Rowmotion: Classical & Birational

Tom Roby (University of Connecticut)
*Describing joint research with Darij Grinberg*

Stanley@70
MIT
Cambridge, MA USA

26 June 2014

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html
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If $P$ is a finite poset, (classical) rowmotion (aka the Fon-der-Flaass map aka Panyushev complementation) is a certain permutation of the set of order ideals (or equivariantly the antichains) of $P$. Various surprising properties of rowmotion have been exhibited in work of Brouwer/Schrijver, Cameron/Fon der Flaass, Panyushev, Armstrong/Stump/Thomas, Striker/Williams, and Propp/R. For example, its order is $p + q$ when $P$ is the product $[p] \times [q]$ of two chains, and several natural statistics have the same average over every rowmotion orbit (i.e., are "homomesic"). Recent work of Einstein/Propp generalizes rowmotion twice: first to the piecewise-linear setting of a poset’s "order polytope”, defined by Stanley in 1986, and then via detropicalization to the birational setting.

In these latter settings, generalized rowmotion no longer has finite order in the general case. Results of Grinberg and the speaker, however, show that it still has order $p + q$ on the product $[p] \times [q]$ of two chains, and still has finite order for a wide class of forest-like ("skeletal") graded posets and for some triangle-shaped posets. Our methods of proof are partly based on those used by Volkov to resolve the type $AA$ (rectangular) Zamolodchikov Periodicity Conjecture.
Acknowledgments

This seminar talk discusses recent work with Darij Grinberg, including ideas and results from Arkady Berenstein, David Einstein, Jim Propp, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Darij Grinberg & Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.
Overview: What to expect in this talk

- Way cool map on $J(P)$ called “rowmotion”, and some of unexpected properties of its order and orbits;
- Great animations by Mike LaCroix to illustrate the above;
- Generalizations of the above to (1) the order polytope of $P$ and (2) arbitrary $\mathbb{K}$-labeling of the nodes of $P$.
- Theorems about the order of these maps for certain classes of posets;
- Allusions to other work that there won’t be time to discuss;
- Several jokes; and
- Appearances of the name “Stanley” in certain key places.

Please interrupt with questions!
Classical rowmotion is the rowmotion studied by Striker-Williams (arXiv:1108.1172). It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or “nonnesting partitions”, with relations to Lie theory).
Let \( P \) be a finite poset. **Classical rowmotion** is the map \( r : J(P) \to J(P) \) which sends every order ideal \( S \) to the order ideal obtained as follows:

Let \( M \) be the set of minimal elements of the complement \( P \setminus S \).

Then, \( r(S) \) shall be the order ideal generated by these elements (i.e., the set of all \( w \in P \) such that there exists an \( m \in M \) such that \( w \leq m \)).

**Example:**
Let \( S \) be the following order ideal (● = inside order ideal):

```

``
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ which sends every order ideal $S$ to the order ideal obtained as follows: Let $M$ be the set of minimal elements of the complement $P \setminus S$. Then, $r(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

**Example:**
Mark $M$ (= minimal elements of complement) blue.
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ which sends every order ideal $S$ to the order ideal obtained as follows: Let $M$ be the set of minimal elements of the complement $P \setminus S$. Then, $r(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

**Example:**
Forget about the old order ideal:
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ which sends every order ideal $S$ to the order ideal obtained as follows: Let $M$ be the set of minimal elements of the complement $P \setminus S$. Then, $r(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

**Example:**
$r(S)$ is the order ideal generated by $M$ (“everything below $M$”):
Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.
Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large. However, for some types of $P$, the order can be explicitly computed or bounded from above. See Striker-Williams for an exposition of known results.

- If $P$ is a $p \times q$-rectangle:

  \[
  (2,3) \quad (2,2) \quad (1,3) \\
  (2,1) \quad (1,2) \\
  (1,1)
  \]

  (shown here for $p = 2$ and $q = 3$), then $\text{ord}(r) = p + q$. 
Example:
Let $S$ be the order ideal of the $2 \times 3$-rectangle given by:
Example:
\( r(S) \) is

```
\[
\begin{array}{cccc}
(2,3) & & (2,2) & (1,3) \\
& (2,1) & \text{ } & (1,2) \\
(1,1) & & &
\end{array}
\]
```
Example:
$r^2(S)$ is
Example:
$r^3(S)$ is
Example:
\( r^4(S) \) is

\[
\begin{array}{c}
(2, 3) \\
(2, 2) \\
(2, 1) \\
(1, 1) \\
(1, 2) \\
(1, 3) \\
\end{array}
\]
Example:

$r^5(S)$ is

which is precisely the $S$ we started with.

$\text{ord}(r) = p + q = 2 + 3 = 5.$
Next we’ll take a look at an interesting property of the *orbits* of rowmotion acting on a product of two chains. For the animations which follow, please temporarily take the point of view that: *the elements of the poset are the squares below* So we would map:

\[ \text{Area} = 8 \quad \xrightarrow{r} \quad \text{Area} = 10 \]

Area = 0
Area = 1
Area = 3
Area = 5
Area = 7
Area = 8

\[
(0 + 1 + 3 + 5 + 7 + 8) / 6 = 4
\]

Rowmotion

\[
\frac{(2+4+6+6+4+2)}{6} = 4
\]

Rowmotion

Area = 3

Area = 5

Area = 4

(3+5+4+3+5+4) / 6 = 4
What is... a Homomesy?
What is ... a Homomesy?
**DEF:** Given an (invertible) action $\tau$ on a finite set of objects $S$, call a statistic $\varphi : S \to \mathbb{C}$ **homomesic** [Gk., “same middle”] with respect to $(S, \tau)$ iff the average of $\varphi$ over each $\tau$-orbit $O$ is the same for all $O$, i.e., $\frac{1}{\#O} \sum_{s \in O} \varphi(s)$ does not depend on the choice of $O$.

We call the triple $(S, \tau, \varphi)$ a **homomesy**.
**DEF:** Given an (invertible) action $\tau$ on a finite set of objects $S$, call a statistic $\varphi : S \to \mathbb{C}$ **homomesic** [Gk., “same middle”] with respect to $(S, \tau)$ iff the average of $\varphi$ over each $\tau$-orbit $O$ is the same for all $O$, i.e.,

$$\frac{1}{\#O} \sum_{s \in O} \varphi(s)$$

does not depend on the choice of $O$.

We call the triple $(S, \tau, \varphi)$ a **homomesy**.

For example, the statistic $\#I$ (cardinality of the ideal) is **homomesic** with respect to rowmotion, $r$, acting on $J([4] \times [2])$. 
Let $\mathcal{O}$ be an arbitrary $r$-orbit in $J([p] \times [q])$. Then

$$\frac{1}{\# \mathcal{O}} \sum_{I \in \mathcal{O}} \# I = \frac{pq}{2},$$

i.e., the cardinality statistic is homomesic with respect to the action of rowmotion on order ideals.
Classical rowmotion: homomesies

**Theorem (Propp, R.)**

Let $\mathcal{O}$ be an arbitrary $r$-orbit in $J([p] \times [q])$. Then

$$\frac{1}{\# \mathcal{O}} \sum_{I \in \mathcal{O}} \# I = \frac{pq}{2},$$

i.e., the cardinality statistic is homomesic with respect to the action of rowmotion on order ideals.

It turns out that to show a similar statement for rowmotion acting on the antichains of $P$, the right tool is an equivariant bijection from Stanley’s “Promotion and Evacuation” paper, as rephrased by Hugh Thomas.

See Jim Propp’s talk next week at FPSAC’14 for more information about homomesies in various settings.
Classical rowmotion: homomesies

Theorem (Propp, R.)

Let \( O \) be an arbitrary \( r \)-orbit in \( J([p] \times [q]) \). Then

\[
\frac{1}{\#O} \sum_{I \in O} \#I = \frac{pq}{2},
\]

i.e., the cardinality statistic is homomesic with respect to the action of rowmotion on order ideals.

It turns out that to show a similar statement for rowmotion acting on the antichains of \( P \), the right tool is an equivariant bijection from Stanley’s “Promotion and Evacuation” paper, as rephrased by Hugh Thomas.

See Jim Propp’s talk next week at FPSAC’14 for more information about homomesies in various settings.
There is an alternative definition of classical rowmotion, which splits it into many small operations, each an involution.

- Define $t_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - $S$ otherwise.
There is an alternative definition of classical rowmotion, which splits it into many small operations, each an involution.

- Define $t_v (S)$ as:
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(“Try to add or remove $v$ from $S$, as long as the result remains within $J(P)$; otherwise, leave $S$ fixed.”)
There is an alternative definition of classical rowmotion, which splits it into many small operations, each an involution.

1. Define $t_v(S)$ as:
   - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
   - $S$ otherwise.
   
   ("Try to add or remove $v$ from $S$, as long as the result remains within $J(P)$; otherwise, leave $S$ fixed.")

2. More formally, if $P$ is a poset and $v \in P$, then the $v$-toggle is the map $t_v : J(P) \rightarrow J(P)$ which takes every order ideal $S$ to:

   - $S \cup \{v\}$, if $v$ is not in $S$ but all elements of $P$ covered by $v$ are in $S$ already;
   - $S \setminus \{v\}$, if $v$ is in $S$ but none of the elements of $P$ covering $v$ is in $S$;
   - $S$ otherwise.
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

- Cameron and Fon-der-Flaass showed that

\[
r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

**Example:**
Start with this order ideal \(S\):

```
(2, 2)    (1, 2)
/\        /\        \\
(2, 1)    (1, 2)    \\
/          /        \\
(1, 1)    (1, 1)
```

Classical rowmotion: the toggling definition

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- Cameron and Fon-der-Flaass showed that

\[
\mathbf{r} = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

**Example:**
First apply \(t_{(2,2)}\), which changes nothing:

```
(2, 2)  
/       
(2, 1)  (1, 2)
/       /     
(1, 1) (1, 2)
```
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

- Cameron and Fon-der-Flaass showed that

\[
  r = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}.
\]

**Example:**
Then apply \(t_{(1,2)}\), which adds \((1, 2)\) to the order ideal:

```
(2, 2)
   /   \
(2, 1)  (1, 2)
        /   \
(1, 1)
```
Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[ r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}. \]

**Example:**
Then apply \(t_{(2,1)}\), which removes \((2, 1)\) from the order ideal:
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).
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\[ r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}. \]

**Example:**
Finally apply \(t_{(1,1)}\), which changes nothing:

\[
\begin{array}{c}
(2, 2) \\
(2, 1) & & (1, 2) \\
(1, 1) & &
\end{array}
\]
Let \((v_1, v_2, ..., v_n)\) be a linear extension of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[ r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}. \]

Example:
So this is \(r(S)\):

```
(2, 2)
(2, 1)       (1, 2)
(1, 1)
```
We can generalize this idea of composition of toggles to define a **continuous piecewise-linear (CPL)** version of rowmotion on an infinite set of functions on a poset.
We can generalize this idea of composition of toggles to define a **continuous piecewise-linear (CPL)** version of rowmotion on an infinite set of functions on a poset.

Let $P$ be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined. The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$. 

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot >x} f(z) + \max_{w \cdot <x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot >x$ means $z$ covers $x$ and $w \cdot <x$ means $x$ covers $w$.

Note that the interval $[\min_{z \cdot >x} f(z), \max_{w \cdot <x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot >x} f(z) + \max_{w \cdot <x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.
Example of flipping at a node

\[
\begin{align*}
\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) &= .7 + .2 = .9 \\
 f(x) + f'(x) &= .4 + .5 = .9
\end{align*}
\]
Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

\[
\begin{array}{c}
\sigma_N \\
\downarrow \\
.8 \\
\quad .4 \\
\quad .1 \\
\sigma_W \\
\downarrow \\
.6 \\
\quad .4 \\
\quad .1 \\
\sigma_S \\
\downarrow \\
.6 \\
\quad .3 \\
\quad .4 \\
\sigma_E \\
\downarrow \\
.6 \\
\quad .3 \\
\quad .1 \\
\end{array}
\]

(Here we successively flip values at the North, West, East, and South.)
Let $\mathbb{K}$ be a field.

A $\mathbb{K}$-labelling of $P$ will mean a function $\hat{P} \to \mathbb{K}$.

The values of such a function will be called the labels of the labelling.

We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\hat{P}$.

**Example:** This is a $\mathbb{Q}$-labelling of the $2 \times 2$-rectangle:

```
    14
     |
    10
   /   \     \   /   \
-2     7   1/3
  /     \    |
 1/3    12
```
For any \( v \in P \), define the **birational** \( v \)-**toggle** as the rational map \( T_v : \mathbb{K}^\hat{P} \rightarrow \mathbb{K}^\hat{P} \) defined by

\[
(T_v f)(w) = \begin{cases} 
  f(w), & \text{if } w \neq v; \\
  \frac{1}{f(v)} \cdot \frac{\sum_{u \in \hat{P}; \ u \lessdot v} f(u)}{\sum_{u \in \hat{P}; \ u \gtrdot v} \frac{1}{f(u)}}, & \text{if } w = v 
\end{cases}
\]

for all \( w \in \hat{P} \).

That is,
- **invert** the label at \( v \),
- **multiply** by the **sum** of the labels at vertices **covered by** \( v \),
- **multiply** by the **harmonic sum** of the labels at vertices **covering** \( v \).
For any $v \in P$, define the **birational $v$-toggle** as the rational map $T_v : \mathbb{K}\hat{P} \rightarrow \mathbb{K}\hat{P}$ defined by

$$
(T_v f)(w) = \begin{cases} 
  f(w), & \text{if } w \neq v; \\
  \frac{1}{f(v)} \cdot \frac{\sum_{u \in \hat{P}; u \preceq v} f(u)}{\sum_{u \in \hat{P}; u \succeq v} \frac{1}{f(u)}}, & \text{if } w = v
\end{cases}
$$

for all $w \in \hat{P}$.

Notice that this is a **local change** to the label at $v$; all other labels stay the same.

We have $T_v^2 = \text{id}$ (on the range of $T_v$), and $T_v$ is a birational map.
We define **birational rowmotion** as the rational map

\[ R := T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_n} : \mathbb{K}^\hat{P} \rightarrow \mathbb{K}^\hat{P}, \]

where \((v_1, v_2, ..., v_n)\) is a linear extension of \(P\).

This is indeed independent on the linear extension, because:
We define **birational rowmotion** as the rational map

\[ R := T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_n} : \mathbb{K}^\hat{P} \rightarrow \mathbb{K}^\hat{P}, \]

where \((v_1, v_2, ..., v_n)\) is a linear extension of \(P\).

This is indeed independent on the linear extension, because:

- \(T_v\) and \(T_w\) commute whenever \(v\) and \(w\) are incomparable (even when one doesn’t cover another other);
- we can get from any linear extension to any other by switching incomparable adjacent elements.
We define **birational rowmotion** as the rational map

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This is indeed independent on the linear extension, because:

- \(T_v\) and \(T_w\) commute whenever \(v\) and \(w\) are incomparable (even when one doesn’t cover another other);
- we can get from any linear extension to any other by switching incomparable adjacent elements.

For more information about the lifting of rowmotion from classical to CPL to birational, see, Einstein-Propp [EiPr13], where \(R\) is denoted \(\rho_B\).
Example:
Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

<table>
<thead>
<tr>
<th>poset</th>
<th>labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( b )</td>
</tr>
<tr>
<td></td>
<td>( z )</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>( x )</td>
</tr>
<tr>
<td></td>
<td>( w )</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>( y )</td>
</tr>
<tr>
<td></td>
<td>( a )</td>
</tr>
<tr>
<td>(1, 2)</td>
<td></td>
</tr>
<tr>
<td>(1, 1)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

We have \( R = T(1, 1) \circ T(1, 2) \circ T(2, 1) \circ T(2, 2) \) (using the linear extension \((1, 1), (1, 2), (2, 1), (2, 2)\)). That is, toggle in the order “top, left, right, bottom”.
Example:
Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

<table>
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<tr>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( w )</td>
</tr>
<tr>
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</tbody>
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That is, toggle in the order “top, left, right, bottom”.
Example:
Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>original labelling $f$</th>
<th>labelling $T_{(2,2)}f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>$z$</td>
<td>$\frac{b(x+y)}{z}$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$y$</td>
<td>$y$</td>
</tr>
<tr>
<td>$w$</td>
<td>$w$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
</tr>
</tbody>
</table>

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. 
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<table>
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<th>Labelling $T_{(2,1)} T_{(2,2)} f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>$T_{(2,1)} T_{(2,2)} f$</td>
</tr>
</tbody>
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We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. 
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Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>$z$</td>
<td>$b(x+y)$</td>
</tr>
<tr>
<td>$x$</td>
<td>$bw(x+y)$</td>
</tr>
<tr>
<td>$y$</td>
<td></td>
</tr>
<tr>
<td>$w$</td>
<td>$bw(x+y)$</td>
</tr>
<tr>
<td>$a$</td>
<td></td>
</tr>
</tbody>
</table>

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. 
Example:
Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>original labelling $f$</th>
<th>labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>$z$</td>
<td>$b(x+y)$</td>
</tr>
<tr>
<td>$x$</td>
<td>$bw(x+y)$</td>
</tr>
<tr>
<td>$y$</td>
<td>$bw(x+y)$</td>
</tr>
<tr>
<td>$a$</td>
<td>$ab$</td>
</tr>
</tbody>
</table>

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. 
Let \( \text{ord} \phi \) denote the order of a map or rational map \( \phi \). This is the smallest positive integer \( k \) such that \( \phi^k = \text{id} \) (on the range of \( \phi^k \)), or \( \infty \) if no such \( k \) exists.

The above shows that \( \text{ord}(r) \mid \text{ord}(R) \) for every finite poset \( P \).

Do we have equality?
Let \( \text{ord} \phi \) denote the order of a map or rational map \( \phi \). This is the smallest positive integer \( k \) such that \( \phi^k = \text{id} \) (on the range of \( \phi^k \)), or \( \infty \) if no such \( k \) exists.

The above shows that \( \text{ord}(r) | \text{ord}(R) \) for every finite poset \( P \).

Do we have equality? 

**No!** Here are two posets with \( \text{ord}(R) = \infty \):

\[
\begin{array}{c}
\circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\quad
\begin{array}{c}
\circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\]
Let \( \text{ord} \phi \) denote the order of a map or rational map \( \phi \). This is the smallest positive integer \( k \) such that \( \phi^k = \text{id} \) (on the range of \( \phi^k \)), or \( \infty \) if no such \( k \) exists.

The above shows that \( \text{ord}(r) \mid \text{ord}(R) \) for every finite poset \( P \).

Do we have equality?

**No!** Here are two posets with \( \text{ord}(R) = \infty \):

Nevertheless, equality holds for many special types of \( P \).
Example:
Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$$R^0 f =$$

\[\begin{array}{c}
\begin{array}{c}
 b \\
 \downarrow \\
 z \\
 \downarrow \\
 x \\
 \downarrow \\
 w \\
 \downarrow \\
 a \\
\end{array}
\end{array}\]
Example:
Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$R^1f =$

```

```

```
Example:
Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$$R^2 f =$$

\[
\begin{array}{c}
\text{b} \\
\mid \\
\text{bw}(x+y) \\
\text{xy} \\
\text{y} \quad \text{x} \\
\text{ab} \quad \text{ab} \\
\text{az} \\
\text{x+y} \\
\text{a}
\end{array}
\]
Example:
Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$$R^3 f =$$

```
  b
 /|
 a b
 |  w
 a y z
 /|
 a y z
 w(x+y)
 /|
 a x z
 w(x+y)
 /|
 a x z
 w(x+y)
 /|
 x y
 w(x+y)
 /|
 a y
 w(x+y)
 /|
 a
```
Example:
Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$R^4 f =$

```
  b
 /\ \
/   \
/     \
/       \
/         \
/           \
/             \
/               \
/                 \
/                   \
/                     \
/                       \
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/                                                                                                                                            
```
**Example:**

Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$$R^4 f =$$

$$\begin{array}{c}
  b \\
  \downarrow \\
  z \\
  \downarrow \\
  x \\
  \downarrow \\
  w \\
  \downarrow \\
  a \\
\end{array}$$

So we are back where we started.

$$\text{ord}(R) = 4.$$
Theorem. Assume that $n \in \mathbb{N}$, and $P$ is a poset which is a forest (made into a poset using the “descendant” relation) having all leaves on the same level $n$ (i.e., each maximal chain of $P$ has $n$ vertices). Then,

$$\text{ord}(R) = \text{ord}(r) \mid \text{lcm}(1, 2, \ldots, n + 1).$$

Example:
This poset

![Diagram](attachment:poset.png)

has $\text{ord}(R) = \text{ord}(r) \mid \text{lcm}(1, 2, 3, 4) = 12.$
Theorem (periodicity): If $P$ is the $p \times q$-rectangle (i.e., the poset $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$ with coordinatewise order), then

$$\text{ord} (R) = p + q.$$ 

Example: For the $2 \times 2$-rectangle, this claims $\text{ord} (R) = 2 + 2 = 4$, which we have already seen.
Theorem (periodicity): If $P$ is the $p \times q$-rectangle (i.e., the poset $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ with coordinatewise order), then

$$\text{ord } (R) = p + q.$$ 

Example: For the $2 \times 2$-rectangle, this claims $\text{ord } (R) = 2 + 2 = 4$, which we have already seen.

Theorem (reciprocity): If $P$ is the $p \times q$-rectangle, and $(i, k) \in P$ and $f \in \mathbb{K}\hat{P}$, then

$$f \left( (p + 1 - i, q + 1 - k) \right) = \frac{f(0)f(1)}{(R^{i+k-1}f)((i, k))}.$$ 

These were conjectured (independently) by James Propp and R.
Example: Here is the generic $R$-orbit on the $2 \times 2$-rectangle again:
**Example:** Here is the generic $R$-orbit on the $2 \times 2$-rectangle again:
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We reparametrize our assignments $f : \hat{P} \to \mathbb{K}$ through $p \times (p + q)$-matrices in such a way that birational rowmotion corresponds to “cycling” the columns of the matrix.

This uses a 3-term Plücker relation.

Lots of technicalities to be managed, particularly around birational maps not necessarily being defined everywhere.
Birational rowmotion: the $\Delta$-triangle case

- **Theorem (periodicity):** If $P$ is the triangle $\Delta(p) = \{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i + k > p + 1\}$ with $p > 2$, then

$$\text{ord} (R) = 2p.$$

**Example:** The triangle $\Delta(4)$:

```
    o
   / \  /
  o   o o
 /\  /\ /
 o o o o
```
Theorem (periodicity): If $P$ is the triangle
$$\Delta(p) = \{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i + k > p + 1\}$$
with $p > 2$, then
$$\text{ord}(R) = 2p.$$  

Example: The triangle $\Delta(4)$:

![Diagram of a triangle with labeled vertices]

Theorem (reciprocity): $R^p$ reflects any $\mathbb{K}$-labelling across the vertical axis.

These are precisely the same results as for classical rowmotion.
Theorem (periodicity): If \( P \) is the triangle
\[
\Delta(p) = \{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i + k > p + 1\}
\]
with \( p > 2 \), then
\[
\text{ord} (R) = 2p.
\]

Example: The triangle \( \Delta(4) \):

![Diagram of \( \Delta(4) \)]

Theorem (reciprocity): \( R^p \) reflects any \( \mathbb{K} \)-labelling across the vertical axis.

These are precisely the same results as for classical rowmotion.

The proofs use a “folding”-style argument to reduce this to the rectangle case.
Birational rowmotion: the △-triangle case

- **Theorem (periodicity):** If $P$ is the triangle 
  \[ \{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i \leq k\}, \] 
  then 
  \[ \text{ord} (R) = 2p. \]

**Example:** For $p = 4$, this $P$ has the form:

\[\text{Diagram of a triangle with specific structure.}\]
Birational rowmotion: the ▷-triangle case

**Theorem (periodicity):** If $P$ is the triangle 
\[ \{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i \leq k\} \], then
\[ \text{ord} (R) = 2p. \]

**Example:** For $p = 4$, this $P$ has the form:

```
          .
         /\   .
        /  \  
       /\   \  
      /  \  /   
     /\   /     
    /  \ /\     
   /\  /  \     
  /  \ /\  \     
 /\  /  \ /\  \  
/  \ /\  /  \ /\ 
```

- Again this is reduced to the rectangle case.
**Conjecture (periodicity):** If $P$ is the triangle
\[ \{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i \leq k; \ i + k > p + 1\}, \]
then
\[ \text{ord}(R) = p. \]

**Example:** For $p = 4$, this $P$ has the form:

![Diagram](attachment:image.png)
Conjecture (periodicity): If $P$ is the triangle
\[\{(i, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\} \mid i \leq k; \ i + k > p + 1\}\],
then
\[\text{ord}(R) = p.\]

Example: For $p = 4$, this $P$ has the form:

```
  .
 /\
/  \\
```

We proved this for $p$ odd.

Note that for $p$ even, this is a type-B positive root poset. Armstrong-Stump-Thomas did this for classical rowmotion.
• **Conjecture (periodicity):** If $P$ is the trapezoid 
$$\{(i, k) \in \{1, 2,..., p\} \times \{1, 2,..., p\} \mid i \leq k; \ i + k > p + 1; \ k \geq s\}$$ 
for some $0 \leq s \leq p$, then

$$\text{ord } (R) = p.$$ 

**Example:** For $p = 6$ and $s = 5$, this $P$ has the form:

- This was observed by Nathan Williams and verified for $p \leq 7$.  
- Motivation comes from Williams’s “Cataland” philosophy.
References


I’m happy to talk about this further with anyone who’s interested.

Slides for this talk are available online (or will be soon) at

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