See-Saw Swap Solitaire $(S^4(4G)^2)$

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Slides available at:
http://www.math.uconn.edu/~troby/research.html
[or google "Tom Roby" for links from my website]

Who is Tom Roby?

- Mathematician and educator, currently at MIT on sabbatical from UConn;
- Interests in Combinatorics, Algebra, & Math Ed;
- Worked with programs for K-12 teachers, high-ability HS students, and ugrads who want help with math-intensive courses.
- Other interests include: folkdancing, Japanese culture & linguistics, Bulgarian singing, ...
- More at `http://www.math.uconn.edu/~troby`

Rules of the Game

Shuffle the cards 1 through 6 and deal them out in a row.

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EG: May we interchange 4 and 2 above?

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EG: May we interchange 4 and 2 above? NO.

EG: May we interchange 3 and 6 above?

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RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: $1, 2, 3, 4, 5, 6$.

EG: May we interchange 4 and 2 above? NO.

EG: May we interchange 3 and 6 above? YES, to get

6 1 4 3 5 2

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

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$$123456$$

See-Saw Swap Solitaire Examples

213456 124356 314652

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213456	124356	314652
243156	624351	514632
243651	654321	512634
643251	634521	542631
143256	631524	245631
123456	136524	265431
5 moves	436521	263451
	436125	213456
	136425	...
	156423	12 moves
	153426	
	123456	
	11 moves	

Natural Questions

Lots of questions naturally arise:

1. Are the number of moves listed above shortest possible?
2. Can the player always win or is it possible to get stuck?
3. What's the longest (# moves) an optimally played game can take?
4. What are good strategies or heuristics for winning?
5. Does the number of cards matter? Who thinks the game gets easier with five cards? With seven?

How hard is reversing?

One interesting case: Solve the Reverse Hand

7654321

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$$\underline{7}65\hat{4}32\underline{1}$$

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3 moves

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$$12\underline{5}\hat{4}\underline{3}67 \quad 124356$$

$$1234567$$

3 moves

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$\underline{7}65\hat{4}32\underline{1}$	$\underline{6}5\hat{4}32\underline{1}$
$1\underline{6}5\hat{4}3\underline{2}7$	$1\underline{5}\hat{4}3\underline{2}6$
$12\underline{5}\hat{4}\underline{3}67$	124356
1234567	Now what?
3 moves	

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$\underline{7}65\hat{4}32\underline{1}$ $\underline{6}5\hat{4}32\underline{1}$

$1\underline{6}5\hat{4}3\underline{2}7$ $1\underline{5}\hat{4}3\underline{2}6$

$12\underline{5}\hat{4}\underline{3}67$ 124356

1234567 Now what?

3 moves

So for n odd, reversing is quick, taking $\dfrac{n-1}{2}$ steps, but for n even it appears to be harder. Can one do better than $2+11$ for $n=6$?

Natural Questions Answered

Lots of questions arise:

1. Can the player always win or is it possible to get stuck? Yes if have at least **six** cards.
2. For an optimal player, what's the maximum number of moves to win? Don't know.
3. What are good strategies or heuristics for winning? Don't know.
4. Does the number of cards matter? Yes! (More positions for seven, but for five. . .)
5. (New) Are there interesting (more fun? better?) variations on this game?

A Bar Bet

For five cards, the game can only be solved from 24 of the 120 possible starting positions. This leads to the following bar bet:

- Demonstrate your skill at the game with six cards.
- Bet someone that they won't be able to do it.
- When they get stuck (quite likely), take pity on them and remove a card (say 6) to give them a "easier" one.
- There's an 80% chance that their current hand is unsolvable.
- CHALLENGE: Are there positions with 6 cards from which the removal of **any** card leaves an unsolvable position?

Math Research Connection

- S^4 comes out of more general work on such permutation games, which generalize "Knuth Relations".
- Variations on the game lead to interesting counting numbers: Fibonacci, Catalan, binomial coefficients
- Full details at `http://arxiv.org/abs/1111.3920`;
- Better yet, google "Tom Roby" for links from my website (including this talk)

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