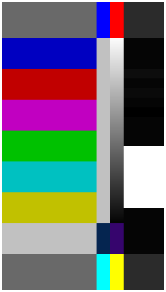
See-Saw Swap Solitaire and Other Games on Permutations

Tom (“sVen”) Roby (UConn)

Joint research with Steve Linton, James Propp, & Julian West

Canada/USA Mathcamp
Lewis & Clark College
Portland, OR USA



29 July 2014

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

See-Saw Swap Solitaire and Other Games on Permutations

Tom ("sVen") Roby (UConn)

Joint research with Steve Linton, James Propp, & Julian West

Canada/USA Mathcamp
Lewis & Clark College
Portland, OR USA

29 July 2014

Slides for this talk are available online (or will be soon) at

`http://www.math.uconn.edu/~troby/research.html`

Outline

- The intermediary game
- Right superior game
- Typical results
- The general framework
- Table of results
- Open Problems

Who is sVen?

- Mathematician and educator at UConn (University of Connecticut),
- specializing in Combinatorics, Algebra, & Math Ed;
- Worked with programs for K-12 teachers, high-ability HS students, and ugrads who want help with math-intensive courses.
- Other interests include: folkdancing, Japanese culture & linguistics, Bulgarian singing, . . .
- More at `http://www.math.uconn.edu/~troby`

See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: $1, 2, 3, 4, 5, 6$.

See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: $1, 2, 3, 4, 5, 6$.

See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: $1, 2, 3, 4, 5, 6$.

EG: May we interchange 4 and 2 above?

See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: $1, 2, 3, 4, 5, 6$.

EG: May we interchange 4 and 2 above? NO.

EG: May we interchange 3 and 6 above?

See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: $1, 2, 3, 4, 5, 6$.

EG: May we interchange 4 and 2 above? NO.

EG: May we interchange 3 and 6 above? YES, to get

6 1 4 3 5 2

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

$$243\underline{1}\hat{5}\underline{6}$$

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

$$243\underline{1}\hat{5}\underline{6}$$

$$\underline{2}\widehat{43}\underline{6}51$$

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

$$243\underline{1}\hat{5}\underline{6}$$

$$\underline{2}\widehat{43}\underline{6}51$$

$$\underline{6}4\hat{3}25\underline{1}$$

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

$$243\underline{1}\hat{5}\underline{6}$$

$$\underline{2}\widehat{43}\underline{6}51$$

$$\underline{6}4\hat{3}25\underline{1}$$

$$1\underline{4}\hat{3}\underline{2}56$$

A Short Game

$$2\underline{1}\hat{3}\underline{4}56$$

$$2431\underline{1}\hat{5}\underline{6}$$

$$\underline{2}\widehat{43}\underline{6}51$$

$$\underline{6}4\hat{3}25\underline{1}$$

$$1\underline{4}\hat{3}\underline{2}56$$

$$123456$$

See-Saw Swap Solitaire Examples

213456 124356 314652

See-Saw Swap Solitaire Examples

213456
243156
243651
643251
143256
123456
5 moves

124356
624351
654321
634521
631524
136524
436521
436125
136425
156423
153426
123456
11 moves

314652
514632
512634
542631
245631
265431
263451
213456
. . .

See-Saw Swap Solitaire Examples

213456
243156
243651
643251
143256
123456
5 moves

124356
624351
654321
634521
631524
136524
436521
436125
136425
156423
153426
123456
11 moves

314652
514632
512634
542631
245631
265431
263451
213456
. . .
12 moves

Natural Questions

Lots of questions naturally arise:

1. Are the number of moves listed above shortest possible?
2. Can the player always win or is it possible to get stuck?
3. What's the longest (# moves) an optimally played game can take?
4. What are good strategies or heuristics for winning?
5. Does the number of cards matter?
   Who thinks the game gets easier with five cards? With seven?

How hard is reversing?

One interesting case: Solve the Reverse Hand

7654321

How hard is reversing?

One interesting case: Solve the Reverse Hand

$$\underline{7}65\hat{4}32\underline{1}$$

$$1\underline{6}5\hat{4}3\underline{2}7$$

How hard is reversing?

One interesting case: Solve the Reverse Hand

$$\underline{7}65\hat{4}32\underline{1}$$

$$1\underline{6}5\hat{4}3\underline{2}7$$

$$12\underline{5}\hat{4}\underline{3}67$$

How hard is reversing?

One interesting case: Solve the Reverse Hand

$$\underline{7}65\hat{4}32\underline{1}$$

$$1\underline{6}5\hat{4}3\underline{2}7$$

$$12\underline{5}\hat{4}\underline{3}67$$

$$1234567$$

3 moves

How hard is reversing?

One interesting case: Solve the Reverse Hand

$$\underline{7}65\hat{4}32\underline{1}$$

$$1\underline{6}5\hat{4}3\underline{2}7$$

$$12\underline{5}\hat{4}\underline{3}67$$

$$1234567$$

3 moves

How hard is reversing?

One interesting case: Solve the Reverse Hand

654321

$\underline{7}65\hat{4}32\underline{1}$

$1\underline{6}5\hat{4}3\underline{2}7$

$12\underline{5}\hat{4}\underline{3}67$

1234567

3 moves

How hard is reversing?

One interesting case: Solve the Reverse Hand

$\underline{7}65\hat{4}32\underline{1}$ $\underline{6}5\hat{4}32\underline{1}$

$1\underline{6}5\hat{4}3\underline{2}7$

$12\underline{5}\hat{4}\underline{3}67$

1234567

3 moves

How hard is reversing?

One interesting case: Solve the Reverse Hand

$\underline{7}65\hat{4}32\underline{1}$ $\underline{6}5\hat{4}32\underline{1}$

$1\underline{6}5\hat{4}3\underline{2}7$ $1\underline{5}\hat{4}3\underline{2}6$

$12\underline{5}\hat{4}\underline{3}67$

1234567

3 moves

How hard is reversing?

One interesting case: Solve the Reverse Hand

$\underline{7}65\hat{4}32\underline{1}$ $\underline{6}5\hat{4}32\underline{1}$

$1\underline{6}5\hat{4}3\underline{2}7$ $1\underline{5}\hat{4}3\underline{2}6$

$12\underline{5}\hat{4}\underline{3}67$ 124356

1234567

3 moves

How hard is reversing?

One interesting case: Solve the Reverse Hand

$\underline{7}65\hat{4}32\underline{1}$	$\underline{6}5\hat{4}32\underline{1}$
$1\underline{6}5\hat{4}3\underline{2}7$	$1\underline{5}\hat{4}3\underline{2}6$
$12\underline{5}\hat{4}\underline{3}67$	124356
1234567	Now what?
3 moves	

How hard is reversing?

One interesting case: Solve the Reverse Hand

$\underline{7}65\hat{4}32\underline{1}$ $\underline{6}5\hat{4}32\underline{1}$

$1\underline{6}5\hat{4}3\underline{2}7$ $1\underline{5}\hat{4}3\underline{2}6$

$12\underline{5}\hat{4}\underline{3}67$ 124356

1234567 Now what?

3 moves

So for n odd, reversing is quick, taking $\frac{n-1}{2}$ steps, but for n even it appears to be harder. Can one do better than $2+11$ for $n=6$?

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as "words" with no repeated letters made from this alphabet.

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as "words" with no repeated letters made from this alphabet.

For $n = 3$, there are six permutations: 123, 132, 213, 231, 312, 321.

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as "words" with no repeated letters made from this alphabet.

For $n = 3$, there are six permutations: 123, 132, 213, 231, 312, 321.

How many are there for $n = 4$?

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as "words" with no repeated letters made from this alphabet.

For $n = 3$, there are six permutations: 123, 132, 213, 231, 312, 321.

How many are there for $n = 4$? 24

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as "words" with no repeated letters made from this alphabet.

For $n = 3$, there are six permutations: 123, 132, 213, 231, 312, 321.

How many are there for $n = 4$? 24

How many for $n = n$?

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as "words" with no repeated letters made from this alphabet.

For $n = 3$, there are six permutations: 123, 132, 213, 231, 312, 321.

How many are there for $n = 4$? 24

How many for $n = n$? n!

Let S_n denote the set of all permutations on n.

Permutations

A *permutation* (on n) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as “words” with no repeated letters made from this alphabet.

For $n = 3$, there are six permutations: 123, 132, 213, 231, 312, 321.

How many are there for $n = 4$? 24

How many for $n = n$? n!

Let S_n denote the set of all permutations on n.

One can also think of permutations as being “bijective functions from $\{1, 2, \ldots n\}$ to itself”, with a group structure given by composition of functions, but we don’t need that here.

See-Saw Swap Solitaire on Permutations

So another way of expressing our game is as follows:

View permutations in S_n as words: $a_1a_2 \cdots a_n$, e.g., $314652 \in S_6$, and allow moves of the following type:

If $a_i < a_j < a_k$ or $a_i > a_j > a_k$ for some $i < j < k$, then we may interchange (swap) a_i and a_k.

See-Saw Swap Solitaire Examples

213456
243156
243651
643251
143256
123456

124356
624351
654321
634521
631524
136524
436521
436125
136425
156423
153426
123456

314652
514632
512634
542631
245631
265431
263451
213456
. . .

Graphs & Equivalence Relations

We can think of this game as creating a *graph*, whose vertex set is S_n, and with edges between any two permutations connected by a legal move. We are interested in questions about the connected components of this graph.

From a slightly more high-falutin' perspective, the transitive closure of the relations defined by these moves is an equivalence relation on the set of permutations. We are interested in the sizes and character of these equivalence classes.

$n = 3, 4, 5, 6$

What happens with our game in small cases?

$n = 3, 4, 5, 6$

What happens with our game in small cases?

For $n = 3$ it's clear that the only legal move is $123 \leftrightarrow 321$, so there are 5 distinct equivalence classes under this relation.

$n = 3, 4, 5, 6$

What happens with our game in small cases?

For $n = 3$ it's clear that the only legal move is $123 \leftrightarrow 321$, so there are 5 distinct equivalence classes under this relation.

For $n = 4$ we get ten equivalence classes including $\{1234, 3214, 1432, 4321\}$.

$n = 3, 4, 5, 6$

What happens with our game in small cases?

For $n = 3$ it's clear that the only legal move is $123 \leftrightarrow 321$, so there are 5 distinct equivalence classes under this relation.

For $n = 4$ we get ten equivalence classes including $\{1234, 3214, 1432, 4321\}$.

For $n = 5$ we get 3 equivalence classes, and the class containing the identity contains 24 elements.

$n = 3, 4, 5, 6$

What happens with our game in small cases?

For $n = 3$ it's clear that the only legal move is $123 \leftrightarrow 321$, so there are 5 distinct equivalence classes under this relation.

For $n = 4$ we get ten equivalence classes including $\{1234, 3214, 1432, 4321\}$.

For $n = 5$ we get 3 equivalence classes, and the class containing the identity contains 24 elements.

For $n \geq 6$ we get a single equivalence class (the "fully mixed" case), which of course contains the identity.

A Bar Game

This leads to the following bar game:

- Demonstrate your skill at obtaining the identity from random permutations in S_6 using only the legal moves.
- Bet someone that they won't be able to do it.
- When they get stuck (quite likely), take pity on them and give them a "easier" permutation in S_5.
- There's an 80% chance that a randomly chosen $\sigma \in S_5$ is NOT legally obtainable by this set of moves.

Basic questions

This example illustrates the basic questions we will be interested in, not just for this game, but for ones with other sets of rules P:

- A Compute the number of equivalence classes $\#\mathrm{Classes}^{\star}(n, P)$ into which S_n is partitioned.
- B Compute the size of $\#\mathrm{Eq}^{\star}(\iota_n, P)$ of the equivalence class containing the identity, ι_n.
- C (More generally) characterise the set $\mathrm{Eq}^{\star}(\iota_n, P)$ of permutations equivalent to the identity.

Basic questions

This example illustrates the basic questions we will be interested in, not just for this game, but for ones with other sets of rules P:

- A Compute the number of equivalence classes $\#\mathrm{Classes}^{\star}(n, P)$ into which S_n is partitioned.
  5,10,3,1,1,1,. . .
- B Compute the size of $\#\mathrm{Eq}^{\star}(\iota_n, P)$ of the equivalence class containing the identity, ι_n.
  2,4,24,720,5040,40320,. . .
- C (More generally) characterise the set $\mathrm{Eq}^{\star}(\iota_n, P)$ of permutations equivalent to the identity.
  all permutations for $n \geq 6$. . . .

Basic questions

This example illustrates the basic questions we will be interested in, not just for this game, but for ones with other sets of rules P:

- A Compute the number of equivalence classes $\#\mathrm{Classes}^\star(n, P)$ into which S_n is partitioned.
  5,10,3,1,1,1,. . .
- B Compute the size of $\#\mathrm{Eq}^\star(\iota_n, P)$ of the equivalence class containing the identity, ι_n.
  2,4,24,720,5040,40320,. . .
- C (More generally) characterise the set $\mathrm{Eq}^\star(\iota_n, P)$ of permutations equivalent to the identity.
  all permutations for $n \geq 6$. . . .

So the relation given by $P = \{123 \leftrightarrow 321\}$ (with no adjacency constraints) is not a particularly interesting example from this standpoint.

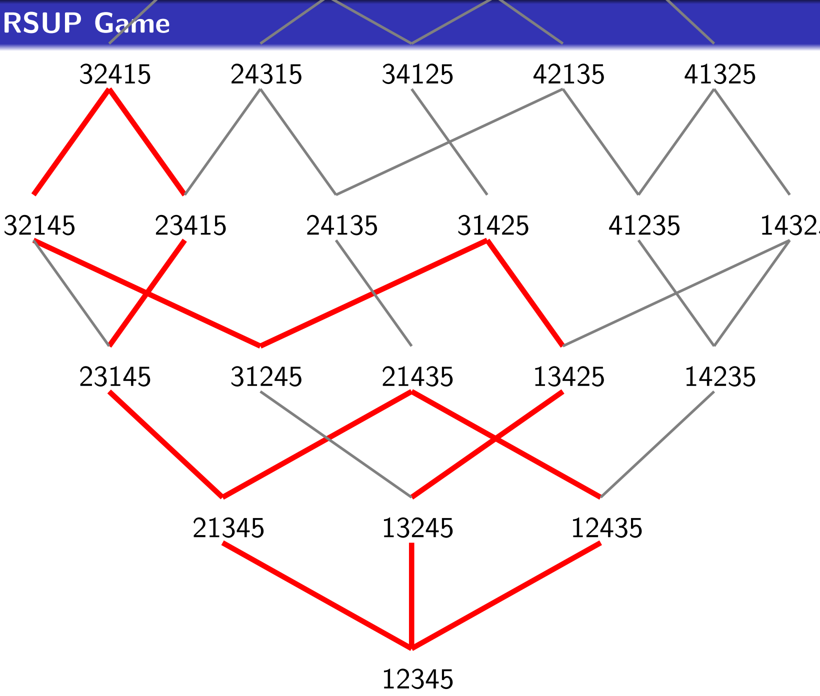
The Right Superior Game

The Right Superior Game

Say that two n-permutations are equivalent if they differ by an **adjacent** transposition $a_i a_{i+1} \leftrightarrow a_{i+1} a_i$, where both inequalities $a_i < a_{i+2}$ and $a_{i+1} < a_{i+2}$ hold.

$P_2^{||} = \{123 \leftrightarrow 213\}$.

Graph of RSUP Game



Size of the Class of ι_n

How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

Size of the Class of ι_n

How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

n	3	4	5	6	7	8	9	10
$\#\mathrm{Eq}^{\circ}(\iota_n, P)$	2	4	12	36	144	576	2880	14400

Do you see a pattern in these numbers?

Size of the Class of ι_n

How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

n	3	4	5	6	7	8	9	10
$\#\mathrm{Eq}^\circ(\iota_n, P)$	2	4	12	36	144	576	2880	14400

Do you see a pattern in these numbers?

Theorem 1. For the Right Superior Game, the number of n-permutations in the equivalence class of the identity is

$$\lfloor n/2 \rfloor ! \lceil n/2 \rceil !$$

Size of the Class of ι_n

How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

n	3	4	5	6	7	8	9	10
$\#\mathrm{Eq}^{\circ}(\iota_n, P)$	2	4	12	36	144	576	2880	14400

Do you see a pattern in these numbers?

Theorem 1. For the Right Superior Game, the number of n-permutations in the equivalence class of the identity is

$$\lfloor n/2 \rfloor ! \lceil n/2 \rceil !$$

i.e., for $n = 2r$, $\#\{\pi : \pi \leftrightarrow 123 \ldots n\} = r!r!$.

and for $n = 2r + 1$, $\#\{\pi : \pi \leftrightarrow 123 \ldots n\} = r!(r + 1)!$.

Proof that this is an upper bound.

Proof that this is an upper bound:
The largest element must be in the rightmost position. (Why?)

This implies that the second-largest element must be in one of the three rightmost positions. (Why?)

This implies that the third-largest element . . .

Now, placing the elements from largest to smallest, we have the following number of choices for each placement:

$1 \cdot 2 \cdot 3 \cdot \cdots \cdot \lceil n/2 \rceil \cdot \lfloor n/2 \rfloor \cdot \cdots \cdot 3 \cdot 2 \cdot 1$

Proof that the upper bound is attained

Proof of equality.
It remains to show that all permutations meeting these constraints are in fact reachable.

Imagine a target permutation meeting the constraints. That is, the first element (even case) or first two elements (odd case) are less than $\lceil n/2 \rceil + 1$, the next two elements are less than $\lceil n/2 \rceil + 2$, etc.

`Target:   kgfOdiPahNQcbTRjUmSVWXelYZ`

Step one. Advance all the “large” elements as far as they will go by rippling them forward:

Proof of Attainability

```
.............NOPQRSTUVWXYZ
............N.OPQRSTUVWXYZ
............NO.PQRSTUVWXYZ
............NOP.QRSTUVWXYZ
............NOPQ.RSTUVWXYZ
            :
............NOPQRSTUVWXY.Z
            :
            :
...........NOPQRSTUVWX.Y.Z
            :
            :
..........NOPQRSTUVW.X.Y.Z
            :
            :
            :
.N.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

Proof of Attainability 2

`Target:    kgfOdiPahNQcbTRjUmSVWXelYZ`

`.N.O.P.Q.R.S.T.U.V.W.X.Y.Z`

Now observe that the "small" elements can be permuted freely while leaving the "large" elements in place.

$$\texttt{fRjs} \mapsto \qquad\qquad \mapsto \texttt{jRfS}\,.$$

Proof of Attainability 2

`Target:    kgfOdiPahNQcbTRjUmSVWXelYZ`

`.N.O.P.Q.R.S.T.U.V.W.X.Y.Z`

Now observe that the “small” elements can be permuted freely while leaving the “large” elements in place.

$$\mathtt{fRjs} \mapsto \mathtt{fjRS} \mapsto \mathtt{jfRS} \mapsto \mathtt{jRfS}\,.$$

Proof of Attainability 2

Target: kgfOdiPahNQcbTRjUmSVWXelYZ

.N.O.P.Q.R.S.T.U.V.W.X.Y.Z

Now observe that the “small” elements can be permuted freely while leaving the “large” elements in place.

$$\mathtt{fRjs} \mapsto \mathtt{fjRS} \mapsto \mathtt{jfRS} \mapsto \mathtt{jRfS}\,.$$

Step two. Using this observation, move the correct element into the first position. (In the odd case, move the two correct elements into the first two positions.) Because the target permutation obeys the constraints, this element (or pair of elements) will be small compared with the fixed skeleton of large elements which is facilitating their movement.

kN.O.P.Q.R.S.T.U.V.W.X.Y.Z

Proof of Attainability 3

`Target:    kgfOdiPahNQcbTRjUmSVWXelYZ`

Continue to place elements two at a time:

```
kN.O.P.Q.R.S.T.U.V.W.X.Y.Z
kgfO.P.Q.R.S.T.U.V.W.X.Y.Z
kgfOdP.Q.R.S.T.U.V.W.X.Y.Z
kgfOdiPQ.R.S.T.U.V.W.X.Y.Z
kgfOdiPahR.S.T.U.V.W.X.Y.Z
            :
kgfOdiPahNQcbTRjUmSVWXelYZ
```

□

$n = 5$ Example 1

As one one might expect, this algorithm constructs some permutations efficiently, but not others.

Construction of the “furthest” permutation 32145:

```
1 2 3 4 5
1 3 2 4 5
1 3 4 2 5
3 1 4 2 5  <-- build the skeleton

3 1 2 4 5  <-- 3 is already at front, so advance 2
3 2 1 4 5  <-- we're there, stop!
```

$n = 5$ Example 2

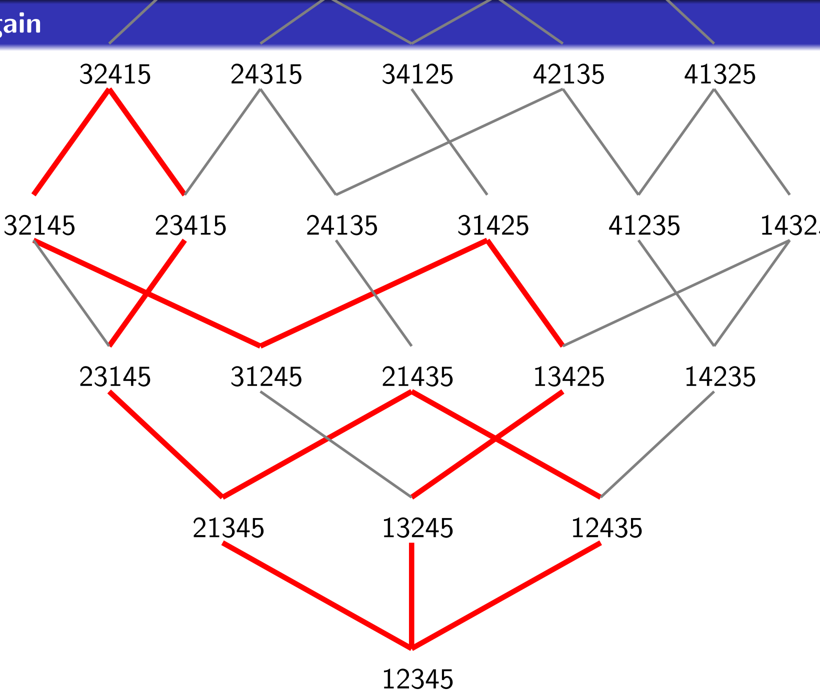
An example where the algorithm is inefficient, 12435:

```
1 2 3 4 5
1 3 2 4 5
1 3 4 2 5
3 1 4 2 5  <-- build the skeleton

1 3 4 2 5  <-- advance 1 (it just came from there!)

1 3 2 4 5  )
1 2 3 4 5  ) <-- 3-step procedure for advancing 2
1 2 4 3 5  )
```

Graph Again


32415
24315
34125
42135
41325
32145
23415
24135
31425
41235
1432
23145
31245
21435
13425
14235
21345
13245
12435
12345

Propp's Proposition

```
From: James Propp <jpropp@cs.uml.edu>
Date: Wed, 8 Jul 2009 17:07:07 -0400
Subject: two hundred and ten questions

:
:

I'd like to know the partition of n! determined by the transitive
closure of each of the following seven relations on S_n:

:

The two most interesting
numbers are probably the number of components and the size of
the component containing the permutation 1,2,3,...,n.
:

I should say that I want this information for _three_ distinct
interpretations of what "123 <--> 213" means:
(a) In the narrowest sense, it could mean that if pi(i+1) = pi(i)+1
    and pi(i+2) = pi(i)+2, then you can swap the values of pi(i) and
    pi(i+1).
(b) More broadly, it could mean that if pi(i) < pi(i+1) < pi(i+2),
    then you can swap the values of pi(i) and pi(i+1).
(c) More broadly still, it could mean that if pi(i) < pi(j) < pi(k)
    for i < j < k, then you can swap the values of pi(i) and pi(j).

Jim
```

General Framework

- Consider interchanges of subwords of "type" $\sigma_1 \leftrightarrow \sigma_2$, where $\sigma_i \in S_3$.
- As Jim described, this can be taken in three sense: (a) both indices and values must be adjacent; (b) entries must be in adjacent positions; (c) unrestricted in value or position
- Restricting entries to be adjacent values (but not necessarily positions) is equivalent to (b) by the map that sends $\pi \rightarrow \pi^{-1}$.
- In theory one could consider any of the $B(6) = 203$ partitions of S_3 as defining a relation (or three) of this type, although some of these will be trivially equivalent.
- To keep the problem within bounds, we currently consider only sets of relations of the form $\iota_3 \leftrightarrow \sigma$, where $\sigma \in S_3$. Equivalently, these are partitions of S_3 with a single nontrivial block (containing ι_3).

Number of Classes

How many equivalences classes for each relation?

$\#\mathrm{Classes}(n, P)$

Transpositions	general	indices adjacent	indices & values adjacent
123 ↔ 132 123 ↔ 213	[5, 14, 42, 132, 429] Catalan	[5, 16, 62, 284, 1507, 9104]	[5, 20, 102, 626, 4458, 36144]
123 ↔ 321	[5, 10, 3, 1, 1, 1] trivial	[5, 16, 60, 260, 1260, 67442]	[5, 20, 102, 626, 4458, 36144]
123 ↔ 132 ↔ 213	[4, 8, 16, 32, 64, 128] powers of 2	[4, 10, 26, 76, 232, 764] involutions/Chinese Monoid	[4, 17, 89, 556, 4011, 32843]
123 ↔ 132 ↔ 321 123 ↔ 213 ↔ 321	[4, 2, 1, 1, 1, 1] trivial	[4, 8, 14, 27, 68, 159, 496]	[4, 16, 84, 536, 3912, 32256]
123 ↔ 132 ↔ 213 ↔ 321	[3, 2, 1, 1, 1, 1] trivial	[3, 4, 5, 8, 11, 20, 29, 57]	[3, 13, 71, 470, 3497]

Size of class containing identity

Size of class containing identity: $\#\mathrm{Eq}^*(\iota, P)$

Transpositions	general	indices adjacent	indices & values adjacent
123 $\leftrightarrow$ 132	[2, 6, 24, 120, 720] (n-1)!	[2, 4, 12, 36, 144, 576, 2880] product of two factorials	[2, 3, 5, 8, 13, 21, 34, 55] Fibonacci numbers
123 $\leftrightarrow$ 213			
123 $\leftrightarrow$ 321	[2, 4, 24, 720] trivial	[2, 3, 6, 10, 20, 35, 70, 126] central binomial coefficients	[2, 3, 4, 6, 9, 13, 19, 28] A000930
123 $\leftrightarrow$ 132 $\leftrightarrow$ 213	[3, 13, 71, 461] connected A003319	[3, 7, 35, 135, 945, 5193] Chinese Monoid	[3, 4, 8, 12, 21, 33, 55, 88] A052952
123 $\leftrightarrow$ 132 $\leftrightarrow$ 321	[3, 23, 120, 720] trivial	[3, 9, 54, 285, 2160, 15825] proven for odd terms	[3, 5, 9, 17, 31, 57, 105, 193] tribonacci numbers A000213
123 $\leftrightarrow$ 213 $\leftrightarrow$ 321			
123 $\leftrightarrow$ 132 $\leftrightarrow$ 213 $\leftrightarrow$ 321	[3, 23, 120, 720] trivial	[4, 21, 116, 713, 5030] $n!-$central Catalan	[4, 6, 13, 23, 44, 80, 149, 273] tribonacci A000073 $-[n \text{ even}]$

This slide intentionally left blank

Further Work & Open Problems

We've just begun the more general study of these kinds of relations. Plenty of open problems remain, including:

1. Find formulae for the unknown data in the table.
2. Recall our initial "Intermediary in-between" rule, but in the adjacent context. We prove that

   $$\#\mathrm{Eq}^{||}(\iota_n, \{\{123, 321\}\}) = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$$

   in a fairly indirect way. Is there a simple combinatorial proof?
3. Understand the structure of the graphs one gets on these relations. Are the (like Bruhat order in the unconstrained case) posets?
4. Is there a useful length (distance from the identity) function?

Further Work & Open Problems 2

5. Answer more generally what the sizes of all the equivalence classes are, or whether there's a simple way to characterize them (as insertion tableaux characterizes all permutations which are Knuth equivalent).
6. Consider more general relations, defined by partitioning S_3 in different ways (more general block structures or connecting non-transpositions. Or even using relations within S_4?
7. Pierrot, Rossin, & West (FPSAC 2011) handle the other case of including non-transpositions within a unique non-singleton block containing ι_3 of a partition of S_3 (e.g., $\{123, 231\}$).

Further Work & Open Problems 3

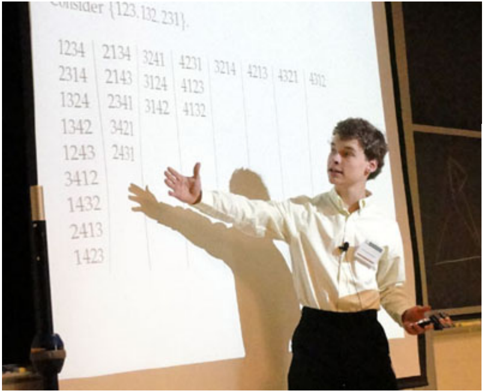
8. Pierrot, Rossin, & West (FPSAC 2011) handle the other case of including non-transpositions within a unique non-singleton block containing ι_3 of a partition of S_3 (e.g., $\{123, 231\}$).
9. Kuszmaul & Zhou consider the case of moves of adjacent elements generated by cyclic shifts, e.g., for $k = 5$ allowing replacements within

   $$\{12345, 23451, 34512, 45123, 51234\},$$

   and characterize the non-singleton classes induced in S_n.
10. In recent work Kuszmaul [3] finds answers for many of the "doubly-adjacent" cases for which we failed to find formulae. He also provides several interesting generalizations.
11. Consider more general relations, defined by partitioning S_3 in different ways (more general block structures or connecting non-transpositions. Or even using relations within S_4?

William Kuszmaul

Kuszmaul started working in this area as a high-school student via the PRIMES program at MIT, working with Darij Grinberg. He's gone on to do other prize-winner work, e.g., 3rd place at Intel 2014 (for a different project).



DAVIDSON FELLOWS - 2013 FELLOW WILLIAM KUSZMAUL

William Kuszmaul

$10,000 Scholarship Recipient

Age: 17
Lexington, MA
Category: Mathematics
Project Title: "Equivalence Classes of Permutations Created Under Pattern-Re
Relations"

Natural Questions Answered

1. Can the player always win or is it possible to get stuck? Yes if have at least **six** cards.
2. For an optimal player, what's the maximum number of moves to win? Don't know.
3. What are good strategies or heuristics for winning? Don't know.
4. Does the number of cards matter? Yes! (More positions for seven, but for five. . .)
5. Are there interesting (more fun? better?) variations on this game?

References 1

- J. Cassaigne, M. Espie, D. Krob, J.-C. Novelli, & F. Hivert, *The Chinese Monoid*, Int'l. J. Algebra and Comp., **11** #3 (2001), 301–334.
- William Kuszmaul, *Counting Permutations Modulo Pattern-Replacement Equivalences for Three-Letter Patterns.*, *Electronic Journal of Combinatorics*, 20(4) (2013), #P10.
- William Kuszmaul & Ziling Zhou, *Equivalence Classes in S_n for Three Families of Pattern-Replacement Relations.* MIT PRIMES, 2013. `http://web.mit.edu/primes/materials/2012/Kuszmaul-Zhou.pdf`.
- William Kuszmaul, *New Results on Doubly Adjacent Pattern-replacement Equivalences*, `arXiv:1402.3881v2`.
- A. Pierrot, D. Rossin, J. West, *Adjacent transformations in permutations.* FPSAC 2011 Proceedings, Discrete Math. Theor. Comput. Sci. Proc., 2011. `http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAO0167/3638`.

References 2

- S. Linton, J. Propp, T. Roby, J. West, *Equivalence Relations of Permutations Generated by Constrained Transpositions.* DMTCS Proceedings, North America, July 2010. `http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAN0168`, also `arXiv:1111.3920v1`.
- S. Linton, J. Propp, T. Roby, & J. West.*Equivalence Classes of Permutations under Various Relations Generated by Constrained Transpositions*, J. Integer Sequences **15** (2012), Article 12.9.1.
- R. Stanley, *An equivalence relation on the symmetric group and multiplicity-free flag h-vectors.* `arXiv:1208.3540`, 2012.
- R. Stanley, *Enumerative Combinatorics Volume 1, 2nd Ed.*, no. 49 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2011.

Thanks!

Thanks for your attention!

Thanks!

Thanks for your attention!

Any questions?