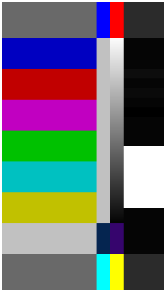
Dynamical Algebraic Combinatorics and the Homomesy Phenomenon

Tom Roby (UConn)

Describing joint research with
James Propp & Gregg Musiker

Discrete Mathematics Seminar
Worcester Polytechnic Institute
Worcester, MA USA



8 February 2018 (Thursday), 10:00

Slides for this talk are available online (or will be soon) at

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Abstract

Abstract: Dynamical Algebraic Combinatorics explores actions on sets of discrete combinatorial objects, many of which can be built up by small local changes, e.g., Schutzenberger's promotion and evacuation, or the rowmotion map on order ideals. There are strong connections to the combinatorics of representation theory and with Coxeter groups. Birational liftings of these actions are related to the Y-systems of statistical mechanics, thereby to cluster algebras, in ways that are still relatively unexplored. The term "homomesy" (coined by Jim Propp and the speaker) describes the following widespread phenomenon. Given a group action on a set of combinatorial objects, a statistic on these objects is called homomesic if its average value is the same over all orbits. Along with its intrinsic interest as a kind of "hidden invariant", homomesy can be used to prove certain properties of the action, e.g., facts about the orbit sizes. Proofs of homomesy often involve developing tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection.

This talk will focus on the combinatorial side, giving a number of examples of homomesy due to the author and others.

Acknowledgments

This seminar talk discusses work with Jim Propp and Gregg Musiker including ideas and results from Arkady Berenstein, David Einstein, Darij Grinberg, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Drew Armstrong, Anders Björner, Barry Cipra, Darij Grinberg, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to interrupt with questions or comments.

Outline

- Homomesy: definition and examples;
- Rowmotion on rectangular posets (products of two chains);
- Liftings to piecewise-linear and birational actions;

Main Definition: Homomesy = "Constant Averages over Orbits"

For many actions τ on a finite set S of combinatorial objects, and for many natural real-valued statistics φ on S, one finds that the ergodic average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\tau^i(x))$$

is **independent** of the starting point $x \in S$.

We say that φ is **homomesic** (from Greek: "same middle") with respect to the combinatorial dynamical system (S, τ).

Introductory examples

1. Rotation of bit-strings;
2. Bulgarian solitaire;
3. Promotion of Near-Standard Young Tableaux; and
4. Suter's dihedral symmetries on Young's lattice.

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

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$0110 \mapsto 2$	AVG $= \frac{4}{2} = 2$
$0011 \mapsto 0$	
AVG$= \frac{8}{4} = 2$	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001

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110000	010001	010010
011000	101000	001001
001100	010100	
000110	001010	
000011	000101	

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EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
$100001 \mapsto 4$	$100010 \mapsto 5$	$100100 \mapsto 6$
$110000 \mapsto 8$	$010001 \mapsto 3$	$010010 \mapsto 4$
$011000 \mapsto 6$	$101000 \mapsto 7$	$001001 \mapsto 2$
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
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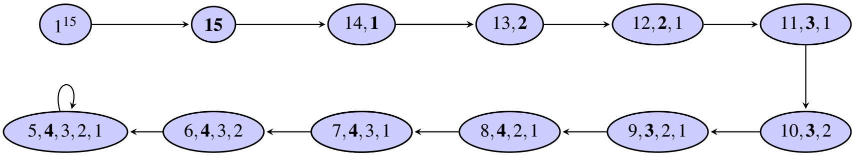
We know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate b (replacing a 01 somewhere in b by 10 or vice versa), or one can write the number of inversions in b as $\sum_{i<j} b_i(1 - b_j)$ and then perform algebraic manipulations.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define б(λ) as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

- First surfaced as a puzzle in Russia around 1980, and a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardner column; see survey of Brian Hopkins [Hop12].
- Initial puzzle: starting from any of 176 partitions of 15, one ends at $(5, 4, 3, 2, 1)$.



Bulgarian solitaire: homomesies

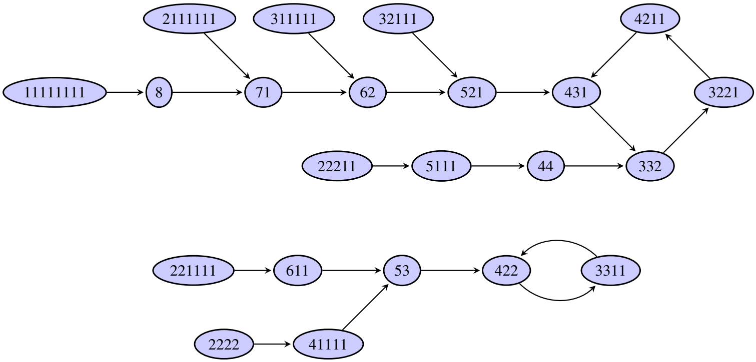
E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).



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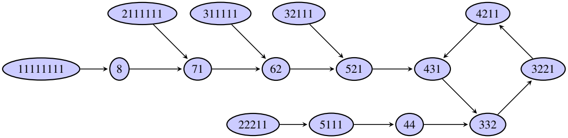

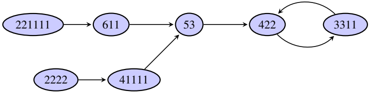

Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5, 3)$, the average value of φ is $(4 + 3)/2 = 7/2$; while for $\lambda = (6, 2)$, the average value of φ is $(3 + 4 + 4 + 3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic б; similarly for "size of (kth) largest part".

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

So an equivalent way of stating our main definition in this case is, φ is homomesic with respect to (S, τ) iff the average of φ over each periodic τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$.

In the rest of this talk, we'll restrict attention to maps τ that are invertible on S, so transience is not an issue.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
(Call f c-mesic for short.)

Example 3: Promotion of Semi-Standard Young Tableaux

Given a partition λ and $N \in \mathbb{P}$, a **Semi-Standard Young Tableau (SSYT)** of shape λ on $[N] = \{1, 2, \ldots, N\}$ is a filling of the diagram of T with entries from $[N]$ which increases weekly in rows, strictly in columns.

For each $i \in [N-1]$, let s_i be the action on SSYT's with ceiling N that replaces i (if it occurs in T) by $i+1$, and vice versa, provided that this does not violate the increasing condition in the definition of Young tableaux, and let ∂ be the composition of the maps:

$$\partial T := s_{N-1} \circ s_{N-2} \circ \cdots \circ s_1 T$$

This gives an operation on SYT introduced by Schützenberger called **promotion**.

For example, applying s_7 transforms the following tableau as shown:

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 7 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 6 & 9 \\ \cline{1-2} \end{array} \quad \xrightarrow{s_7} \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 8 & 10 \\ \hline 2 & 7 & 11 \\ \cline{1-3} 6 & 9 \\ \cline{1-2} \end{array}$$

Promotion of SSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$. (Here the ceiling $N = 11$.)

$$T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} \end{array} \quad ,$$

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Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$. (Here the ceiling $N = 11$.)

$$T = \begin{array}{|c|c|c|c|}\hline {\color{red}1}&4&6&10\\\hline 2&8&11\\\cline{1-3} 7&9\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&4&6&10\\\hline {\color{red}2}&8&11\\\cline{1-3} 7&9\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&4&6&10\\\hline {\color{red}3}&8&11\\\cline{1-3} 7&9\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&6&10\\\hline {\color{red}4}&8&11\\\cline{1-3} 7&9\\\cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&6&10\\\hline {\color{red}5}&8&11\\\cline{1-3} 7&9\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&10\\\hline {\color{red}6}&8&11\\\cline{1-3} 7&9\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&10\\\hline 6&8&11\\\cline{1-3} {\color{red}7}&9\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&10\\\hline 6&7&11\\\cline{1-3} {\color{red}8}&9\\\cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&10\\\hline 6&7&11\\\cline{1-3} 8&{\color{red}9}\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&9\\\hline 6&7&11\\\cline{1-3} 8&{\color{red}10}\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&9\\\hline 6&7&10\\\cline{1-3} 8&{\color{red}11}\\\cline{1-2}\end{array} = \partial T.$$

A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942; http://arxiv.org/abs/1108.1172):

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942 1927

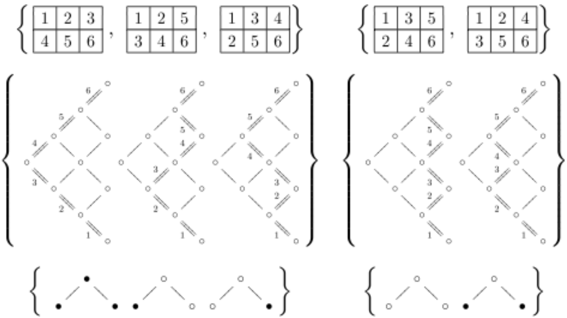

Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

A small example of promotion: centrally symmetric sums

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942 1927

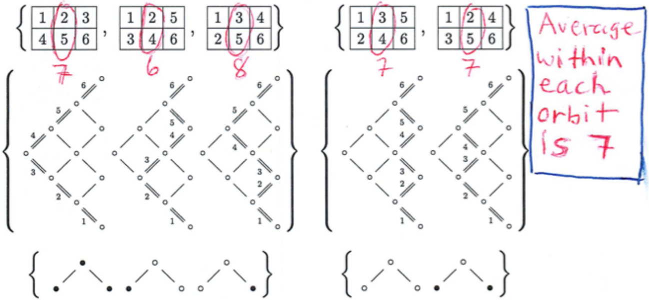

Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

Promotion of Semi-Standard Young Tableaux: homomesies

Theorem (Bloom-Pechenik-Saracino 2016, Conj. Propp-Roby 2013)

Let S be the set of Semi-Standard Young Tableau of rectangular shape λ, and ceiling N. If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center, (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\varphi(T)$ denotes the sum of the numbers in cells c and c', then φ is homomesic with respect to (S, ∂) with average value $N + 1$.

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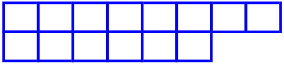
Although rectangular shapes may appear to be a very special case, they are one of the few shapes where the order of promotion on the set of SYT is small, i.e., n or $2n$. Striker & Williams point out that the order of promotion on SYT of shape $(8, 6)$ is 7,554,844,752.



Rowmotion: an invertible operation on antichains

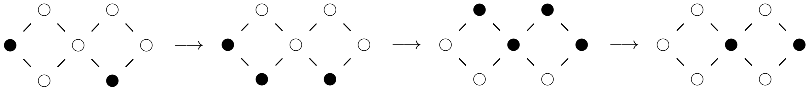
Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_A(A)$ be the set of minimal elements of the complement of the downward-saturation of A.

ρ_A is invertible since it is a composition of three invertible operations:

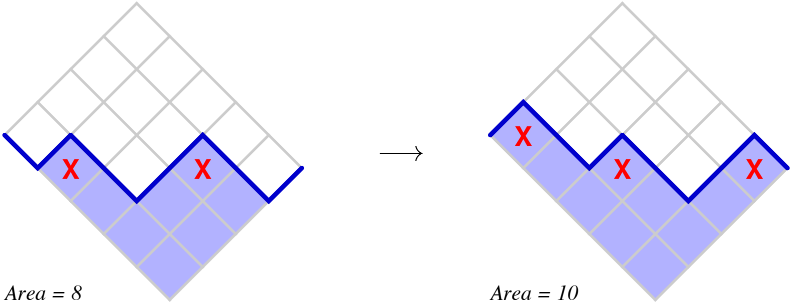
$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

ρ_A :



This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Viewing the elements of the poset as **squares** below, we would map:



Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong-Stump-Thomas [AST11], Conj. [Pan09])

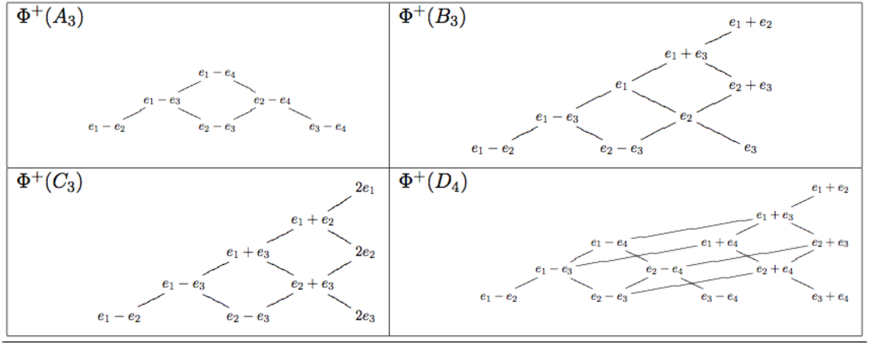
Let $\mathcal{O}$ be an arbitrary ρ_A-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

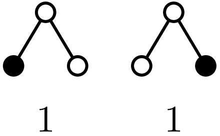
Picture of root posets

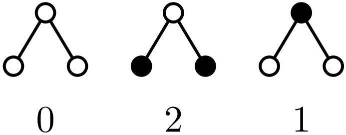
Here are the classes of posets included in Panyushev's conjecture.



(Graphic courtesy of Striker-Williams.)

Here we have just an orbit of size 2 and an orbit of size 3:

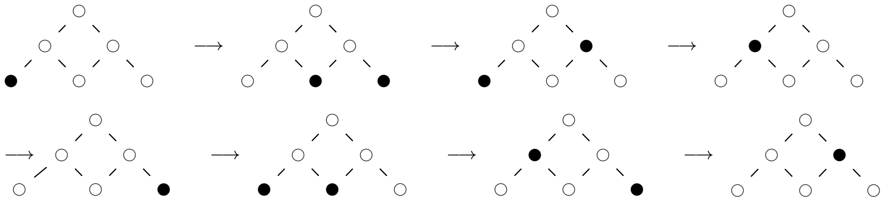


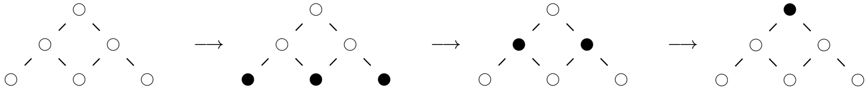


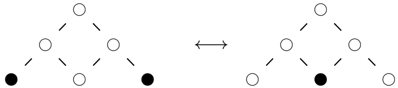
Within each orbit, the average antichain has cardinality $n/2 = 1$.

Example of Rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ_A-orbits, of sizes 8, 4, 2:







Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (the type A *minuscule* poset), where $[k] = \{1, 2, \ldots, k\}$:

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary ρ_A-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

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This proof uses an non-obvious equivariant bijection (the "Stanley-Thomas" word [Sta09, §2]) between order ideals in $[a] \times [b]$ and binary strings, which carries the ρ_J action to cyclic rotation of bitstrings.

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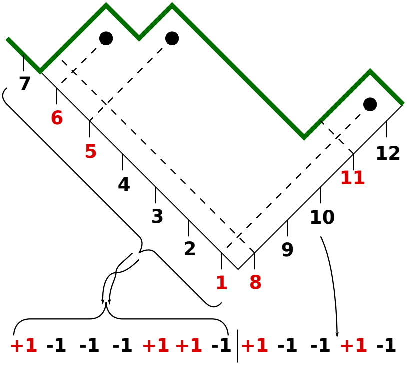
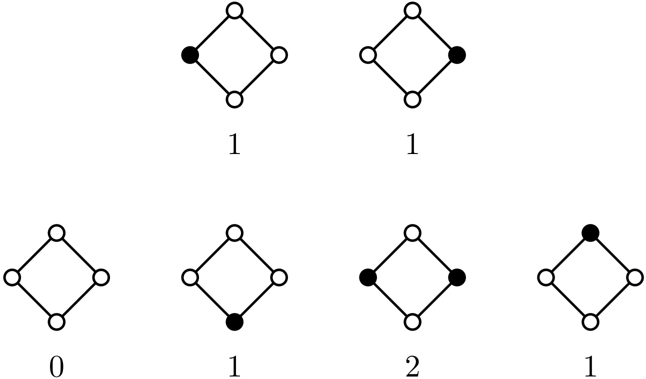


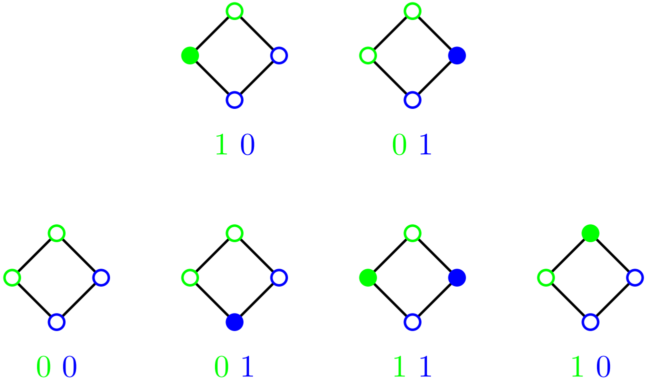
Shows the Stanley-Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$.

Here we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average antichain has cardinality $ab/(a + b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has

$1/2$ a green element and $1/2$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i,j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under ρ_A, even though the indicator functions $1_{i,j}$ aren't.

Antichains in $[a] \times [b]$: centrally symmetric homomesies

Theorem (Propp, R.)

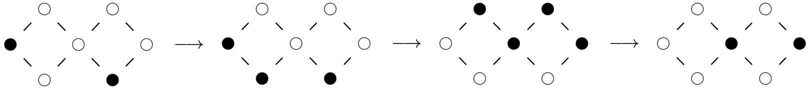
In any orbit, the number of A that contain (i,j) equals the number of A that contain the opposite element $(i',j') = (a+1-i, b+1-j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under ρ_A, with average value 0 in each orbit.

Rowmotion on order ideals

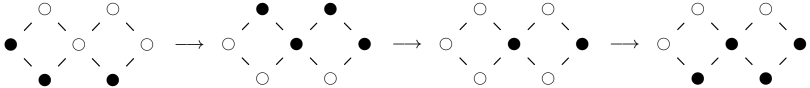
We've already seen examples of Rowmotion on antichains ρ_A:

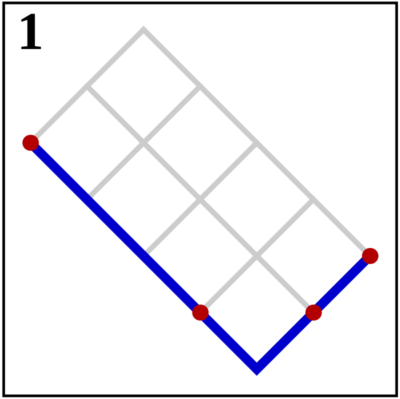
ρ_A :



We can also define it as an operator ρ_J on $J(P)$, the set of order ideals of a poset P, by shifting the waltz beat by 1:

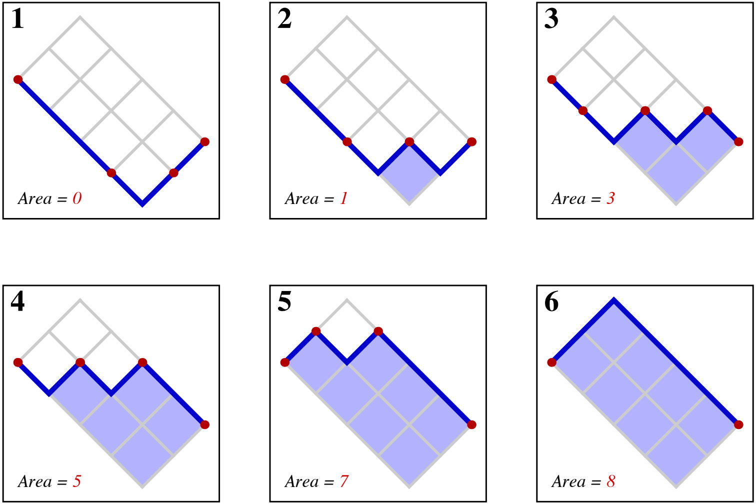
ρ_J :



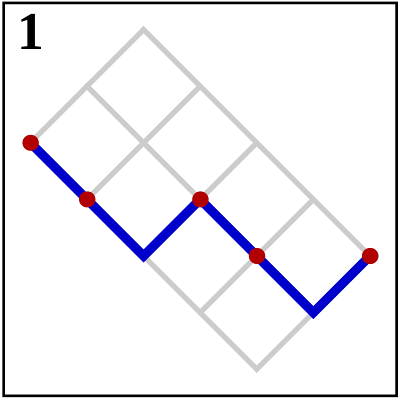


Rowmotion

Rowmotion on $[4] \times [2]$ A

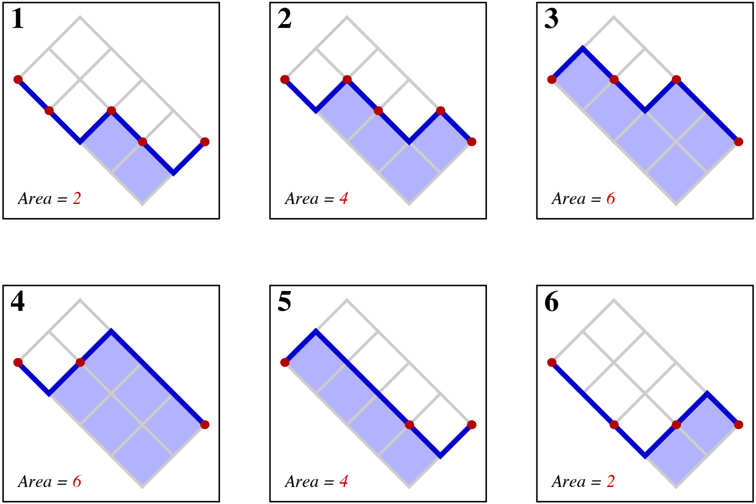


$$(0+1+3+5+7+8)\ /\ 6 = 4$$

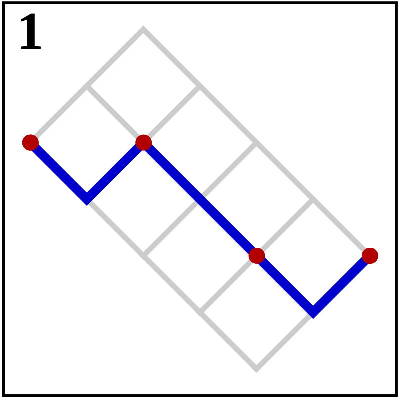


Rowmotion

Rowmotion on $[4] \times [2]$ B

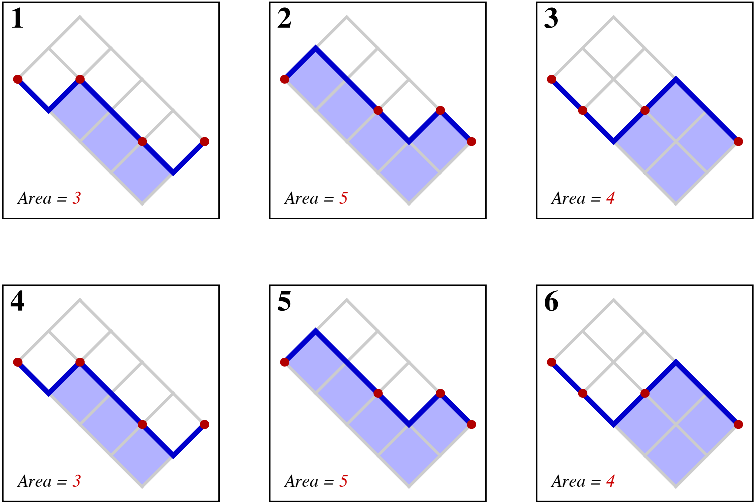


$$(2+4+6+6+4+2)\ /\ 6 = 4$$



Rowmotion

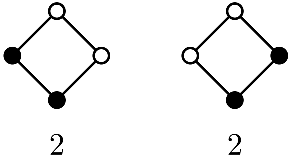
Rowmotion on $[4] \times [2]$ C

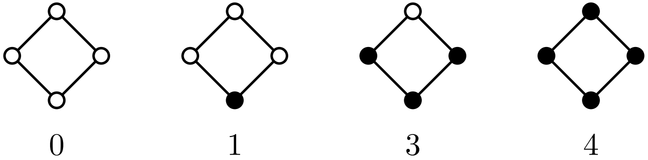


$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

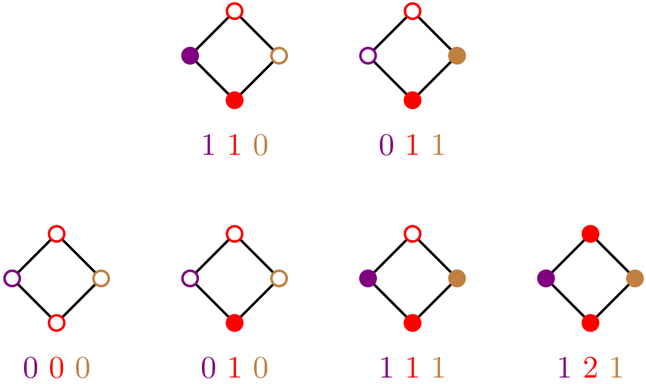
Again we have an orbit of size 2 and an orbit of size 4:





Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \le k \le a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \le i \le a,\ 1 \le j \le b,\ i - j = k\}.$$

For $1 - b \le k \le a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every ρ_J-orbit $\mathcal{O}$ in $J([a] \times [b])$:

- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \ge 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \le 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
    - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.

Classical rowmotion: the toggling definition

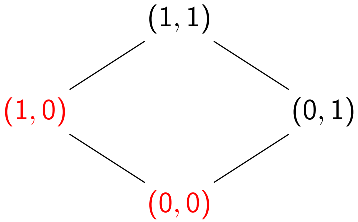
- Let $(v_1, v_2, \ldots, v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \ldots \circ \mathbf{t}_{v_n}.$$

Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Start with this order ideal S:



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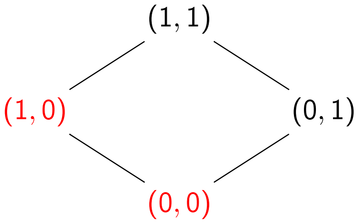
Example: Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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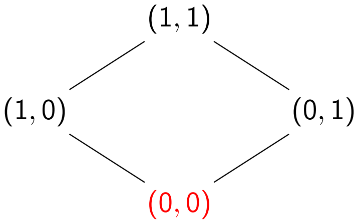
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1, 0)$ from the order ideal:



- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
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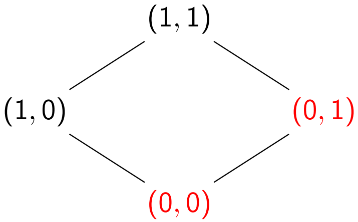

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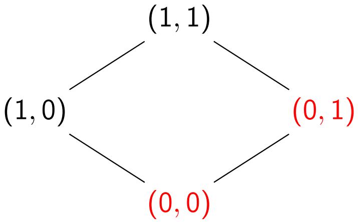
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Finally apply $\mathbf{t}_{(0,0)}$, which changes nothing:



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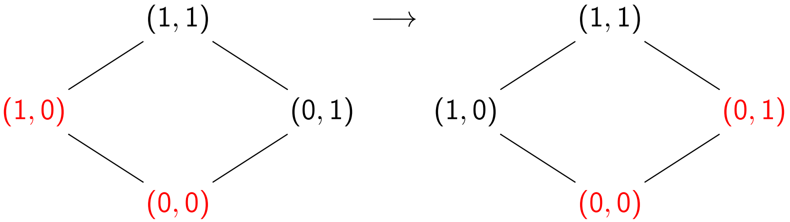
- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

  Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

Example: **Re-coordinatizing** $P = [a] \times [b] = [0, r] \times [0, s]$**, sorry!**

So this is $S \longrightarrow \mathbf{r}(S)$:



Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

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where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

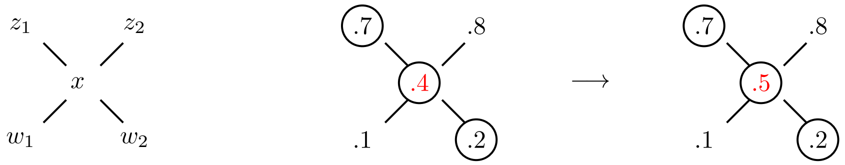
Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot\, x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

Example of flipping at a node

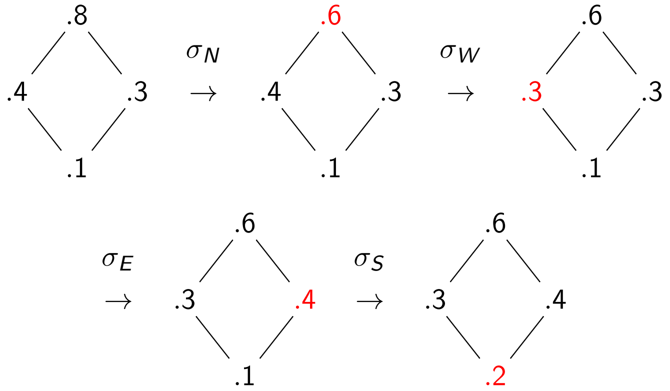


$$\min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

How PL rowmotion generalizes classical rowmotion

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

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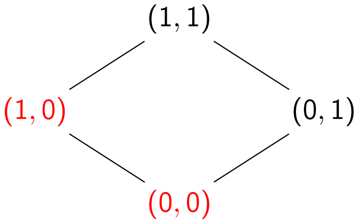
Example:

Start with this order ideal S:



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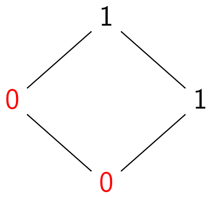
Translated to the PL setting:



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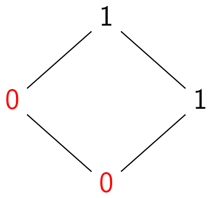
Example:

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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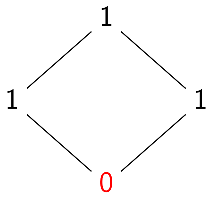
Example:

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1, 0)$ from the order ideal:



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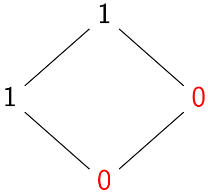
Example:

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0,1)$ to the order ideal:



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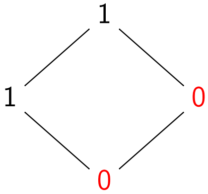

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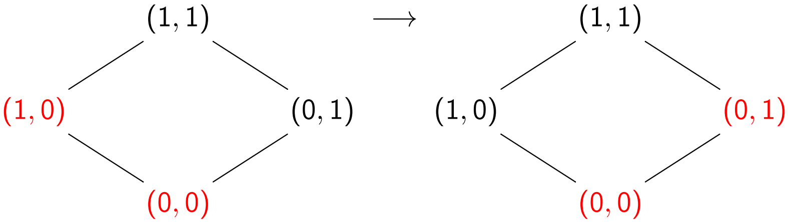

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can"detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements $\widehat{0}$ and $\widehat{1}$ to P and forcing
  - $\widehat{0}$ to be less than every other element, and
  - $\widehat{1}$ to be greater than every other element.
- Let $\mathbb{K}$ be a field.
- A **$\mathbb{K}$-labelling of P** will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.
- For any $v \in P$, define the **birational v-toggle** as the rational map

  $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $(T_v f)(w) = \dfrac{\sum_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.

  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.

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- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.
- We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- This is originally due to Einstein and Propp [EiPr13, EiPr14]. Another exposition of these ideas can be found in [Rob16].

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset	labelling
$\begin{tikzcd}[column sep=small] & \widehat{1} \arrow[d, dash] & \\ & (1,1) \arrow[dl, dash] \arrow[dr, dash] & \\ (1,0) \arrow[dr, dash] & & (0,1) \arrow[dl, dash] \\ & (0,0) \arrow[d, dash] & \\ & \widehat{0} & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

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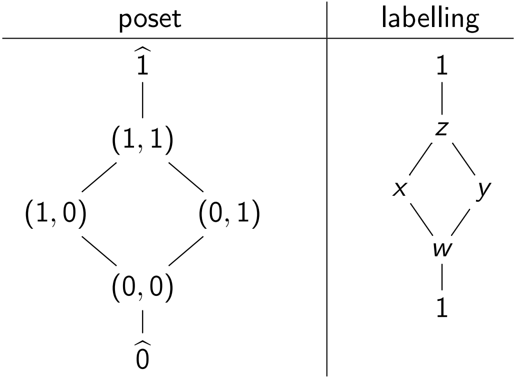

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension

$((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Birational rowmotion: example

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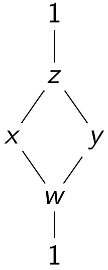
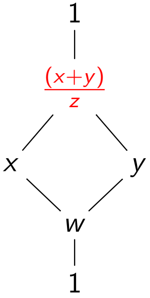
original labelling f	labelling $T_{(1,1)} f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

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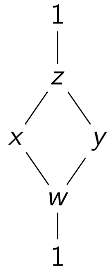	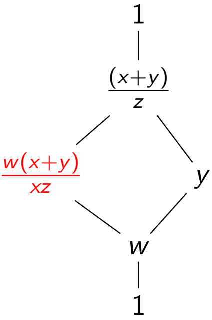

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Example:

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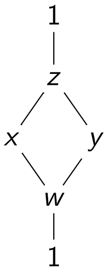	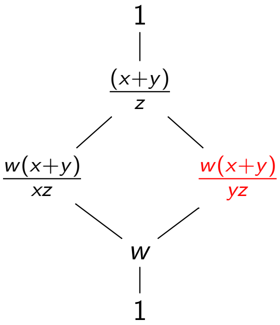

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Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$
$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{w(x+y)}{xz} \arrow[dr, dash] & & \frac{w(x+y)}{yz} \arrow[dl, dash] \\ & \textcolor{red}{\frac{1}{z}} \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

We are using $\rho_B = \textcolor{red}{T_{(0,0)}} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion orbit on a product of chains

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd}[column sep=small] & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{(x+y)w}{xz} \arrow[dr, dash] & & \frac{(x+y)w}{yz} \arrow[dl, dash] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[column sep=small] & \frac{(x+y)w}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{1}{y} \arrow[dr, dash] & & \frac{1}{x} \arrow[dl, dash] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[column sep=small] & \frac{1}{w} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{yz}{(x+y)w} \arrow[dr, dash] & & \frac{xz}{(x+y)w} \arrow[dl, dash] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}.$$

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Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0,r] \times [0,s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Why study this generalization?

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- This generalization implies the results at the PL and combinatorial level (but not vice-versa).
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PrRo15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.

The poset $[0, 1] \times [0, 1]$ has three files, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz}\ \frac{1}{y}\ \frac{yz}{(x+y)w}\ (x) = 1,$$

Birational homomesy on files of $J([0,r] \times [0,s])$

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

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$$\frac{1}{z}\ \frac{x+y}{z}\ \frac{z}{x+y}\ \frac{(x+y)w}{xy}\ \frac{xy}{(x+y)w}\ \frac{1}{w}\ (x)\ \ (z) = 1,$$

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$$\rho_B(f)(0,0)\rho_B(f)(1,1)\rho_B^2(f)(0,0)\rho_B^2(f)(1,1)\rho_B^3(f)(0,0)\rho_B^3(f)(1,1)\rho_B^4(f)(0,0)\rho_B^4(f)(1,1) =$$

$$\frac{1}{z}\ \frac{x+y}{z}\ \frac{z}{x+y}\ \frac{(x+y)w}{xy}\ \frac{xy}{(x+y)w}\ \frac{1}{w}\ (x)\ (z) = 1,$$

$$\rho_B(f)(0,1)\rho_B^2(f)(0,1)\rho_B^3(f)(0,1)\rho_B^4(f)(0,1) = \frac{(x+y)w}{yz}\ \frac{1}{x}\ \frac{xz}{(x+y)w}\ (y) = 1$$

The poset $[0,1]\times[0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz}\ \frac{1}{y}\ \frac{yz}{(x+y)w}\ (x) = 1,$$

$$\rho_B(f)(0,0)\rho_B(f)(1,1)\rho_B^2(f)(0,0)\rho_B^2(f)(1,1)\rho_B^3(f)(0,0)\rho_B^3(f)(1,1)\rho_B^4(f)(0,0)\rho_B^4(f)(1,1) =$$

$$\frac{1}{z}\ \frac{x+y}{z}\ \frac{z}{x+y}\ \frac{(x+y)w}{xy}\ \frac{xy}{(x+y)w}\ \frac{1}{w}\ (x)\ (z) = 1,$$

$$\rho_B(f)(0,1)\rho_B^2(f)(0,1)\rho_B^3(f)(0,1)\rho_B^4(f)(0,1) = \frac{(x+y)w}{yz}\ \frac{1}{x}\ \frac{xz}{(x+y)w}\ (y) = 1$$

Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *is periodic, with period* $r+s+2$*.*

(2) The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *satisfies the following reciprocity:*
$\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem (Musiker-R [MR18])

Given a file F *in* $[0,r] \times [0,s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1.$

Birational homomesy on files of $J([0,r] \times [0,s])$

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ is periodic, with period $r+s+2$.

(2) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ satisfies the following reciprocity: $\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem (Musiker-R [MR18])

Given a file F in $[0,r] \times [0,s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1.$

The proof of this involves constructing a complicated formula for the ρ_B^k in terms of families of non-intersecting lattice paths, from which one can also deduce periodicity and the other geometric homomesies of this action, first proved by Grinberg-R [GrRo15b,].

The final slide of this talk (before the references)

I'm happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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Thanks very much for coming to this talk!

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