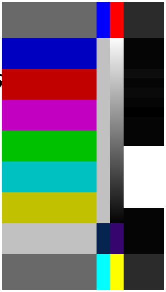
Dynamical Algebraic Combinatorics: Actions, Orbits, and Averages

Tom Roby (UConn)

Describing joint research with
Michael Joseph, James Propp, & Gregg Mus



Discrete Math Seminar
UMass Amherst USA

26 April 2018 (Thursday), 15:00

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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Abstract

Abstract: Dynamical Algebraic Combinatorics explores actions on sets of discrete combinatorial objects, many of which can be built up by small local changes, e.g., Schutzenberger's promotion and evacuation, or the rowmotion map on order ideals. There are strong connections to the combinatorics of representation theory and with Coxeter groups. Birational liftings of these actions are related to the Y-systems of statistical mechanics, thereby to cluster algebras, in ways that are still relatively unexplored. The term "homomesy" (coined by Jim Propp and the speaker) describes the following widespread phenomenon: Given a group action on a set of combinatorial objects, a statistic on these objects is called "homomesic" if its average value is the same over all orbits. Along with its intrinsic interest as a kind of "hidden invariant", homomesy can be used to prove certain properties of the action, e.g., facts about the orbit sizes. Proofs of homomesy often involve developing tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection.

This talk will be a introduction to these ideas, focussing on the combinatorial side and giving a number of examples of such actions.

Acknowledgments

This seminar talk discusses work with Mike Joseph, with Jim Propp and with (if we get that far) Gregg Musiker including ideas and results from Arkady Berenstein, David Einstein, Darij Grinberg, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Drew Armstrong, Anders Björner, Barry Cipra, Darij Grinberg, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to interrupt with questions or comments.

Outline

- Cyclic rotation of binary strings and definition of *homomesy*;
- Bulgarian Solitaire;
- Rowmotion map on antichains and order ideals of posets;
- Toggling independent sets of path graph;

Cyclic rotation of binary strings

Cyclic rotation of binary strings

- Let $\binom{[n]}{k}$ be the set of length n binary strings with k 1s.
- Let $C_R : \binom{[n]}{k} \to \binom{[n]}{k}$ be rightward cyclic rotation.

Example

$n = 6, \; k = 2$

$$101000 \underset{C_R}{\longmapsto} 010100$$

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

$n = 6, \ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R.[PrRo15, §2.3])

Let $\mathcal{I}(s)$ denote the number of inversions of $s \in \binom{[n]}{k}$.

Then the function $\mathcal{I} : \binom{[n]}{k} \to \mathbb{Q}$ is homomesic with average $\frac{k(n-k)}{2}$ with respect to cyclic rotation on $S_{n,k}$.

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Proof.

Consider **superorbits** of length n. Show that replacing “01" with “10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

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$n = 6, \ k = 2$

String	Inv	String	Inv	String	Inv
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Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well, e.g., Let $\mathbb{1}_j(s) := s_j$, the *jth* bit of the string s. Can you see why this is homomesic?

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of SSYT; Rowmotion of "nice" (e.g., minuscule heap) posets [PrRo15, StWi11, Had14, RuWa15+] ;

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- Liftings of homomesy from combinatorial actions to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].
- There are many others, including the next few examples.

Bulgarian Solitaire

There are some cases where we find a similar phenomenon, but where the map no longer has finite orbits. Here is a more general definition of homomesy that is useful for some purposes.

Definition

Let τ be an self-map on a discrete set of objects S, and f be a statistic on S. We say f is **homomesic** if the value of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = c$$

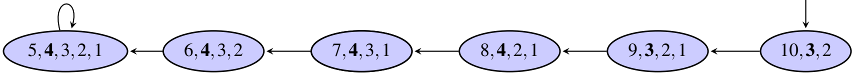
is **independent** of the starting point $x \in S$. (Also, f is c-mesic.)

This clearly reduces to the earlier definition in the case where we have an invertible action with finite orbits.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define б(λ) as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

- First surfaced as a puzzle in Russia around 1980, and a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardiner column; see survey of Brian Hopkins [Hop12].
- Initial puzzle: starting from any of 176 partitions of 15, one ends at $(5, 4, 3, 2, 1)$.



Bulgarian solitaire: “orbits” are now “trajectories”

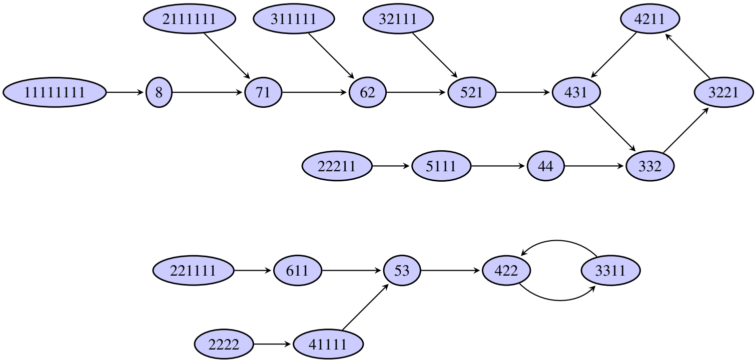
E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

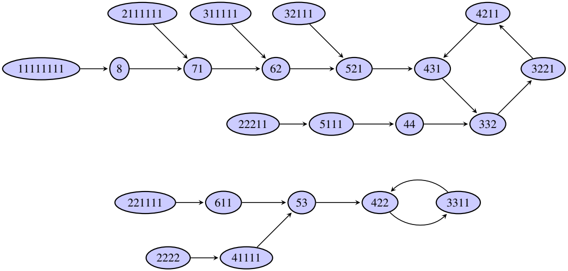
and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).



Bulgarian solitaire: homomesies



Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5,3)$, the average value of φ is $(4+3)/2 = 7/2$; while for $\lambda = (6,2)$, the average value of φ is $(3+4+4+3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic wrt/б; similarly for "size of (kth) largest part".

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

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This definition also works in situations where S is infinite. But for rest of this talk, we'll restrict attention to maps τ that are invertible on S, where S is finite, so our initial definition (below) makes sense.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
(Call f c-**mesic** for short.)

Rowmotion: an invertible operation on antichains

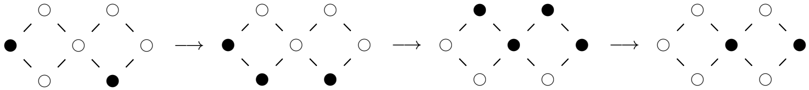
Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_A(A)$ be the set of minimal elements of the complement of the downward-saturation of A.

ρ_A is invertible since it is a composition of three invertible operations:

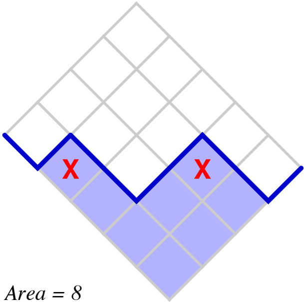
$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

ρ_A :



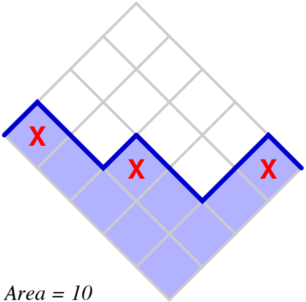
This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Viewing the elements of the poset as **squares** below, we would map:



Area = 8

$\longrightarrow$



Area = 10

Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong-Stump-Thomas [AST11], Conj. [Pan09])

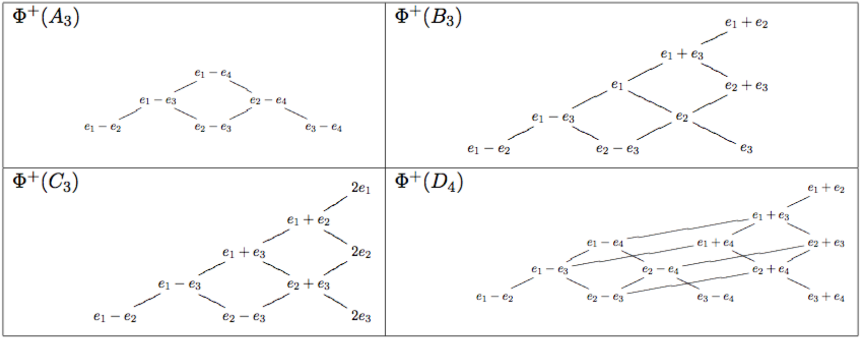
Let $\mathcal{O}$ be an arbitrary ρ_A-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

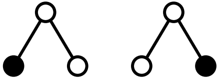
Picture of root posets

Here are the classes of posets included in Panyushev's conjecture.

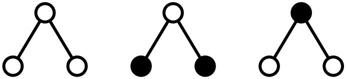


(Graphic courtesy of Striker-Williams.)

Here we have just an orbit of size 2 and an orbit of size 3:



$1 \quad 1$

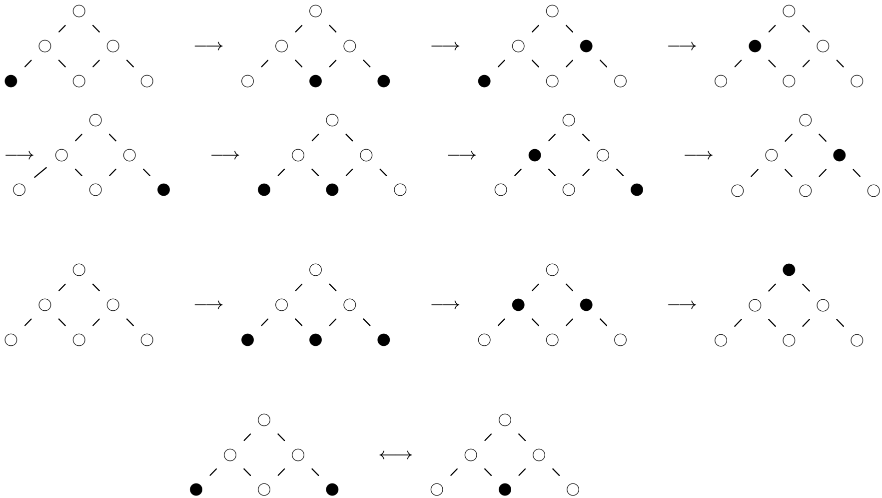


$0 \quad 2 \quad 1$

Within each orbit, the average antichain has cardinality $n/2 = 1$.

Example of Rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ_A-orbits, of sizes 8, 4, 2:



Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (the type A *minuscule* poset), where $[k] = \{1, 2, \ldots, k\}$:

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary ρ_A-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

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$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

This proof uses an non-obvious equivariant bijection (the "Stanley-Thomas" word [Sta09, §2]) between order ideals in $[a] \times [b]$ and binary strings, which carries the ρ_J action to cyclic rotation of bitstrings.

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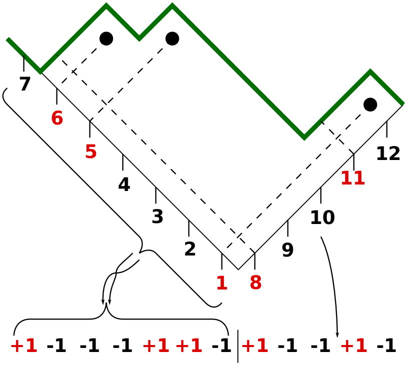
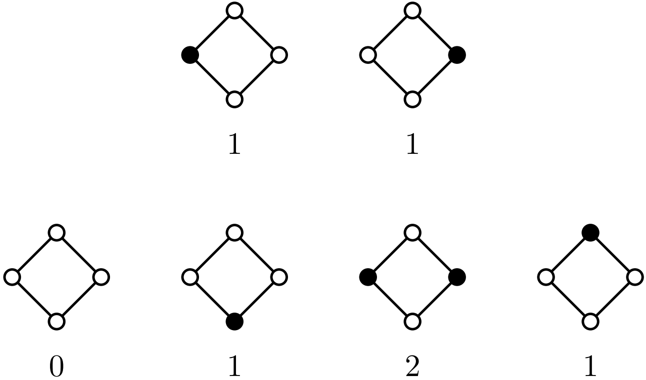


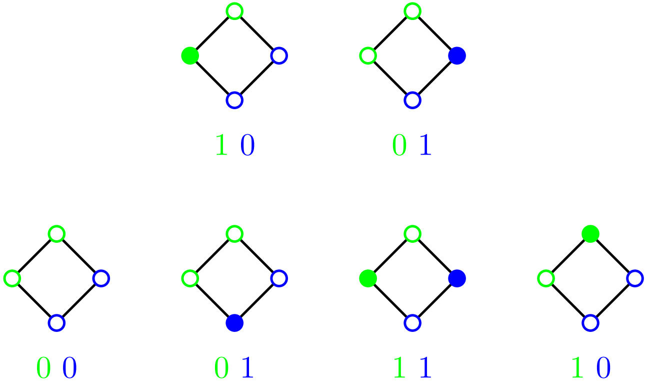
Shows the Stanley-Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$.

Here we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has

$1/2$ a green element and $1/2$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i,j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under ρ_A, even though the indicator functions $1_{i,j}$ aren't.

Antichains in $[a] \times [b]$: centrally symmetric homomesies

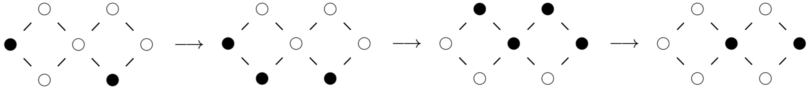
Theorem (Propp, R.)

In any orbit, the number of A that contain (i, j) equals the number of A that contain the opposite element $(i', j') = (a + 1 - i, b + 1 - j)$.

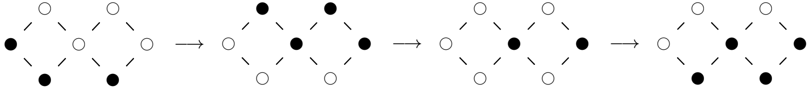
That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under ρ_A, with average value 0 in each orbit.

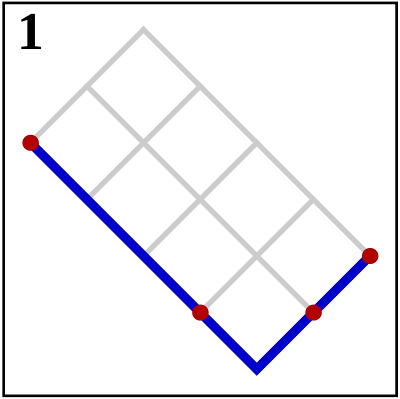
Rowmotion on order ideals

We've already seen examples of Rowmotion on antichains ρ_A:

ρ_A :


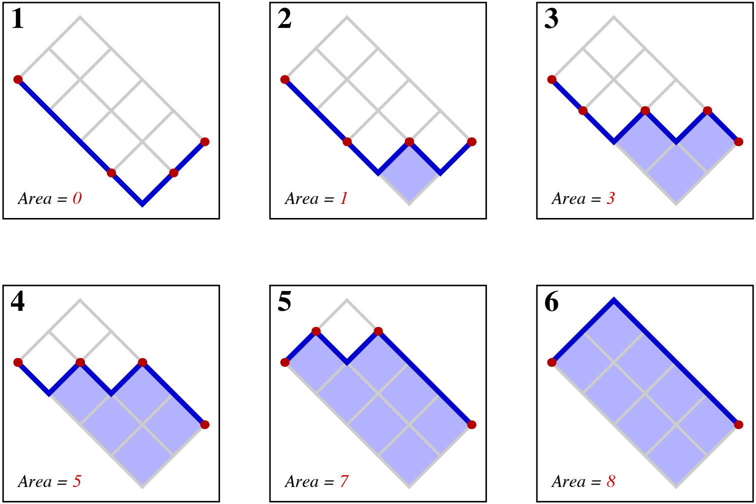
We can also define it as an operator ρ_J on $J(P)$, the set of order ideals of a poset P, by shifting the waltz beat by 1:

ρ_J :


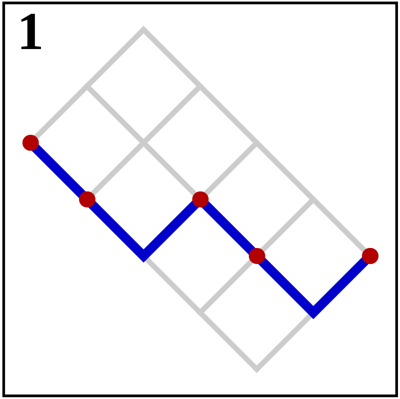


Rowmotion

Rowmotion on $[4] \times [2]$ A



$$(0+1+3+5+7+8)\ /\ 6 = 4$$



Rowmotion

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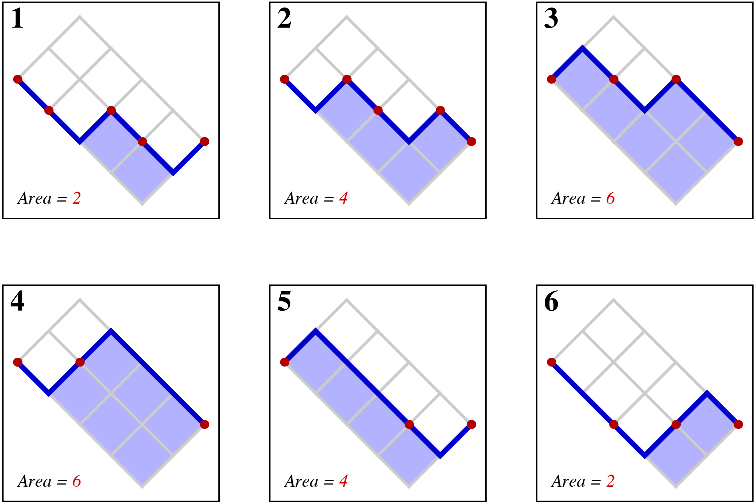
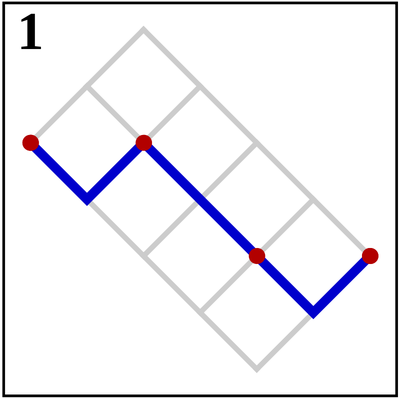
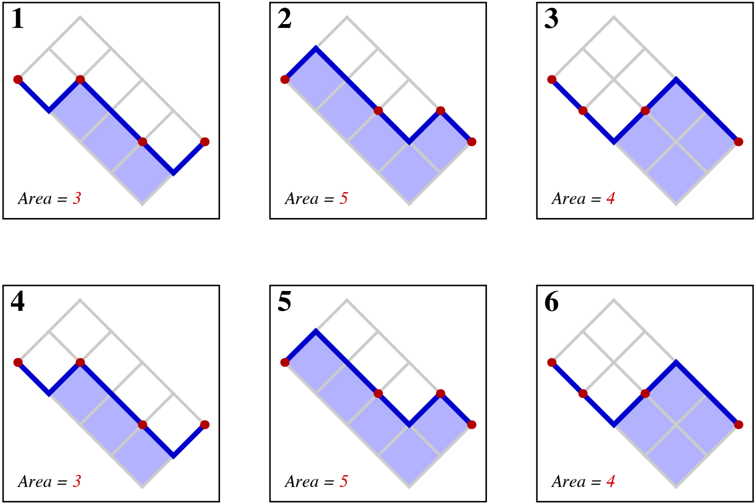


$$(2+4+6+6+4+2)\ /\ 6 = 4$$



Rowmotion

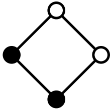
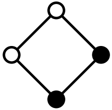
Rowmotion on $[4] \times [2]$ C



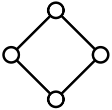
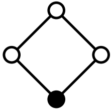
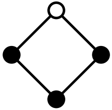
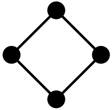
$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:

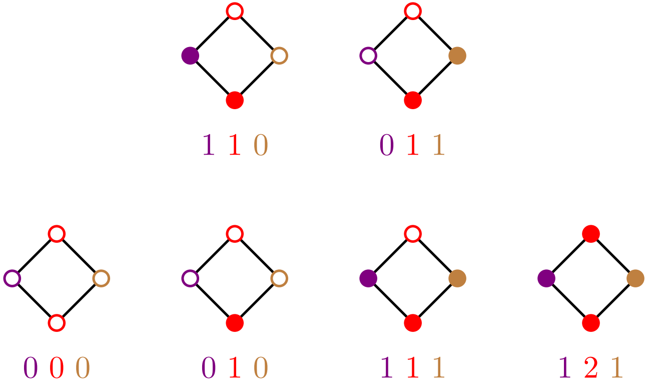
2 2

0 1 3 4

Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1-b \leq k \leq a-1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i-j=k\}.$$

For $1-b \leq k \leq a-1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every ρ_J-orbit $\mathcal{O}$ in $J([a] \times [b])$:

- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

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  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave S fixed.")

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  - S otherwise.

  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.

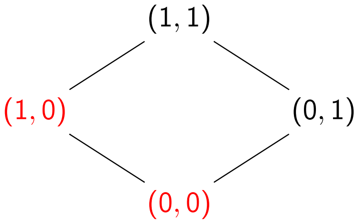
- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

  Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Start with this order ideal S:



- Let $(v_1, v_2, \ldots, v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
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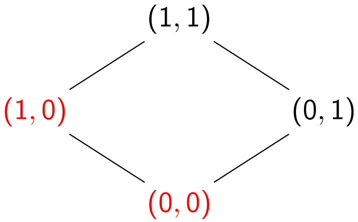
Example: Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



Classical rowmotion: the toggling definition

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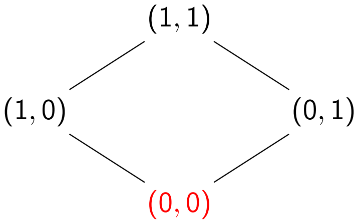
Example: Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1, 0)$ from the order ideal:



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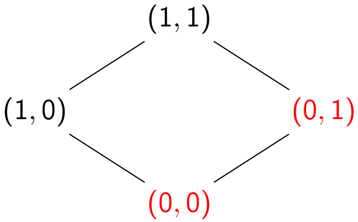
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0, 1)$ to the order ideal:



Classical rowmotion: the toggling definition

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
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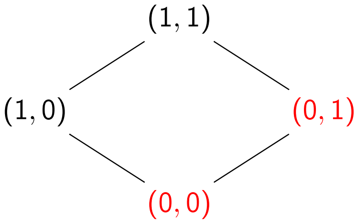
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Finally apply $\mathbf{t}_{(0,0)}$, which changes nothing:



Classical rowmotion: the toggling definition

- Let $(v_1, v_2, \ldots, v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
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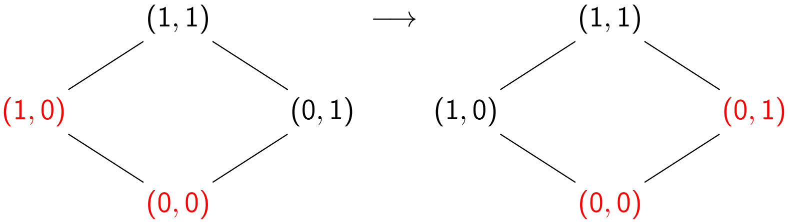
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

So this is $S \longrightarrow \mathbf{r}(S)$:



Toggling Independent Sets of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

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An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the subsequence 11.

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex i** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

Toggles

Proposition

Each toggle τ_i *is an involution, i.e.,* τ_i^2 *is the identity. Also,* τ_i *and* τ_j *commute if and only if* $|i - j| \neq 1$*.*

Definition

The **toggle group** is the group generated by the n toggles.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Homomesy

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph-R.[JR18])

Define $\chi_i : \mathcal{I}_n \to \{0,1\}$ to be the indicator function of vertex i.

For $1 \leq i \leq n$, $\chi_i - \chi_{n+1-i}$ is 0-mesic on φ-orbits of $\mathcal{I}_n$.

Also $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ are 1-mesic on φ-orbits of $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1)$ mod 4.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

The final slide of this talk (before the references)

I'm happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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Thanks very much for coming to this talk!

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Piecewise-linear and birational liftings

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Generalizing to the piecewise-linear setting

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

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where $z \cdot > x$ means z covers x and $w <\cdot x$ means x covers w.

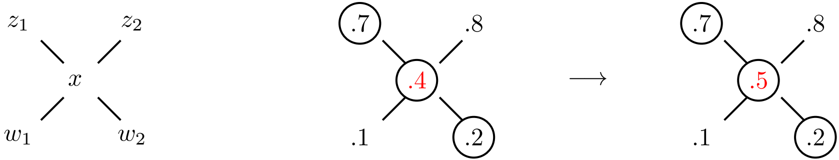
Note that the interval $[\min_{z \cdot > x} f(z), \max_{w <\cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot > x} f(z) + \max_{w <\cdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

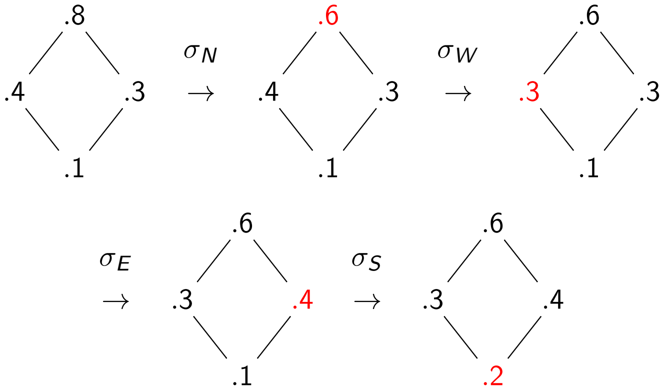
Example of flipping at a node



$$\min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

How PL rowmotion generalizes classical rowmotion

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

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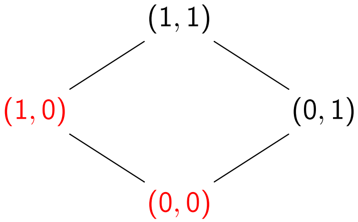
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

Start with this order ideal S:



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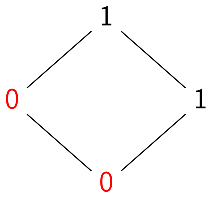
where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

Example:

Translated to the PL setting:



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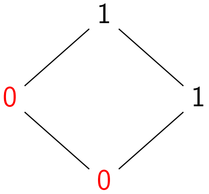
where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

Example:

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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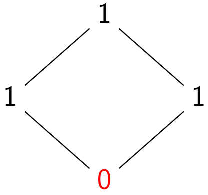
where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

Example:

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1,0)$ from the order ideal:



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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

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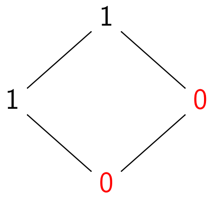
where $z \cdot> x$ means z covers x and $w <\cdot x$ means x covers w.

Example:

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0,1)$ to the order ideal:



How PL rowmotion generalizes classical rowmotion

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

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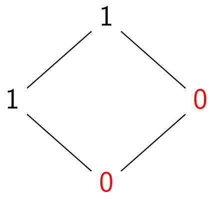
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

Finally apply $\mathbf{t}_{(0,0)}$, which changes nothing:



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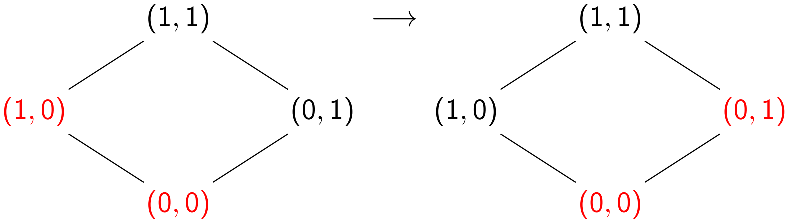
where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can"detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements $\widehat{0}$ and $\widehat{1}$ to P and forcing
  - $\widehat{0}$ to be less than every other element, and
  - $\widehat{1}$ to be greater than every other element.
- Let $\mathbb{K}$ be a field.
- A **$\mathbb{K}$-labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.
- For any $v \in P$, define the **birational** v**-toggle** as the rational map

  $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $(T_v f)(w) = \dfrac{\sum_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.

  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $(T_v f)(w) = \frac{\sum_{u \lessdot v} f(u)}{f(v) \sum_{u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.
- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.

Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $\left(T_v f\right)(w) = \dfrac{\sum_{u \lessdot v} f(u)}{f(v) \sum_{u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.
- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.
- We define **birational rowmotion** as the rational map

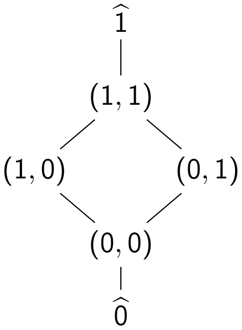
$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

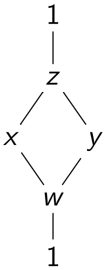
  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- This is originally due to Einstein and Propp [EiPr13, EiPr14]. Another exposition of these ideas can be found in [Rob16].

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset	labelling





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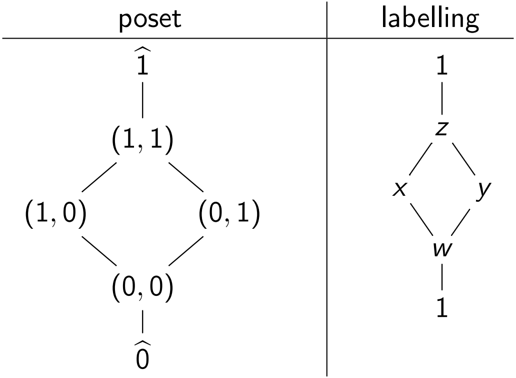


We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension

$((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

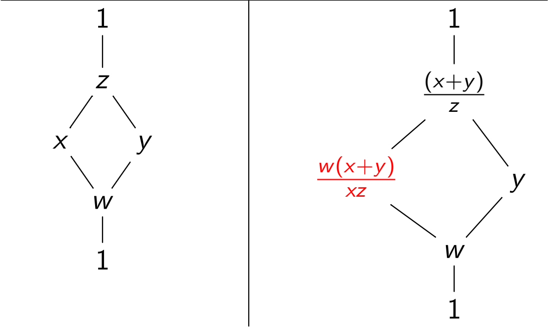
original labelling f	labelling $T_{(1,1)} f$
$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & \color{red}{\frac{(x+y)}{z}} \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ \color{red}{T_{(1,1)}}$.

Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

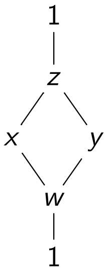
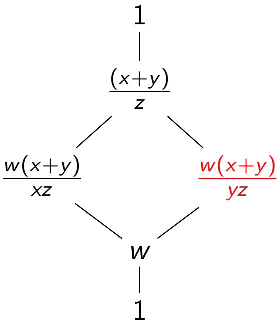
original labelling f	labelling $T_{(1,0)} T_{(1,1)} f$



We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

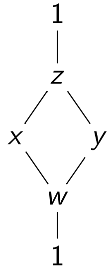
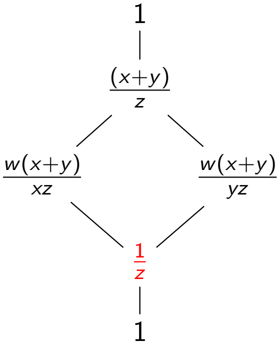
original labelling f	labelling $T_{(0,1)} T_{(1,0)} T_{(1,1)} f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd} & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{(x+y)w}{xz} \arrow[dr, dash] & & \frac{(x+y)w}{yz} \arrow[dl, dash] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd} & \frac{(x+y)w}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{1}{y} \arrow[dr, dash] & & \frac{1}{x} \arrow[dl, dash] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd} & \frac{1}{w} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{yz}{(x+y)w} \arrow[dr, dash] & & \frac{xz}{(x+y)w} \arrow[dl, dash] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}.$$

Birational rowmotion orbit on a product of chains

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd}[column sep=small] & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{(x+y)w}{xz} \arrow[dr, dash] & & \frac{(x+y)w}{yz} \arrow[dl, dash] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[column sep=small] & \frac{(x+y)w}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{1}{y} \arrow[dr, dash] & & \frac{1}{x} \arrow[dl, dash] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[column sep=small] & \frac{1}{w} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{yz}{(x+y)w} \arrow[dr, dash] & & \frac{xz}{(x+y)w} \arrow[dl, dash] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}.$$

Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Why study this generalization?

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- This generalization implies the results at the PL and combinatorial level (but not vice-versa).
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PrRo15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.

The poset $[0, 1] \times [0, 1]$ has three files, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz}\ \frac{1}{y}\ \frac{yz}{(x+y)w}\ (x) = 1,$$

Birational homomesy on files of $J([0,r] \times [0,s]$

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$$\frac{1}{z}\ \frac{x+y}{z}\ \frac{z}{x+y}\ \frac{(x+y)w}{xy}\ \frac{xy}{(x+y)w}\ \frac{1}{w}\ (x)\ (z) = 1,$$

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Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ is periodic, with period $r+s+2$.

(2) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ satisfies the following reciprocity: $\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem (Musiker-R [MR18])

Given a file F in $[0,r] \times [0,s]$, $$\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1.$$

Birational homomesy on files of $J([0,r] \times [0,s])$

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The proof of this involves constructing a complicated formula for the ρ_B^k in terms of families of non-intersecting lattice paths, from which one can also deduce periodicity and the other geometric homomesies of this action, first proved by Grinberg-R [GrRo15b,].