$

489.0836

$$\pi(1 + v p_x) = 608 + 350\, v p_x$$

$$\pi = \frac{608 + 350(.95)\, p_x}{1 + .95\, p_x} = \frac{608 + 350(\cancel{.95})\frac{.835}{\cancel{.95}}}{1 + .95(.835/.95)}$$

Derive p_x from Kevin's



$+ v^2 p_x q_{x+1}'$

$+ v^2 p_x p_{x+1}'$

$$608 + 350 v p_x = \underline{\underline{1000}}\, \underbrace{A_{x:\overline{2|}}'}$$

$(.95)^2$

$$608 + 350 \underset{.95}{v} p_x' = 1000(\underset{.95}{v} \underset{(1-p_x')}{q_x'} + v^2 p_x')$$

$$\underline{608 - 1000v} = p_x'\left[-1000v + 1000v^2 - 350v\right]$$

$v = .95$

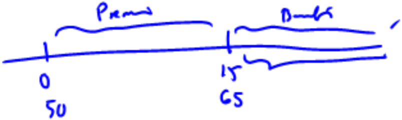
$$p_x = \frac{608 - 1000v}{\underbrace{-1000v + 1000v^2 - 350v}_{(.835/.95)}}$$

Illustrative example 2

An insurance company issues a 15-year deferred life annuity contract to (50). You are given:

- Level monthly premiums of P are paid during the deferred period. monthly
- The annuity benefit of $25,000 is to be paid at the beginning of each year the insured is alive, starting when he reaches the age of 65. annual
- Mortality follows the `Survival Ultimate Life Table` with $i = 0.05$.
- Mortality between integral ages follow the Uniform Distribution of Death (UDD) assumption.

1. Write down an expression for the net future loss, at issue, random variable.
2. Calculate the amount of P.
3. If an additional benefit of $10,000 is to be paid at the moment of death during the deferred period, how much will the increase in the monthly premium be?



① loss at issue

$$L_0 = PVFB_0 - PVFP_0$$

$$= \begin{cases} 0, & K<15 \\ 25000\ {}_{15|}\ddot{a}_{\overline{K+1}|}, & K\geq 15 \end{cases} - 12P \begin{cases} \ddot{a}^{(12)}_{\overline{K^{(12)}+1/12}|}, & K<15 \\ \ddot{a}^{(12)}_{\overline{15}|}, & K\geq 15 \end{cases}$$

P = monthly prem

SULT $i = .05$

② $\underbrace{APV(FP_0)} = \underbrace{APV(FB_0)}$

$$12P\ \underbrace{\ddot{a}^{(12)}_{50:\overline{15}|}}_{\swarrow} = 25000\ \underbrace{{}_{15|}\ddot{a}_{50}}$$

$${}_{15}E_{50}\ \ddot{a}_{65} \quad 13.5498$$

$${}_{10}E_{50} \cdot {}_{5}E_{60}$$

.60182 .76687

$$\boxed{\ddot{a}^{(m)}}$$

UDD $\ddot{a}_x^{(m)} \approx \alpha(m)\,\ddot{a}_x - \beta(m)$

$\alpha(12) = 1.00020$

$\beta(12) = .46651$

$$\ddot{a}^{(12)}_{50:\overline{15|}} = \ddot{a}^{(12)}_{50} - {}_{15}E_{50}\,\ddot{a}^{(12)}_{65}$$

$\ddot{a}^{(12)}_{50} \rightarrow \alpha(12)\,\ddot{a}_{50} - \beta(12)$, $\ddot{a}_{50} = 17.0245$

${}_{15}E_{50} \rightarrow .60182 * .76687$

$\ddot{a}^{(12)}_{65} \rightarrow \alpha(12)\,\ddot{a}_{65} - \beta(12)$, $\ddot{a}_{65} = 13.5498$

$= 10.52197$ ✓

$$12P\,\ddot{a}^{(12)}_{50:\overline{15|}} = 25000\,{}_{15}E_{50}\,\ddot{a}_{65}$$

$$\Rightarrow P = \frac{25000\,(.60182 * .76687)\,13.5498}{12 * 10.52197} = \underline{1{,}238.177}$$

monthly ✓

③

$$APV(FP_0) = APV(FB_0)$$

$$\underline{12P\ddot{a}^{(12)}_{50:\overline{15|}}} = \underbrace{25000\ {}_{15|}\ddot{a}_{65}} + \underbrace{10000\ \bar{A}^{\ 1}_{50:\overline{15|}}}$$

$$\frac{i}{\delta}\underbrace{A^{\ 1}_{50:\overline{15|}}}$$

$$\underbrace{\underset{.18931}{A_{50}} - {}_{15}E_{50}\ \underset{.35471}{A_{65}}}$$

$$= .02357736$$

$i = .05$

$\delta = \log(1.05)$

$P = 1258.137$ (verify) with insurance

$-)\ P = 1238.177$ without

$$\Delta P = \boxed{20.76} \text{ monthly}$$

Different possible combinations

varying premiums
benefits

Premium payment	Benefit payment
annually	at the end of the year of death
	at the end of the $\frac{1}{m}$th year of death
	at the moment of death
m-thly of the year	at the end of the year of death
	at the end of the $\frac{1}{m}$th year of death
	at the moment of death
continuously	at the end of the year of death
	at the end of the $\frac{1}{m}$th year of death
	at the moment of death

$APV(FP_0) = APV(FB_0)$ equivalence principle.

fully continuous <u>whole life</u>

premiums ✓
benefits ✓

$$L_0 = PVFB_0 - PVFP_0$$

$$= \underbrace{v^T - P\,\bar{a}_{\overline{T|}}} \Rightarrow E[L_0] = 0 \Rightarrow P = \boxed{\frac{\bar{A}_x}{\bar{a}_x}}$$

$$v^T - P\left(\frac{1-v^T}{\delta}\right) = v^T\underbrace{\left(1+\frac{P}{\delta}\right)} - \frac{P}{\delta}$$

$$Var(L_0) = \underbrace{\left(1+\frac{P}{\delta}\right)}^2 \underbrace{var(v^T)}_{\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right]} = \ldots\ldots = \frac{{}^2A_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2}$$

$$\bar{A}_x = 1 - \delta\,\bar{a}_x$$

Fully continuous premiums - whole life insurance

Consider a fully continuous level annual premiums for a unit whole life insurance payable immediately upon death of (x).

- The insurer's net random future loss is expressed as

$$L_0 = \underset{\sim}{v}^{\boxed{T}} - \overset{\to \text{ P per year}}{P}\,\bar{a}_{\overline{T|}}. \quad = \text{Benefit} - \text{Premium}$$

loss at issue
present values

- By the principle of equivalence,

$$P = \frac{\bar{A}_x}{\bar{a}_x} = \frac{1}{\bar{a}_x} - \delta = \frac{\delta \bar{A}_x}{1 - \bar{A}_x}.$$

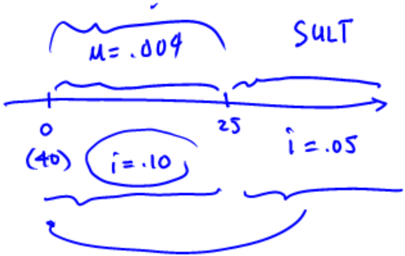
- The variance of the insurer's net random future loss can be expressed as

$$\begin{aligned} \mathrm{Var}[L_0] &= \left[1 + (P/\delta)\right]^2 \left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right] \\ &= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{(\delta \bar{a}_x)^2} = \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{(1 - \bar{A}_x)^2}. \end{aligned}$$

Q#11 Fall 2018

$\ddot{a}_{40} = ?$

$_tp_x = e^{-\mu t}$



$$\ddot{a}_{40} = \underbrace{\ddot{a}_{40:\overline{25|}}} + \underbrace{{}_{25}E_{40}}\ \ddot{a}_{65} \longrightarrow \text{SULT} \Rightarrow 13.5498$$

$${}_{25}E_{40} \rightarrow v^{25}\,{}_{25}p_{40} \rightarrow \underbrace{\left(\frac{1}{1.10}\right)^{25} e^{-25(.004)}}$$

$= \underline{\underline{10.82594}}$

CPT

$$\sum_{k=0}^{24} v^k\,{}_kp_{40} \rightarrow \underbrace{\left(\frac{1}{1.10}\right)^k e^{-.004k}}$$

$$\sum_{k=0}^{24}\left(\frac{e^{-.004}}{1.1}\right)^k = \frac{1-\left(\frac{e^{-.004}}{1.1}\right)^{25}}{1-\left(\frac{e^{-.004}}{1.1}\right)} = 9.694359$$

Q#10

Woolhouse of three terms

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x+\delta)$$

	l_x	μ_x
60	.	.
:	.	.
63		

$i = .05$

$m = 12$

$m^2 = 12^2 = 144$

$$\ddot{a}^{(12)}_{60:\overline{3|}} = \ddot{a}^{(12)}_{60} - {}_3E_{60}\,\ddot{a}^{(12)}_{63}$$

$${}_3E_{60}\left[\ddot{a}_{63} - \frac{11}{24} - \frac{143}{1728}(\mu_{63}+\delta)\right]$$

$$\ddot{a}_{60} - \frac{11}{24} - \frac{143}{1728}(\mu_{60}+\delta)$$

$$= \ddot{a}_{60:\overline{3|}} - \frac{11}{24}(1 - {}_3E_{60}) - \frac{143}{1728}\left[(\mu_{60}+\delta) - {}_3E_{60}(\mu_{63}+\delta)\right]$$

$\delta = \log 1.05$

$\mu_{60} = .0024$

$\mu_{63} = .0045$

2.771183

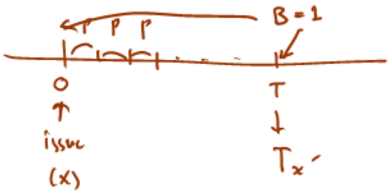
$$\ddot{a}_{60:\overline{3|}} = 1 + v\,p_{60} + v^2\,p_{60}\,p_{61} = 2.842812$$

$\frac{l_{61}}{l_{60}}$: l_{61} – 994, l_{60} – 1000

$\frac{l_{62}}{l_{60}}$: l_{62} – 988, l_{60} – 1000

$${}_3E_{60} = v^3\,{}_3p_{60} = \left(\frac{1}{1.05}\right)^3 \frac{l_{63}}{l_{60}} = \boxed{.8448332}$$

l_{63} – 978, l_{60} – 1000



$$L_0 = v^T - P\bar{a}_{\overline{T}|}$$

$$E[L_0] = E[v^T] - P\,E[\bar{a}_{\overline{T}|}]$$

$$\bar{A}_x - P\bar{a}_x = 0$$ — e.p. $\Rightarrow$ $P = \dfrac{\bar{A}_x}{\bar{a}_x}$

if $i=0$

$\underline{\bar{a}_x = E[T_x]}$

spread over expected lifetime

$$Var(L_0) = Var\Big(\underbrace{v^T - P\,\frac{1-v^T}{\delta}}\Big)$$

$$= Var\left(\left(1+\frac{P}{\delta}\right)v^T \boxed{-\frac{P}{\delta}}\right)$$ — constant

$$P = \bar{A}_x/\bar{a}_x$$

$$= \left(1+\frac{P}{\delta}\right)^2 \underbrace{Var(v^T)}_{\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right]} = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2}$$

A simple illustration

relations A to a

$$\bar{A}_x = \frac{\mu}{\mu+\delta}, \quad \bar{a}_x = \frac{1}{\mu+\delta}$$

$$\bar{A}_x = 1 - \delta\,\bar{a}_x$$

$$\underbrace{\frac{\bar{A}_x}{\bar{a}_x} = \mu}$$

$P = .08$

For a fully continuous whole life insurance of $1, you are given:

- Mortality follows a constant force of $\mu = 0.04$.
- Interest is at a constant force $\delta = 0.08$.
- L_0 is the loss-at-issue random variable with the benefit premium calculated based on the equivalence principle.

Calculate the annual benefit premium and $\mathrm{Var}[L_0]$.

$$\bar{A}_x = \frac{4}{12} = \frac{1}{3}$$

$$^2\bar{A}_x = \frac{.04}{.04+.16} = \frac{.04}{.20} = \frac{1}{5}$$

$$\mathrm{Var}(L_0) = \frac{\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right]}{(1-\bar{A}_x)^2} = \frac{\frac{1}{5} - \left(\frac{1}{3}\right)^2}{\left(1-\frac{1}{3}\right)^2} = \frac{1}{5} = 0.20$$

Published SOA question #14

$$^2\bar{A}_x = \frac{\mu}{\mu+2\delta} = \frac{.03}{.03+2\delta} = .20$$

$$\delta = .06$$

$$\bar{A}_x = \frac{.03}{.03+.06} = \frac{1}{3}$$

For a fully continuous whole life insurance of \$1 on (x), you are given:

- The forces of mortality and interest are constant.
- $^2\bar{A}_x = 0.20$
- The benefit premium is 0.03.
- L_0 is the loss-at-issue random variable based on the benefit premium.

$P = \mu = .03$

benefit

Calculate Var$[L_0]$.

$$\frac{^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2} = \frac{.20 - (\frac{1}{3})^2}{(1-\frac{1}{3})^2} = \underline{0.20}$$

Endowment insurance

Consider an n-year endowment insurance with benefit of \$1:

- The net random future loss is

$$L = \begin{cases} v^T - P\,\bar{a}_{\overline{T}|}, & T \le n \\ v^n - P\,\bar{a}_{\overline{n}|}, & T > n \end{cases}$$

- Net annual premium formulas:

$$P = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} = \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta = \frac{\delta\,\bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}}$$

$$\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2}$$

- The variance of the net random future loss:

$$\begin{aligned} \text{Var}\left[L_0\right] &= \left[1 + (P/\delta)\right]^2 \left[{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2\right] \\ &= \frac{{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2}{(\delta\,\bar{a}_{x:\overline{n}|})^2} = \boxed{\frac{{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2}{(1 - \bar{A}_{x:\overline{n}|})^2}} \end{aligned}$$

Illustrative example 3

$$\frac{{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2}{\left(1-\bar{A}_{x:\overline{n}|}\right)^2} = \frac{a - .5198^2}{(1-.5198)^2} = \underline{.4631556}$$

For a fully continuous n-year endowment insurance of \$1 issued to (x), you are given:

- Z is the present value random variable of the benefit for this insurance.
- $\mathsf{E}[Z] = 0.5198$
- $\mathsf{Var}[Z] = 0.1068$
- Level annual premiums are paid on this insurance, determined according to the equivalence principle.

$$\longrightarrow \bar{A}_{x:\overline{n}|} = 0.5198$$

$$\underline{{}^2\bar{A}_{x:\overline{n}|}} - \underbrace{\left(\bar{A}_{x:\overline{n}|}\right)^2}_{(.5198)^2} = .1068 \Rightarrow {}^2\bar{A}_{x:\overline{n}|} = \underset{a}{\boxed{\underbrace{.1068 + (.5198)^2}}}$$

Calculate $\mathsf{Var}[L_0]$, where L_0 is the net random future loss at issue.

For a fully continuous whole life of 1 on (x):

equivalence principle $\Leftarrow$ ✓ · Π = benefit premium

✓ · L_0 = loss at issue with premium Π

· L_0^* = loss at issue with premium $\underbrace{1.25\Pi}$

· $\bar{a}_x = 5.0$

· $\delta = .08$

· $\mathrm{Var}(L_0) = \overset{\checkmark}{0.5625}$

$$L_0 = v^T - \Pi\bar{a}_{\overline{T|}} = \left(1+\frac{\Pi}{\delta}\right)v^T - \frac{\Pi}{\delta}$$

$$L_0^* = \underbrace{\left(1+\frac{1.25\Pi}{\delta}\right)}v^T - \frac{1.25\Pi}{\delta}$$

$$\bar{A}_x = 1-\delta\bar{a}_x = 1-.08(5) = 1-.4 = .6$$

$$\underbrace{E[L_0]=0}\quad \text{e.p.}$$

$$\Pi = \bar{A}_x/\bar{a}_x = .60/5.0 = \boxed{.12}$$

Calculate $E[L_0^*] + \sqrt{\mathrm{Var}(L_0^*)}$.

$$E[L_0^*] = E[v^T - 1.25\Pi\bar{a}_{\overline{T|}}] = \underset{.6}{\bar{A}_x} - 1.25\underset{.12}{\Pi}\,\underset{5.0}{\bar{a}_x} = \underline{\underline{-.15}}$$

$$\mathrm{Var}[L_0^*] = \left(1+\frac{1.25\Pi}{\delta}\right)^2\left[\underbrace{{}^2\bar{A}_x - (\bar{A}_x)^2}\right]$$

$$Var[L_0] = \left(1+\frac{\pi}{\delta}\right)^2\left[\,{}^2\bar{A}_x - (\bar{A}_x)^2\right] = .5625$$

$$Var[L_0^*] = \left(1+\frac{1.25\pi}{\delta}\right)^2\left[\,{}^2\bar{A}_x - (\bar{A}_x)^2\right] = b$$

$\pi = .12$

$\delta = .08$

$$\frac{b}{\cancel{.5625}} = \frac{\left(1+\frac{1.25\pi}{\delta}\right)^2}{\left(1+\frac{\pi}{\delta}\right)^2} * .5625$$

$$b = \frac{\left(1+\frac{1.25(0.12)}{.08}\right)^2}{\left(1+\frac{.12}{.08}\right)^2} * .5625$$

Study this again

Illustrative example 4

x=30

For a fully discrete whole life insurance of 100 on (30), you are given:

- π denotes the annual premium and $L_0(\pi)$ denotes the net random future loss-at-issue random variable for this policy.
- Mortality follows the `Survival Ultimate Life Table` with $i = 0.05$.

discrete

Calculate the smallest premium, π^*, such that the probability is less than ~~0.5~~ 0.25 that the loss $L_0(\pi^*)$ is positive.

$$L_0 = \underbrace{100 v^{K+1} - \pi^* \ddot{a}_{\overline{K+1}|}}_{K=?} > 0$$

$$\Pr[L_0 > 0] < .25$$

$$L_0 > 0 \iff \underbrace{100\,v^{K+1} - \pi^* \ddot{a}_{\overline{K+1}|}}_{\downarrow\ \frac{1-v^{K+1}}{d}} > 0$$

$i = .05$

$d = .05/1.05$

$$\iff \left(100 + \frac{\pi^*}{d}\right)v^{K+1} - \frac{\pi^*}{d} > 0$$

$$\iff v^{K+1} > \frac{\pi^*/d}{100 + \pi^*/d}$$

$$\iff (K+1)\underbrace{\log v}_{-\delta} > \log\left(\frac{\pi^*/d}{100 + \pi^*/d}\right)$$

$$\iff K < a = \boxed{\frac{\log\left(\frac{\pi^*/d}{100+\pi^*/d}\right)}{-\delta} - 1}$$

$$\Pr[L_0 > 0] = \Pr[K < a]$$

$$= 1 - \underbrace{\Pr[K \geq a]}_{< \ .25}$$

$$\underbrace{\Pr[K \geq a]} > .75$$

$$\underbrace{\Pr[K_x \geq a]} = \frac{l_{30+a}}{l_{30}} > .75$$

$$l_{30+a} > .75\, l_{30}$$

$$\underbrace{\overset{99727.3}{\text{SULT}}}$$

$$l_{30+a} > 74795.48$$

$\boxed{79 \quad 77927.4}$

x	l_x
80	75657.2
81	73186.3

80 + above

$30 + a = 80 \Rightarrow \underline{a = 50}$

Solve for π^* from $a = 50$

$$a = \frac{\log \frac{\pi^*/d}{100 + \pi^*/d}}{-\delta} - 1 = 50$$

$i = .05$

$\delta = \log 1.05$

$d = .05/1.05$

$$\frac{\pi^*/d}{100 + \pi^*/d} = e^{51(-\delta)}$$

net loss L_0^n

$$\boxed{\pi^* = 0.4313019}$$

$$L_0^g = PVFB - PVFP + \underbrace{PVFE}_{\text{add expenses (+ profits)}}$$

Types of life insurance contract expenses

- Investment-related expenses (e.g. analysis, cost of buying, selling, servicing)
- Insurance-related expenses:
    - acquisition (agents' commission, underwriting, preparing new records)
    - maintenance (premium collection, policyholder correspondence)
    - general (research, actuarial, accounting, taxes)
    - settlement (claim investigation, legal defense, disbursement)

First year vs. renewal expenses

- Most life insurance contracts incur large losses in the first year because of large first year expenses:
    - agents' commission
    - preparing new policies, contracts
    - records administration
- These large losses are hopefully recovered in later years.
- How then do these first year expenses spread over the policy life?
- Anything not first year expense is called renewal expense (used for maintaining and continuing the policy).

first year vs renewal year

↓ ↓

% of premium

\+

% of insurance benefit

\+

fixed expense

net premium ⇒ benefit only

$\underbrace{\text{gross premium}}$ ⇒ benefit + expenses

expense-loaded premium

Gross premium calculations

- Treat expenses as if they are a part of benefits. The gross random future loss at issue is defined by

$$\boxed{L_0^g} = \underline{\text{PVFB}_0} + \overset{\text{expenses-}}{\underline{\text{PVFE}_0}} - \underline{\underline{\text{PVFP}_0}},$$

where PVFE_0 is the present value random variable associated with future expenses incurred by the insurer.

- The gross premium, generically denoted by G, may be determined according to the principle of equivalence by setting

$$\mathsf{E}\left[L_0^g\right] = 0. \rightarrow \text{same idea -}$$

- This is equivalent to setting $\underline{\mathsf{E}\left[\text{PVFB}_0\right]} + \underline{\mathsf{E}\left[\text{PVFE}_0\right]} = \mathsf{E}\left[\text{PVFP}_0\right]$. In other words, at issue, we have

$$\underline{\text{APV}(\text{FP}_0)} = \underline{\text{APV}(\text{FB}_0)} + \underline{\text{APV}(\text{FE}_0)}.$$

Consider where expenses are fixed at e, same every year.

fully discrete whole life.

$e =$ same each year

$G =$ gross annual premium

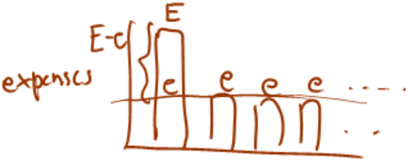
$B =$ benefit $= 1$

$$APFP_0 = AVFB_0 + AVFE_0$$

$$G\ddot{a}_x = 1\cdot A_x + e\ddot{a}_x$$

$$\overbrace{G-P}^{\text{expense loading}} = e$$

$$G = \frac{A_x + e\ddot{a}_x}{\ddot{a}_x} = \underbrace{\frac{A_x}{\ddot{a}_x}}_{P = \text{net annual prem}} + e \Rightarrow G = P + e$$



B = 1

fully discrete whole life.

G = gross premium

E−e = extra first year expense.

$$\underbrace{APFP_0}_{G\,\ddot{a}_x} = \underbrace{APVFB_0}_{A_x} + \underbrace{APVFE_0}_{\underbrace{E + e\,a_x}_{(E-e) + e\,\ddot{a}_x}}$$

$$a_x \longrightarrow (\ddot{a}_x - 1)$$

$$\frac{G\cancel{\ddot{a}_x}}{\cancel{\ddot{a}_x}} = \frac{A_x}{\ddot{a}_x} + \frac{E-e}{\ddot{a}_x} + \frac{e\cancel{\ddot{a}_x}}{\cancel{\ddot{a}_x}}$$

$$G = P + \overbrace{\underbrace{\frac{E-e}{\ddot{a}_x}}_{\text{spread over life of policy.}} + e}^{\text{expense loading}}$$

Illustration of gross premium calculation

A 1,000 fully discrete whole life policy issued to (45) with level annual premiums is priced with the following expense assumptions:

	% of Premium	Per 1,000	Per Policy
First year	40% 30%	1.0 .5	5.0 2.5
Renewal years	10% 10%	0.5 .5	2.5 2.5

In addition, assume that mortality follows the *Survival Ultimate Life Table* with interest rate $i = 0.05$.

Calculate the expense-loaded annual $\overset{G}{\text{premium}}$.

$$APVFP_0 = APVFB_0 + APVFE_0$$

$$\underbrace{G\ddot{a}_{45}} = 1000\,A_{45} +$$

% of premium $\quad .10\,G\,\ddot{a}_{45} + .30\,G$

% of 1000 $\quad \frac{.5 \times 1000}{1000}\,\ddot{a}_{45} + .5$

fixed $\quad 2.5\,\ddot{a}_{45} + 2.5$

$$G(\underbrace{.90\,\ddot{a}_{45} - .30}) = 1000A_{45} + 3\,\ddot{a}_{45} + 3$$

$$G = \frac{1000\,A_{45} + 3\,\ddot{a}_{45} + 3}{.90\,\ddot{a}_{45} - .30} = \underline{\underline{13.22302}}$$

$$P = 1000\,\frac{A_{45}}{\ddot{a}_{45}} = 8.509671$$

$A_{45} = .15161$

$\ddot{a}_{45} = 17.8162$

$$G - P = 13.22 - 8.51 = \underline{\underline{???}}$$

SOA MLC Fall 2015 Question #7

Exam MLC = Exam LTAM

Cathy purchases a fully discrete whole life insurance policy of 100,000 on her 35th birthday.

You are given:

- The annual gross premium, calculated using the equivalence principle, is 1770.

  40% extra + 10% maint.

- The expenses in policy year 1 are 50% of premium and 200 per policy.
- The expenses in policy years 2 and later are 10% of premium and 50 per policy.
- All expenses are incurred at the beginning of the policy year. ✓
- $i = 0.035$

Calculate $\ddot{a}_{35}$. = ?

$$APVFP_1 = APVFB_0 + APVFE_0 \quad ✓$$

$$\underset{\substack{\swarrow \\ 1770}}{G}\,\ddot{a}_{35} = 100000\,\underset{\substack{\downarrow \\ 1-d\ddot{a}_{35}}}{A_{35}} + \underbrace{.10\,G\ddot{a}_{35} + .30\,G}_{\%\text{ of premium}} + \underbrace{50\,\ddot{a}_{35} + 150}_{\text{fixed}}$$

$$d = \frac{.035}{1.035}$$

$$\ddot{a}_{35}\left[\underbrace{1770 + 100000\,d - .10(1770) - 50}\right] = 100{,}000 + .30(1770) + 150$$

$$\boxed{\ddot{a}_{35} = 20.48027}$$

SOA MLC Fall 2015 Question #8

$L_0^g = PVFB_0 + PVFE_0 - PVFP_0$

For a fully discrete whole life insurance of 100 on (x), you are given:

- The first year expense is 10% of the gross annual premium.
- Expenses in subsequent years are 5% of the gross annual premium.
- $i = 0.04$
- $\ddot{a}_x = 16.50$
- ${}^2A_x = 0.17$

equivalence premium principle
(always implied)

Calculate the variance of the loss at issue random variable.

$$L_0^g = 100v^{K+1} + .05G\ddot{a}_{\overline{K+1|}} + .05G - G\ddot{a}_{\overline{K+1|}}$$

$$= \underbrace{100v^{K+1} - .95G\ddot{a}_{\overline{K+1|}}}_{\frac{1-v^{K+1}}{d}} + .05G$$

$$d = \frac{.04}{1.04}$$

$$= \left(100 + \frac{.95G}{d}\right)\underset{\sim}{v}^{K+1} + \text{constant}$$

$$\mathrm{Var}[L_0^g] = \left(100 + \frac{.95G}{d}\right)^2 \left[\, \underset{\downarrow \atop 0.17}{{}^2A_x} - \left(A_x\right)^2 \right]$$

$$\xrightarrow{} 1 - d\ddot{a}_x$$

$$= \underbrace{1 - \frac{.04}{1.04}(16.50)}$$

$$=$$

$$\downarrow$$

$$= \underline{\underline{908.1414}}$$

$$E[L_0^g] = 0 \Rightarrow \underbrace{G\ddot{a}_x = 100A_x + .05G\ddot{a}_x + .05G}$$

$$G = \frac{100A_x}{.95\ddot{a}_x - .05} \boxed{= \underline{2{,}338}}$$

Extend with expenses $\quad \Pr[L_i^g > 0] = ?$

- fully discrete whole life of 100 to (40):
- SULT mortality at $i = 5\%$
- expenses: $\begin{array}{lll} \text{1st year} & 5\% \text{ of } G \\ \text{renewal} & 1\% \text{ of } G \end{array}$ $\quad \begin{array}{ll} 4\% \text{ of } G & \text{extra 1st yr} \\ 1\% \text{ of } G & \text{maintenance} \end{array}$
- $G = 1$

Calculate $\Pr[L_i^S > 0]$.

$$L_0^g = PVFB_0 + PVFE_0 - PVFP_0$$

$$= 100v^{K+1} + \underline{.01G}\,\ddot{a}_{\overline{K+1}|} + .04G - \underline{G}\,\ddot{a}_{\overline{K+1}|}$$

$$\ddot{a}_{\overline{K+1}|} \downarrow \frac{1-v^{K+1}}{d}$$

$$= \underbrace{\left(100 - \frac{.99G}{d}\right)}_{}v^{K+1} \underbrace{- \frac{.99G}{d} + .04G}_{} > 0$$

$$L_i^g = \left(100 - \frac{.99}{d}\right) v^{K+1} \underbrace{- \frac{.99}{d} + .04}_{} > 0$$

$G = 1$

$$v^{K+1} > \frac{\frac{.99}{d} + .04}{100 - \frac{.99}{d}}$$

take log

divide by $\log v = -\delta$

⋮

$$\Pr[K < \dots] = ????$$

Portfolio percentile premium principle

N

Suppose insurer issues a portfolio of N "identical" and "independent" policies where the PV of loss-at-issue for the i-th policy is $L_{0,1}$.

The total portfolio (aggregate) future loss is then defined by

$$L_{\text{agg}} = L_{0,1} + L_{0,2} + \cdots + L_{0,N} = \sum_{i=1}^{N} L_{0,i}$$

aggregate loss

L_{agg}

Its expected value is therefore

$$\mathsf{E}[L_{\text{agg}}] = \sum_{i=1}^{N} \mathsf{E}\left[L_{0,i}\right] = N \cdot E[L_{0,1}]$$

and, by "independence", the variance is

$$\mathsf{Var}[L_{\text{agg}}] = \sum_{i=1}^{N} \mathsf{Var}\left[L_{0,i}\right]. = N \cdot Var[L_{0,1}]$$

Portfolio percentile premium principle

The portfolio percentile premium principle sets the premium P so that there is a probability, say α with $0 < \alpha < 1$, of a positive gain from the portfolio.

In other words, we set P so that

loss positive small

$$\boxed{\mathsf{Pr}[L_{\mathsf{agg}} < 0] = \alpha.} \iff \Pr[L_{agg} > 0] = 1-\alpha$$

Note that loss could include expenses.

$\alpha = 0.95$

Consider Example 6.12 (2nd edition)

$$\Pr[L_{agg} < 0] = \alpha \iff \Pr\left[\underbrace{\frac{L_{agg} - E[L_{agg}]}{\sqrt{Var(L_{agg})}}}_{N = \text{standard Normal}} < \boxed{\frac{-E[L_{agg}]}{\sqrt{Var(L_{agg})}}} \right] = \alpha$$

= α^{th} percentile

$$\Pr[N \le Z_\alpha] = \alpha$$

$\alpha = 0.95$

$Z_\alpha = 1.645$

$$\Rightarrow \quad Z_\alpha = \frac{-E[L_{agg}]}{\sqrt{Var(L_{agg})}}$$

Illustrative example 5

An insurer sells 100 fully discrete whole life insurance policies of \$1, each of the same age 45. You are given:

- All policies have independent future lifetimes.
- Mortality follows the `Survival Ultimate Life Table` with $i = 0.05$.

Using the Normal approximation:

$\Pr[\text{Loss} > 0] < 1 - \alpha = 1 - .95 = .05$

1. Calculate the annual contract premium according to the portfolio percentile premium principle with $\alpha = 0.95$. $\Pr[\text{Loss} < 0] = 0.95$
2. Suppose the annual contract premium is set at 0.02 per policy. Determine the smallest number of policies to be sold so that the insurer has at least a 95% probability of a gain from this portfolio of policies.

$$L_{0,1} = v^{K+1} - P\ddot{a}_{\overline{K+1|}}$$

$$E[L_{0,1}] = E[v^{K+1}] - P\,E[\ddot{a}_{\overline{K+1|}}] = A_{45} - P\ddot{a}_{45} = \underbrace{.15161 - P(17.8162)}$$

$$= v^{K+1} - P\left(\frac{1-v^{K+1}}{d}\right) = \left(1+\frac{P}{d}\right)v^{K+1} - \frac{P}{d}$$

$${}^2A_{45} = .03463$$

$$A_{45} = .15161$$

$$Var[L_{0,1}] = \left(1+\frac{P}{d}\right)^2 \underbrace{Var(v^{K+1})}$$

$$\underbrace{\left[{}^2A_{45} - (A_{45})^2\right]}_{.01164441}$$

$N = 100$

$\alpha = .95$

$Z_\alpha = 1.645$

$$E[L_{agg}] = 100\left(.15161 - P(17.8162)\right) \Rightarrow 1.645 = \frac{-\overset{10}{\cancel{100}}\left(.15161 - P(17.8162)\right)}{\sqrt{100\left(1+\frac{P}{d}\right)^2(.01164441)}}$$

$$Var(L_{agg}) = 100\left(1+\frac{P}{d}\right)^2(.01164441)$$

$$\cancel{10}\left(1+\frac{P}{d}\right)\left(\sqrt{.01164441}\right)$$

$$-10\left(.15161 - P(17.8162)\right) = 1.645\left(1+\frac{P}{d}\right)\sqrt{.01164441}$$

linear in P $\longrightarrow$ $P = .009709163$

$d = .05/1.05$

$P = .02$ Known   Find $N =$ number of polices

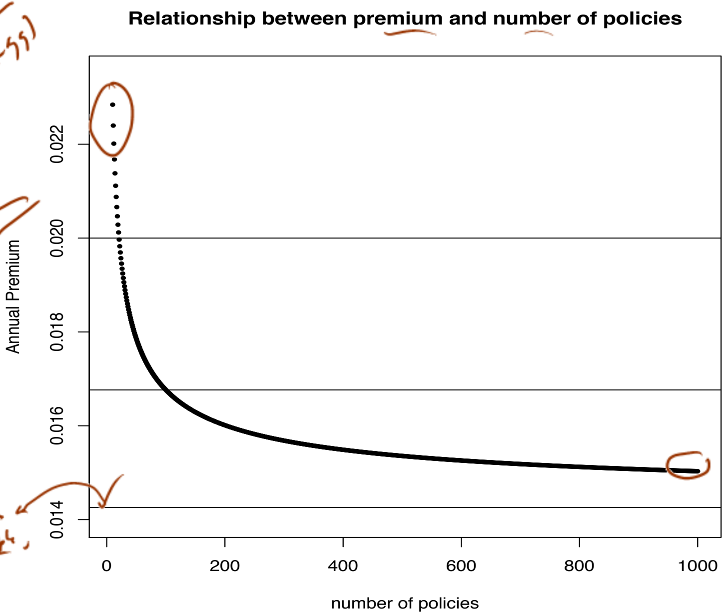
$$\frac{-E[L_{agg}]}{\sqrt{Var[L_{agg}]}} = \frac{-N\,E[\overset{.15161 - 17.8162\overset{.02}{P}}{L_{0,1}}]}{\sqrt{N\cdot \underset{\left(1+\frac{\overset{.02}{P}}{\underset{.05/1.05}{d}}\right)^2(.01164441)}{Var[L_{0,1}]}}}$$

$$= \frac{\overset{\sqrt{N}}{\cancel{-N}}[\cancel{-}.204714]}{\cancel{\sqrt{N}}\sqrt{\cancel{N}(.02347978)}} = 1.645$$

$$(\sqrt{N})^2 = \left(\frac{1.645(.204714)}{\sqrt{.02347978}}\right)^2 \Rightarrow N = 1.5161611$$

at least 2 policies

$\frac{n\,E(L_S)}{\sqrt{n}\sqrt{Var(L_S)}} = 2\alpha$



equivalence principle,

$\to \infty$

Profit

Consider a fully discrete whole life insurance to (x) with benefit equal to $\$B$ and annual premiums of $\$P$. The net loss-at-issue can be expressed as

$$L_0 = B\,v^{K+1} - P\,\ddot{a}_{\overline{K+1}|},$$

where $K = K_x$ is the curtate future lifetime of (x).

The probability that the insurer makes a profit on the policy is

$$\begin{aligned} \text{Pr}[L_0 < 0] &= \text{Pr}\left[B\,v^{K+1} - P\,\ddot{a}_{\overline{K+1}|}\right] \\ &= \text{Pr}[K > \tau - 1] = 1 - \text{Pr}[K \leq \tau - 1] \\ &= 1 - \text{Pr}[K \leq \lfloor \tau \rfloor - 1] = 1 - {}_{\lfloor \tau \rfloor}q_x = {}_{\lfloor \tau \rfloor}p_x \end{aligned}$$

where

$$\tau = -\frac{1}{\delta}\log\left(\frac{P/d}{B + P/d}\right).$$

Profit for a single policy $\underbrace{\Pr[L_0 < 0]}$ = profit ⇔ negative loss

fully discrete whole life B, (x)

P = Known

B = 100,000

P = 1088.779

x = 40

SULT i = .05

$\delta = \log 1.05$

$d = .05/1.05$

$$L_0 < 0 \iff B \cdot v^{K+1} - P\ddot{a}_{\overline{K+1|}} < 0$$

$$\iff B \cdot v^{K+1} - P\left(\frac{1-v^{K+1}}{d}\right) < 0$$

$$\iff \left(B + \frac{P}{d}\right)v^{K+1} - \frac{P}{d} < 0$$

$$\iff v^{K+1} < \frac{P/d}{B+P/d}$$

$$\iff K > a \rightarrow \underbrace{\frac{-\log\left(\frac{P/d}{B+P/d}\right)}{\delta} - 1}_{31.30934}$$

$$\Pr[L_0<0] \iff \Pr[K > a]$$

$$\iff \Pr[K \geq 32] = 1 - \underbrace{\Pr[K \leq 31]}$$

$$= {}_{31}p_{40} = \frac{\ell_{71}}{\ell_{40}} = \underline{.687}$$

- continued

Consider the case where $x = 40$, $B = 100,000$, mortality follows the `Survival Ultimate Life Table`, and $i = 0.05$. Thus we have, assuming equivalence principle,

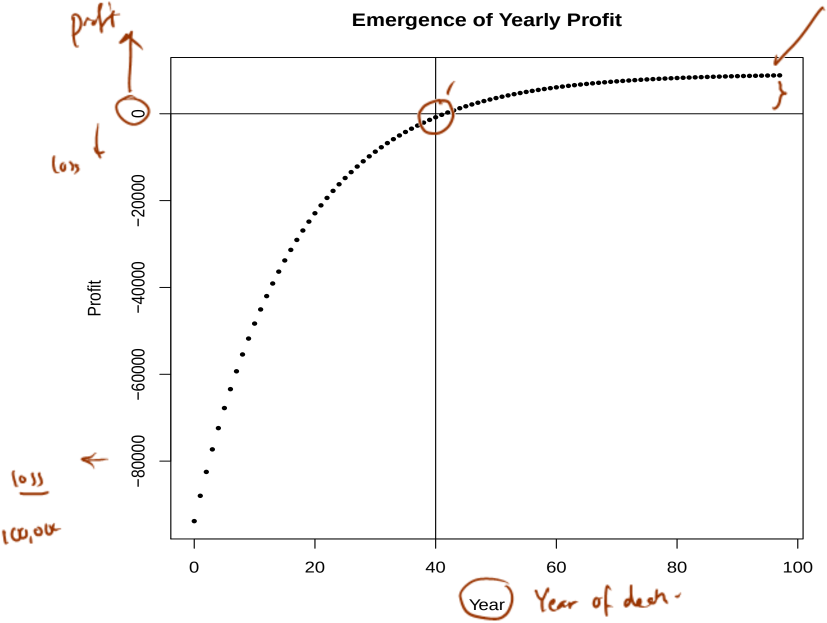
$$P = \frac{100000\, A_{40}}{\ddot{a}_{40}} = \frac{100000(0.16132)}{14.8166} = 1088.779$$

so that

$$\tau = -\frac{1}{\log(1.06)} \log\left(\frac{1088.779/(.06/1.06)}{100000 + 1088.779/(.06/1.06)}\right) = 31.30934$$

The probability that the insurer makes a profit on the policy are

$$\mathsf{Pr}[L_0 < 0] = {}_{31}p_{40} = \frac{\ell_{71}}{\ell_{40}} = \frac{6396609}{9313166} = 0.6868351.$$



profit

loss

loss

100,000

Year of death.

Return of premium policies

Consider a fully discrete whole life insurance to (x) with benefit equal to $\$B$ plus return of all premiums accumulated with interest at rate j.

The net random future loss in this case can be expressed as

$$L_0 = P\,\ddot{s}_{\overline{K+1}|j}\,v^{K+1} + B\,v^{K+1} - P\,\ddot{a}_{\overline{K+1}|},$$

for $K = 0, 1, \ldots$ and $\ddot{s}_{\overline{K+1}|j}$ is calculated at rate j. All other actuarial functions are calculated at rate i.

Consider the following cases:

- Let $j = 0$. This implies $\ddot{s}_{\overline{K+1}|j} = (K+1)$ and the annual benefit premium will be
$$P = \frac{B\,A_x}{\ddot{a}_x - (IA)_x}.$$

- continued

- Let $i = j$. In this case, the loss $L_0 = B\,v^{K+1}$ because $\ddot{s}_{\overline{K+1|}j}\,v^{K+1} = \ddot{a}_{\overline{K+1|}}$. Thus, there is no possible premium because all premiums are returned and yet there is an additional benefit of $\$B$.
- Let $i < j$. Then we have

$$L_0 = P\left(\ddot{s}_{\overline{K+1|}j} - \ddot{s}_{\overline{K+1|}}\right)v^{K+1} + B\,v^{K+1},$$

which is always positive because $\ddot{s}_{\overline{K+1|}j} > \ddot{s}_{\overline{K+1|}}$ when $i < j$. No possible premium.

- continued

- Let $i > j$. Then we can write the loss as

$$L_0 = P\,\frac{v_{j^*}^{K+1} - v^{K+1}}{d_j} + B\,v^{K+1} - P\,\ddot{a}_{\overline{K+1}|}$$

where $d_j = 1 - [1/(1+j)]$ and v_{j^*} is the corresponding discount rate associated with interest rate $j^* = [(1+i)/(1+j)] - 1$. Here,

$$P = \frac{A_x}{\ddot{a}_x - \frac{(A_x)_{j^*} - A_x}{d_j}},$$

where $(A_x)_{j^*}$ is a (discrete) whole life insurance to (x) evaluated at interest rate j^*.

Illustrative example 2

For a whole life insurance on (40), you are given:

- Death benefit, payable at the end of the year of death, is equal to \$10,000 plus the return of all premiums paid without interest.
- Annual benefit premium of 290.84 is payable at the beginning of each year.
- $(IA)_{40} = 8.6179$
- $i = 4\%$

Calculate $\ddot{a}_{40}$.

SOA Question #22 Fall 2012

You are given the following information about a special fully discrete 2-payment, 2-year term insurance on (80):

- Mortality follows the `Survival Ultimate Life Table`.
- $i = 0.0175$
- The death benefit is 1000 plus a return of all premiums paid without interest.
- Level premiums are calculated using the equivalence principle.

Calculate the benefit premium for this special insurance.

For practice: try calculating the benefit premium if the return of all premiums paid comes with an interest of say 0.01.

SOA Question #3 Spring 2013

For a special fully discrete 20-year endowment insurance on (40), you are given:

- The only death benefit is the return of annual benefit premiums accumulated with interest at 5% to the end of the year of death.
- The endowment benefit is 100,000.
- Mortality follows the `Survival Ultimate Life Table`.
- $i = 0.05$

Calculate the annual benefit premium.

SOA Question #7 Fall 2017

For a special 10-year deferred whole life annuity-due of 300 per year issued to (55), you are given:

- Annual premiums are payable for 10 years.
- If death occurs during the deferral period, all premiums paid are returned without interest at the end of the year of death.
- $\ddot{a}_{55} = 12.2758$
- $\ddot{a}_{55:\overline{10}|} = 7.4575$
- $(IA)^{1}_{55:\overline{10}|} = 0.51213$

Calculate the level net premium.

Pricing with extra or substandard risks

An impaired individual, or one who suffers from a medical condition, may still be offered an insurance policy but at a rate higher than that of a standard risk.

Generally there are three possible approaches:

- age rating: calculate the premium with the individual at an older age
- constant addition to the force of mortality: $\mu^s_{x+t} = \mu_{x+t} + \phi$, for $\phi > 0$
- constant multiple of mortality rates: $q^s_{x+t} = \min\left(cq_{x+t}, 1\right)$, for $c > 1$

Read Section 6.9.

Published SOA question #45

Your company is competing to sell a life annuity-due with an APV of \$500,000 to a 50-year-old individual.

Based on your company's experience, typical 50-year-old annuitants have a complete life expectancy of 25 years. However, this individual is not as healthy as your company's typical annuitant, and your medical experts estimate that his complete life expectancy is only 15 years.

You decide to price the benefit using the issue age that produces a complete life expectancy of 15 years. You also assume:

- For typical annuitants of all ages, $\ell_x = 100(\omega - x)$, for $0 \leq x \leq \omega$.
- $i = 0.06$

Calculate the annual benefit that your company can offer to this individual.

Other terminologies and notations used

Expression	Other terms/symbols used
net random future loss	loss at issue
L_0	${}_0L$
net premium	benefit premium
gross premium	expense-loaded premium
equivalence principle	actuarial equivalence principle
generic premium	G P π
~~substandard~~	~~may be superscripted with * or s~~

expense loading = G - P

calculator ✓

+

up to 3 sheets

$i^{(m)}$ ✓ $d^{(m)}$ ✓

① ✓

12 questions

Chap 2	survival models	1
Chap 3	life tables	1
Chap 4	insurance	2
Chap 5	annuities	2
Chap 6	premiums →	6 ✓

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review all the problems ✓

+

review last 2 finals

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Class Test 2