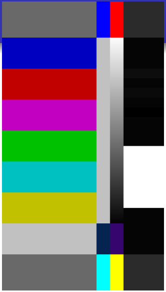
Dynamical Algebraic Combinatorics and the Homomesy Phenomenon on Independent Sets of a Path Graph and on Permutations



Tom Roby (UConn)

Describing joint research with
Michael Joseph & Michael LaCroix

Tsuda University
Kodaira-shi, Tokyo JAPAN

Slides for this talk are available online (or will be soon) at
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2 March 2018 (Friday)

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Abstract: Given a group acting on a finite set of combinatorial objects, one can often find natural statistics on these objects which are *homomesic*, i.e., over each orbit of the action, the average value of the statistic is the same. Since the notion was codified a few years ago, homomesic statistics have been uncovered in a wide variety of situations within dynamical algebraic combinatorics. We discuss a couple of interesting examples in depth, including "Coxeter toggling" the independent sets in a path graph (joint work with Michael Joseph), and "Foatic actions" on $\mathfrak{S}_n$ (joint work with Michael LaCroix).

Acknowledgments

This seminar talk discusses joint work with Michael Joseph and Michael La Croix. Thanks to James Propp for suggesting the study of toggling independent set and of Foatic actions, as well as earlier collaborations on the homomesy phenomenon.

Please feel free to interrupt with questions or comments.

何かご質問コメント等があったら、ご遠慮なくおしゃって下さい。

Outline

- Cyclic Rotation of bitstrings (or k-subsets);
- Actions, orbits, and homomesy;
- Toggling Independent Sets of a Path Graph;
- Foatic actions on $\mathfrak{S}_n$.

Cyclic rotation of binary strings

- Let $\binom{[n]}{k}$ be the set of length n binary strings with k 1s.
- Let $C_R : \binom{[n]}{k} \to \binom{[n]}{k}$ be rightward cyclic rotation.

Example

$n = 6,\ k = 2$

$$101000 \underset{C_R}{\longmapsto} 010100$$

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

$n = 6,\ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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010001	3	100001	4		
Average	4	**Average**	4	**Average**	4

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Homomesy

Theorem (Propp & R.[PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in \binom{[n]}{k}$.

Then the function $\mathrm{inv} : \binom{[n]}{k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Theorem (Propp & R.[PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in \binom{[n]}{k}$*.*

Then the function $\mathrm{inv} : \binom{[n]}{k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$*.*

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Example

$n = 6,\ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
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100010	5	000011	0	010010	4
010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of SSYT; Rowmotion of "nice" (e.g., minuscule heap) posets [PrRo15, StWi11, Had14, RuWa15+] ;

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- In general, composing certain involutions called "toggles" on the set leads to operations with interesting homomesy [Str15+];

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- Liftings of homomesy from combinatorial actions to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].

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- Liftings of homomesy from combinatorial actions to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].
- There are many others, including the next two examples.

Toggling Independent Sets of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

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Definition

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Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the subsequence 11.

Toggles

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex *i*** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

Toggles

Proposition

Each toggle τ_i is an involution, i.e., τ_i^2 is the identity. Also, τ_i and τ_j commute if and only if $|i - j| \neq 1$.

Definition

The **toggle group** is the group generated by the n toggles.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

Homomesy

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph-R.[JoRo18])

Define $\chi_i : \mathcal{I}_n \to \{0,1\}$ to be the indicator function of vertex i.

For $1 \leq i \leq n$, $\chi_i - \chi_{n+1-i}$ is 0-mesic on φ-orbits of $\mathcal{I}_n$.

Also $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ are 1-mesic on φ-orbits of $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
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$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
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Total:	6	3	4	4	4	4	4	4	3	6

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$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
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$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
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Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
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$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1)$ mod 4.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Foatic actions on $\mathfrak{S}_n$

The Rényi-Foata (Drop Parentheses) Map

Definition

For $w \in \mathfrak{S}_n$, its **canonical (disjoint) cycle decomposition (CCD)** satisfies:
(a) each cycle is written with its largest element first; and
(b) the cycles are written in increasing order of largest element.
The map $\mathcal{F} : \mathfrak{S}_n \to \mathfrak{S}_n$ simply removes the parentheses from the CCD of w and regards the resulting word as a permutation in one-line notation.

$$w = 847296513 = (42)(6)(81)(9375) \overset{\mathcal{F}}{\mapsto} 426819375 = (2)(951487369),$$

Note that here w has 4 cycles, and $\mathcal{F}(w)$ has 4 **records** (i.e., left-to-right maxima) viz., 4, 6, 8, and 9.

It is easy to see that $\mathcal{F}$ is a bijection.

Foatic Actions of $\mathfrak{S}_n$

We consider actions $\mathfrak{S}_n$ of the following form:

$$\mathfrak{S}_n \xrightarrow{\mathcal{F}} \mathfrak{S}_n \xrightarrow{\mathcal{A}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{B}} \mathfrak{S}_n$$

where $\mathcal{A}$ and $\mathcal{B}$ are **dihedral involutions**, defined below.

- a) $\mathcal{C} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \dots w_n$ to its **complement** whose value in position i is $n + 1 - w_i$;
- b) $\mathcal{R} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \dots w_n$ to its **reversal** whose value in position i is w_{n+1-i};
- c) $\mathcal{Q}^2 : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \dots w_n$ to its **rotation by 180-degrees**, whose value in position i is $n + 1 - w_{n+1-i}$.
- d) $\mathcal{I} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation w to its **inverse** w^{-1};
- e) $\mathcal{D} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation w to its its **rotated-inverse** $\mathcal{Q}^2(\mathcal{I}(w))$.

We call such fourfold compositions, where $\mathcal{A}$ and $\mathcal{B}$ are from the above list, **Foatic**.

Foatic example

$\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{I}$ gives the Foatic map $\gamma := \mathcal{I} \circ \mathcal{F}^{-1} \circ \mathcal{C} \circ \mathcal{F}$.
If $n = 5$, then $\gamma[(4213)(5)] = (2)(4)(513)$ as follows

$$(4213)(5) \overset{\mathcal{F}}{\mapsto} 42135 \overset{\mathcal{C}}{\mapsto} 24531 \overset{\mathcal{F}^{-1}}{\mapsto} (2)(4)(531) \overset{\mathcal{I}}{\mapsto} (2)(4)(513)$$

The orbit (of size six) generated by (4213)(5) is

$$(4213)(5) \overset{\gamma}{\mapsto} (2)(4)(513) \overset{\gamma}{\mapsto} (412)(53) \overset{\gamma}{\mapsto} (2)(5314) \overset{\gamma}{\mapsto} (431)(52) \overset{\gamma}{\mapsto} (2)(3)(541)$$

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Define the statistic "$\mathrm{Fix}\, w$" to count the number of fixed points (equiv. 1-cycles) of w.

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Define the statistic "$\mathrm{Fix}\, w$" to count the number of fixed points (equiv. 1-cycles) of w.

Theorem (Sheridan-Rossi–R.)

The statistic Fix *is homomesic with respect to the Foatic complement-inversion action on* $\mathfrak{S}_n$.

Reversal-Inversion

The Foatic action with the nicest orbit structures and properties appears to be *Reversal-Inversion*:

$$\overline{\varphi} : \mathfrak{S}_n \xrightarrow{\mathcal{F}} \mathfrak{S}_n \xrightarrow{\mathcal{R}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{I}} \mathfrak{S}_n.$$

$$\varphi : \mathfrak{S}_n \xrightarrow{\mathcal{R}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{I}} \mathfrak{S}_n. \xrightarrow{\mathcal{F}} \mathfrak{S}_n$$

$$\begin{aligned}
w = &(2)(43)(51) \mapsto 24351 \mapsto 15342 \mapsto (1)(5342) \mapsto (1)(5243) &= \overline{\varphi}(w) \\
&(1)(5243) \mapsto 15243 \mapsto 34251 \mapsto (3)(42)(51) \mapsto (3)(42)(51) &= \overline{\varphi}^2(w) \\
&(3)(42)(51) \mapsto 34251 \mapsto 15243 \mapsto (1)(5243) \mapsto (1)(5342) &= \overline{\varphi}^3(w) \\
&(1)(5342) \mapsto 15342 \mapsto 24351 \mapsto (2)(43)(51) \mapsto (2)(43)(51) &= \overline{\varphi}^4(w)
\end{aligned}$$

This example also displays (down the second column) the conjugate orbit of φ, also of size 4.

$$24351 \xrightarrow{\varphi} 15243 \xrightarrow{\varphi} 34251 \xrightarrow{\varphi} 15342 \Lsh$$

Data on Orbit Sizes for Foatic Reversal-Inversion

n	1	2	3	4	5	6	7	8	9	10
# of orbits:	1	1	2	5	19	84	448	2884	21196	174160
LCM of orbit sizes:	1	2	4	8	16	32	64	128	256	512
GCD of orbit sizes:	1	2	2	4	4	4	4	8	8	8
Longest orbit size:	1	2	4	8	16	32	64	128	256	512
Shortest orbit size:	1	2	2	4	4	4	4	8	8	8
Size of id's orbit:	1	2	4	8	16	32	64	128	256	512

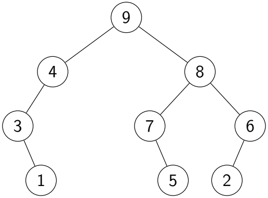
Heap representation of a permutation

Definition

Recursively define the **heap** of $w \in \mathfrak{S}_n$, $H(w)$ as follows: Set $H(\emptyset) = \emptyset$ (the empty word). If $w \neq \emptyset$, let m be the largest element of w, so w can be written uniquely as umv, where u and v are partial permutations (possibly empty). Set m to be the root of $H(w)$, with $H(u)$ its left subtree and $H(v)$ its right subtree.

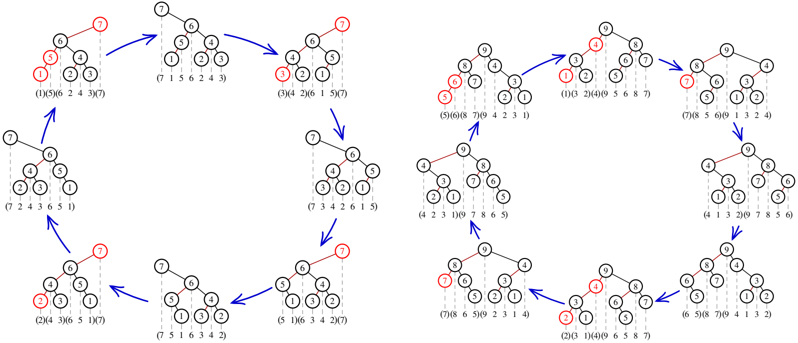
The heap of a permutation will turn out to be a *decreasing binary tree*, (labels decrease along any path from root).

The heap associated with $w = 314975826$ is shown below.



Orbits represented as heaps

Two orbits (one for $\mathfrak{S}_7$, one for $\mathfrak{S}_9$) of the Foatic reversal-inversion map $\overline{\varphi}$ with associated heaps, with fixed points marked in red. Each orbit has an average of one fixed point per permutation.



Theorem (La Croix-R.)

The orbits of the action of $\overline{\varphi}$ (or φ) on $\mathfrak{S}_n$, satisfy the following:

1. *The size of a φ-orbit $\mathcal{O}$ (equivalently $\overline{\varphi}$-orbit) is 2^h, where h is the number of edges in a maximal path from the root (to a leaf) for any $w \in \mathcal{O}$.*
2. *Let $\mathrm{Fix}\, w$ denote the number of fixed points, i.e., 1-cycles, of w. Then the statistic Fix is 1-mesic with respect to the action of $\overline{\varphi}$; (Equivalently, Rasc =#record-ascents is 1-mesic with respect action of φ.)*
3. *For fixed $i \neq j$ in $[n]$, let $\mathbb{1}_{i<j}(u)$ denote the indicator statistic of whether i occurs to the left of j in the one-line notation of u. Then $\mathbb{1}_{i<j}$ is $\frac{1}{2}$-mesic with respect to the action of φ.*
4. *Similarly for fixed $i \in [n]$, let $\mathbb{1}_{(i,n)}$ denote the indicator statistic of whether i and n lie in the same cycle of w. Then $\mathbb{1}_{(i,n)}$ is $\frac{1}{2}$-mesic with respect to the action of $\overline{\varphi}$.*

Reveral-Inversion Recursion (Key Lemma)

All the results listed above follow without difficulty from the following key lemma.

Lemma

Let $w \in \mathfrak{S}_n$ have the form AnB (in one-line notation), where A and B are (possibly empty) partial permutations of n. Then the action of φ satisfies $\varphi(AnB) = \varphi(B)nA$. Thus, $H(\varphi(AnB))$ is the heap interchanging the left and right subtrees at v, leaving the former unchanged and applying φ recursively to the latter. In particular, the action of φ preserves the underlying unlabeled graph of the corresponding heaps.

The final slide of this talk (before the references)

We're happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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Thanks very much for coming to this talk!

どうも有り難う御座いました。

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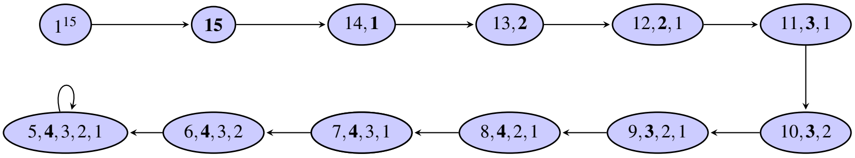
Bulgarian Solitaire
Bonus Slides
(because you asked!)

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define $\text{б}(\lambda)$ as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

- First surfaced as a puzzle in Russia around 1980, and a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardner column; see survey of Brian Hopkins [**?**].
- Initial puzzle: starting from any of 176 partitions of 15, one ends at $(5, 4, 3, 2, 1)$.



Bulgarian solitaire: homomesies

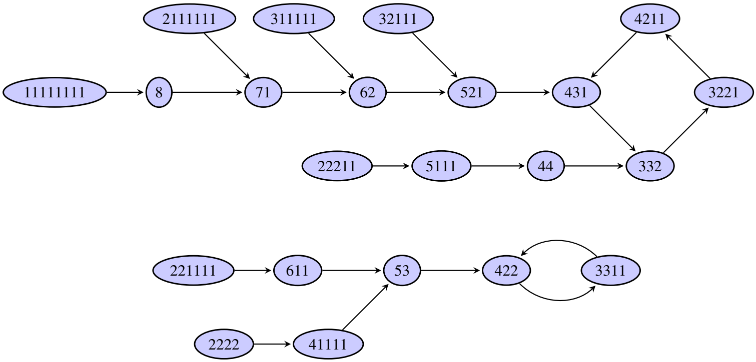
E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).



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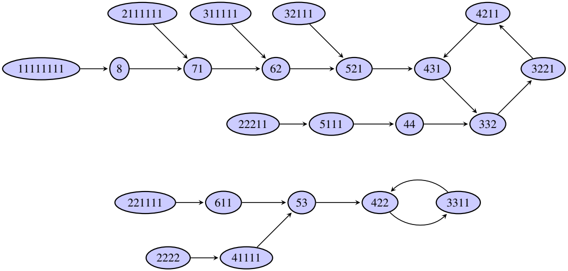

Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5, 3)$, the average value of φ is $(4 + 3)/2 = 7/2$; while for $\lambda = (6, 2)$, the average value of φ is $(3 + 4 + 4 + 3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \le j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic б; similarly for "size of (kth) largest part".

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

So an equivalent way of stating our main definition in this case is, φ is homomesic with respect to (S, τ) iff the average of φ over each periodic τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$.

In the rest of this talk, we'll restrict attention to maps τ that are invertible on S, so transience is not an issue.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
(Call f c-mesic for short.)