Rowmotion orbits in finite posets

Tom Roby (UConn)

Enumerative and Algebraic Combinatorics
Bruce Sagan's 70th Retirement
University of Florida
Gainesville, FL USA

27 February 2024

Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

Abstract: After briefly describing a Bulgaria Scene, we present some older and newer results about the dynamics of the rowmotion map on finite posets, including some by the Incurable Sage, who from his time as a young Genius Cal Bear cub has continually come up with exciting Algebraic Nuse.

Acknowledgments

This talks discusses joint work, mostly with (chronologically) Jim Propp, Mike Joseph, and Matthew Plante.

I'm grateful to Matthew, Mike, and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

Thanks also to Drew Armstrong, Arkady Berenstein, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, David Einstein, Darij Grinberg, Shahrzad Haddadan, Sam Hopkins, Mike La Croix, Svante Linusson, Gregg Musiker, Nathan Williams, Vic Reiner, **Bruce Sagan**, Richard Stanley, Jessica Striker, Ralf Schiffler, Hugh Thomas, and Ben Young.

My collaborations with Bruce

- How I met Ira Gessel;

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;
- Madison, $\sim$ 1996;
- Madison, 1998;

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;
- Madison, $\sim$ 1996;
- Madison, 1998;
- What do Bruce Sagan and Zach Hamaker have in common?

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;
- Madison, $\sim$ 1996;
- Madison, 1998;
- What do Bruce Sagan and Zach Hamaker have in common?
- Third BAD Math Day at Cal State Hayward, 2001.Nov.17;

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;
- Madison, $\sim$ 1996;
- Madison, 1998;
- What do Bruce Sagan and Zach Hamaker have in common?
- Third BAD Math Day at Cal State Hayward, 2001.Nov.17;
- What happened when Bruce & I were finally in same town for an academic year?

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;
- Madison, $\sim$ 1996;
- Madison, 1998;
- What do Bruce Sagan and Zach Hamaker have in common?
- Third BAD Math Day at Cal State Hayward, 2001.Nov.17;
- What happened when Bruce & I were finally in same town for an academic year?
- When did Bruce and I write our first math paper together?

My collaborations with Bruce

- How I met Ira Gessel;
- How I met Bruce;
- Mathematical tangentiality;
- FPSAC IV (1992) in Montreal;
- Madison, $\sim$ 1996;
- Madison, 1998;
- What do Bruce Sagan and Zach Hamaker have in common?
- Third BAD Math Day at Cal State Hayward, 2001.Nov.17;
- What happened when Bruce & I were finally in same town for an academic year?
- When did Bruce and I write our first math paper together?
  Sergi Elizalde, Matthew Plante, Tom Roby, and Bruce Sagan, *Rowmotion on fences*, Algebraic Combinatorics, **6**#1, (2023), pp. 17–36.

1000 Words

1000 Words

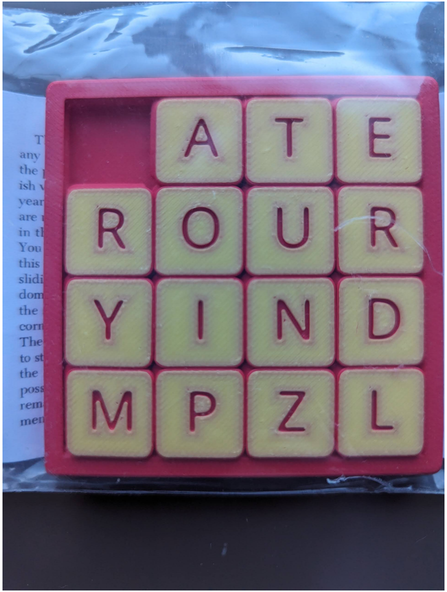
1000 Words

1000 Words

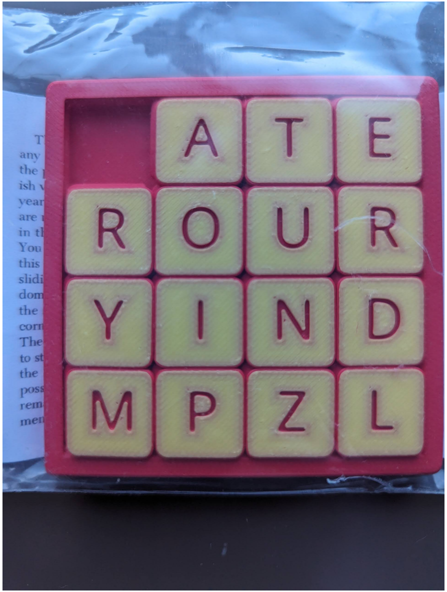
1000 Words

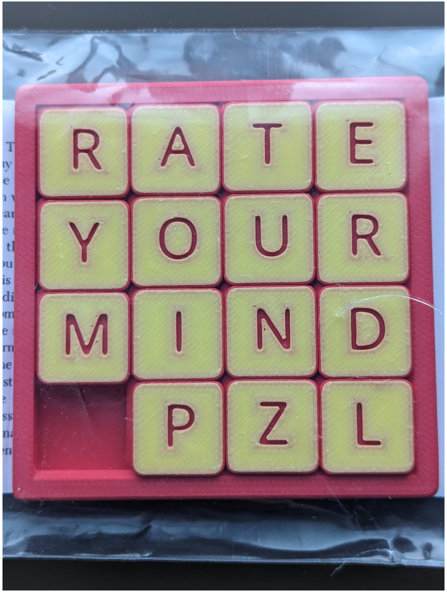
Open in separate window

1000 Words



1000 Words





Some themes in dynamical algebraic combinatorics

1. Periodicity/order;
2. Orbit structure;
3. Homomesy;
4. Equivariant bijections; and
5. Lifting from combinatorial to piecewise-linear and birational settings.

Cyclic rotation of binary strings

"Immer mit den einfachsten Beispielen anfangen." — David Hilbert

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \underset{C_R}{\longmapsto} 010100$$

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \underset{C_R}{\longmapsto} 010100$$

- *Periodicity* is clear here. The map has order $n = 6$.
- *Orbit structure* is very nice; every orbit size must divide n.
- *Homomesy?* Need a statistic, first.
- *Equivariant bijection?* No need.

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6,\ k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6,\ k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$*.*

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$*.*

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Cyclic rotation of binary strings

Example

$n = 6, \; k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

$n = 6, \ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2	100100	6
100010	5	000011	0	010010	4
010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well:
Let $\chi_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic under cyclic rotation?

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];
- Rowmotion of various "nice" posets (e.g., Lie theoretic root and minuscule posets, fences, "Chain of V's") [PrRo15, StWi11, Had14, RuWa15+, EPRS23];

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];
- Rowmotion of various "nice" posets (e.g., Lie theoretic root and minuscule posets, fences, "Chain of V's") [PrRo15, StWi11, Had14, RuWa15+, EPRS23];
- In general, composing certain involutions called "toggles" on the set leads to operations with interesting homomesy [Str18];

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];
- Rowmotion of various "nice" posets (e.g., Lie theoretic root and minuscule posets, fences, "Chain of V's") [PrRo15, StWi11, Had14, RuWa15+, EPRS23];
- In general, composing certain involutions called "toggles" on the set leads to operations with interesting homomesy [Str18];
- Toggling the "arcs" in noncrossing partitions [EFGJMPR16];

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];
- Rowmotion of various "nice" posets (e.g., Lie theoretic root and minuscule posets, fences, "Chain of V's") [PrRo15, StWi11, Had14, RuWa15+, EPRS23];
- In general, composing certain involutions called "toggles" on the set leads to operations with interesting homomesy [Str18];
- Toggling the "arcs" in noncrossing partitions [EFGJMPR16];
- Whirling functions between finite sets: injections, surjections, parking functions, etc. [JPR17+]; and

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];
- Rowmotion of various "nice" posets (e.g., Lie theoretic root and minuscule posets, fences, "Chain of V's") [PrRo15, StWi11, Had14, RuWa15+, EPRS23];
- In general, composing certain involutions called "toggles" on the set leads to operations with interesting homomesy [Str18];
- Toggling the "arcs" in noncrossing partitions [EFGJMPR16];
- Whirling functions between finite sets: injections, surjections, parking functions, etc. [JPR17+]; and
- Liftings of dynamics from combinatorial to piecewise linear and to birational maps (even with noncommuting variables) [EiPr21, GR16, GR15, JR20, JR21, GR23].

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of rectangular SSYT [BlPeSa13];
- Rowmotion of various "nice" posets (e.g., Lie theoretic root and minuscule posets, fences, "Chain of V's") [PrRo15, StWi11, Had14, RuWa15+, EPRS23];
- In general, composing certain involutions called "toggles" on the set leads to operations with interesting homomesy [Str18];
- Toggling the "arcs" in noncrossing partitions [EFGJMPR16];
- Whirling functions between finite sets: injections, surjections, parking functions, etc. [JPR17+]; and
- Liftings of dynamics from combinatorial to piecewise linear and to birational maps (even with noncommuting variables) [EiPr21, GR16, GR15, JR20, JR21, GR23].
- There are many others still being discovered.

Bulgarian Solitaire

Homomesy: A more general definition

There are some cases where we find a similar phenomenon, but where the map no longer has finite orbits. Here is a more general definition of homomesy that is useful for some purposes.

Definition

Let τ be an self-map on a discrete set of objects S, and f be a statistic on S. We say f is **homomesic** if the value of

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = c$$

is **independent** of the starting point $x \in S$. (Also, f is c-mesic.)

This clearly reduces to the earlier definition in the case where we have an invertible action with finite orbits.

Example 2: Bulgarian solitaire

Given a division of n identical pebble into one or more heaps (represented as a partition λ of n), define $\text{б}(\lambda)$ as the partition of n that results from removing a pebble from each heap and putting all the removed pebbles into a new heap.

- First surfaced as a puzzle in Russia around 1980, with a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardner column; see survey of Brian Hopkins [Hop12].
- ***Original puzzle:*** starting from any of 176 partitions of 15, what happens when you iterate this process.
- ***Audience Participation:*** Since time is short, let's start with ten pebbles. Pick your own starting position. (I'll start from $3, 2, 2, 1, 1, 1$.)

Example 2: Bulgarian solitaire

Given a division of n identical pebble into one or more heaps (represented as a partition λ of n), define $\text{б}(\lambda)$ as the partition of n that results from removing a pebble from each heap and putting all the removed pebbles into a new heap.

- First surfaced as a puzzle in Russia around 1980, with a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardner column; see survey of Brian Hopkins [Hop12].
- ***Original puzzle:*** starting from any of 176 partitions of 15, what happens when you iterate this process.
- ***Audience Participation:*** Since time is short, let's start with ten pebbles. Pick your own starting position. (I'll start from $3, 2, 2, 1, 1, 1$.)
- **What happened?**

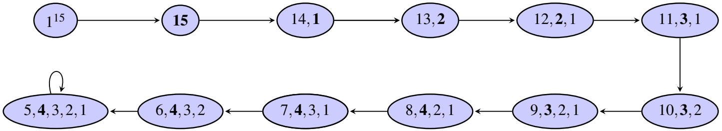
Example 2: Bulgarian solitaire

Given a division of n identical pebble into one or more heaps (represented as a partition λ of n), define $\text{б}(\lambda)$ as the partition of n that results from removing a pebble from each heap and putting all the removed pebbles into a new heap.

- First surfaced as a puzzle in Russia around 1980, with a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardner column; see survey of Brian Hopkins [Hop12].
- ***Original puzzle:*** starting from any of 176 partitions of 15, what happens when you iterate this process.
- ***Audience Participation:*** Since time is short, let's start with ten pebbles. Pick your own starting position. (I'll start from $3, 2, 2, 1, 1, 1$.)
- **What happened?** My sequence was:
  $32111 \mapsto 6211 \mapsto 541 \mapsto 433 \mapsto 3322 \mapsto 42211 \mapsto 5311 \mapsto 442$
  $\mapsto 3331 \mapsto 4222 \mapsto 43111 \mapsto 532 \mapsto 4321 \mapsto 4321 \mapsto$ itself.

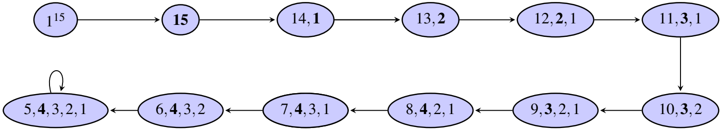
Bulgarian solitaire: “orbits” are now “trajectories”

Here’s one trajectory for the original $n = 15$ puzzle:

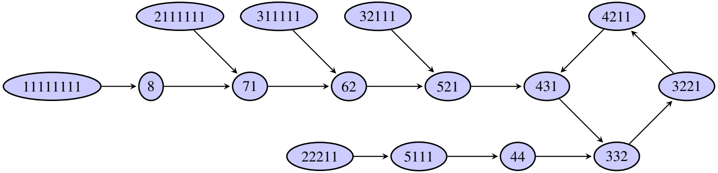


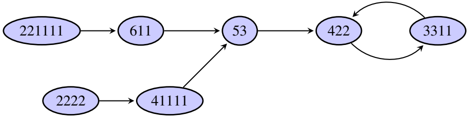
Bulgarian solitaire: "orbits" are now "trajectories"

Here's one trajectory for the original $n = 15$ puzzle:

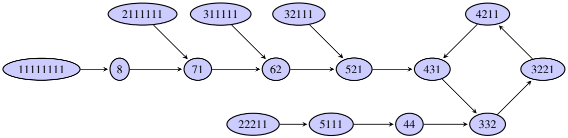


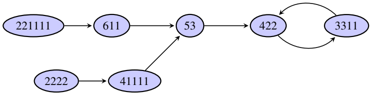
Here's the entire dynamics for $n = 8$, with 2 connected components.





Bulgarian solitaire: homomesies





Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5, 3)$, the average value of φ is $(4 + 3)/2 = 7/2$; while for $\lambda = (6, 2)$, the average value of φ is $(3 + 4 + 4 + 3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \le j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic wrt/б; similarly for "size of (kth) largest part".

Further work on non-invertible maps and possible directions

Colin Defant has done some work on non-invertible maps that are variants of pop-stack sorting (including a dual version called pop-tsack torsing) [Def21+]. In his talk on this at the BIRS-DAC workshop at UBCO, he highlighted the following questions:

Let $f : X \to X$ be a noninvertible map on a finite set X. Define the *forward orbit* of $x \in X$ to be $O_f(X) := \{x, f(x), f^2(x), \dots\}$.

- What are the periodic points of f?
- What is the image of f?
- How can we compute the number of preimages of some $x_0 \in X$ under f?
- What is the maximum number of preimages an element of X can have?
- How many elements have exactly 1 preimage?
- What is $\max_{x \in X} \# O_f(X)$? For which x is the max attained?
- Which elements $x \in X$ maximize $O_f(X)$?
- How big is $O_f(X)$ on average?

Ignore transience (for the rest of this talk)

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

Ignore transience (for the rest of this talk)

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

This definition also works in situations where S is infinite. But for rest of this talk, we'll restrict attention to maps τ that are invertible on S, where S is finite, so our initial definition (below) makes sense.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.

(Call f c-**mesic** for short.)

Rowmotion on Order Ideals of a Poset

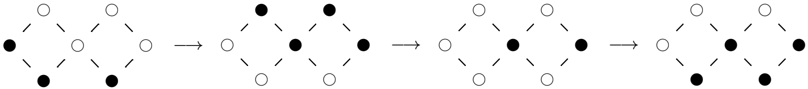
Rowmotion: an invertible operation on order ideals

We define the (cyclic) group action of **rowmotion** on the set of order ideals $\mathcal{J}(P)$ via the map $\mathrm{Row} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the following three-step process.

Start with an order ideal, and

1. Θ: Take the complement (giving an order filter)
2. ∇: Take the minimal elements (giving an antichain)
3. Δ^{-1}: Saturate downward (giving a order ideal)

ρ_J :

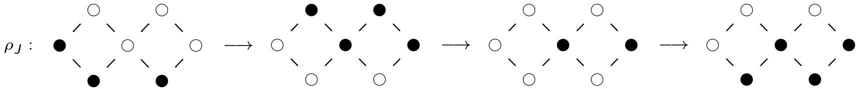


Rowmotion: an invertible operation on order ideals

We define the (cyclic) group action of **rowmotion** on the set of order ideals $\mathcal{J}(P)$ via the map $\mathrm{Row} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the following three-step process.

Start with an order ideal, and

1. Θ: Take the complement (giving an order filter)
2. ∇: Take the minimal elements (giving an antichain)
3. Δ^{-1}: Saturate downward (giving a order ideal)



This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Dynamical properties of rowmotion: cardinality is homomesic

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary rowmotion orbit in $\mathcal{J}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Dynamical properties of rowmotion: cardinality is homomesic

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

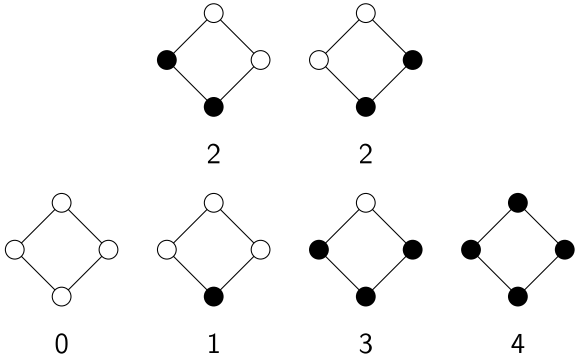
Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary rowmotion orbit in $\mathcal{J}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

We have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order ideals* (equivalently *order filters*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{J}(P)$ be the set of order ideals of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{J}(P) \to \mathcal{J}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{J}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{J}(P), \\ U & \text{otherwise.} \end{cases}$$

Theorem (Cameron and Fon-Der-Flaass 1995)

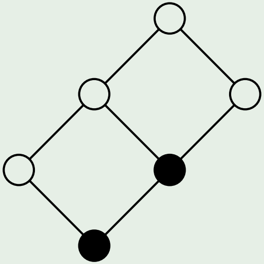
Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order ideals of P.

Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example

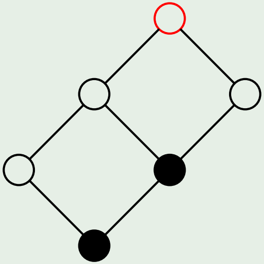


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example

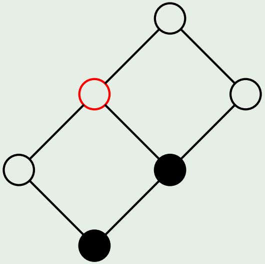


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

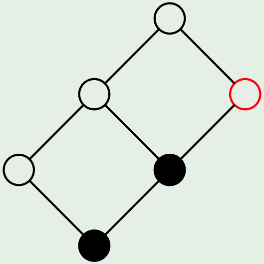
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example

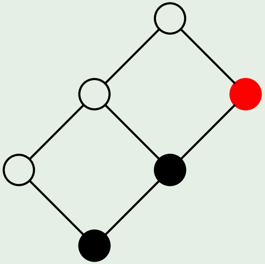


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example

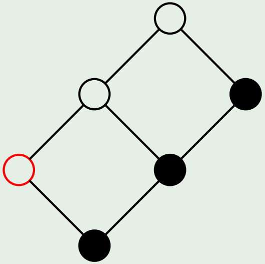


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

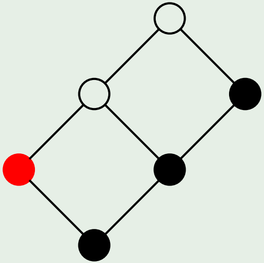
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example

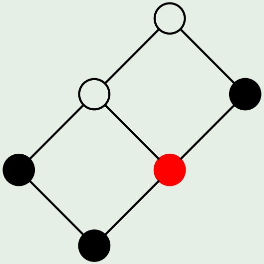


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

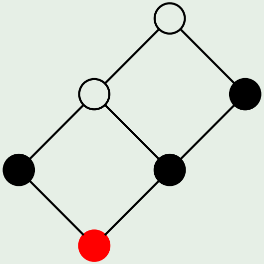
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example

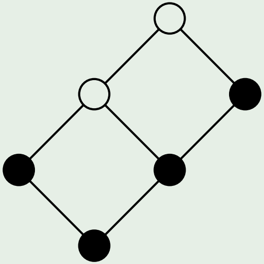


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

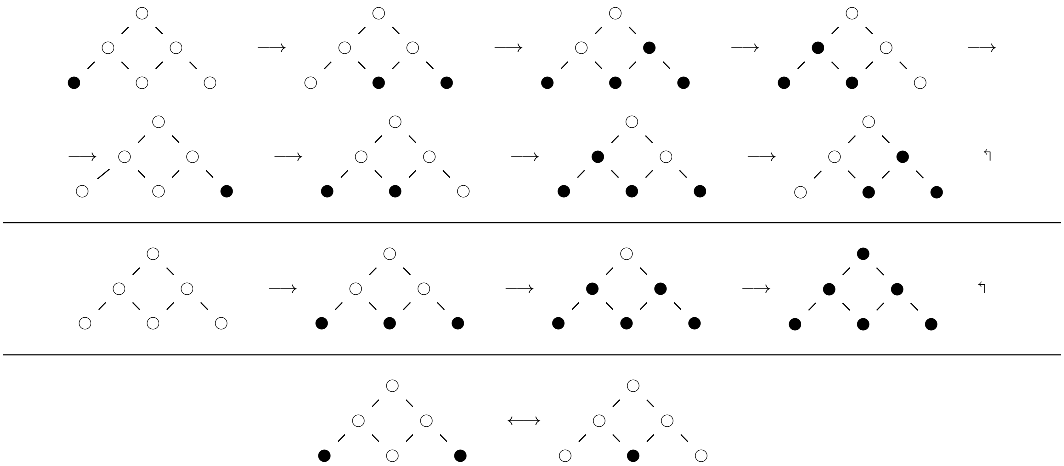
Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example



Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:

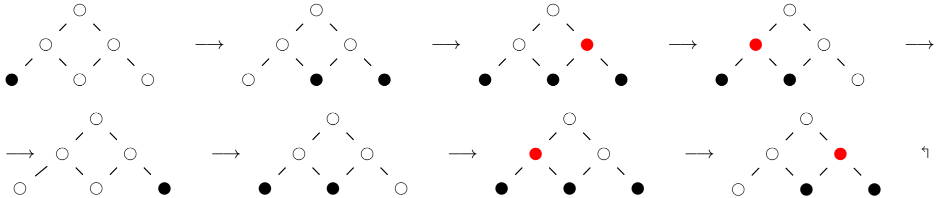


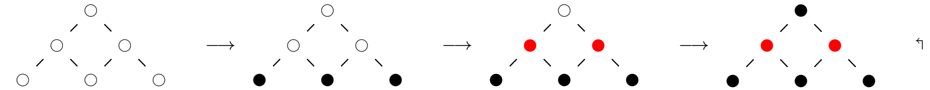
Checking the average cardinality for each orbit we find that

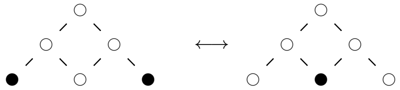
$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}; \quad \frac{0+3+5+6}{4} = \frac{7}{2}; \quad \frac{2+1}{2} = \frac{3}{2}. \text{ Darn!}$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:







Checking the average rank-alternating cardinality for each orbit we find:

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{1+2+2+1}{4} = \frac{2+1}{2} = \frac{3}{2} \text{ Yay!}$$

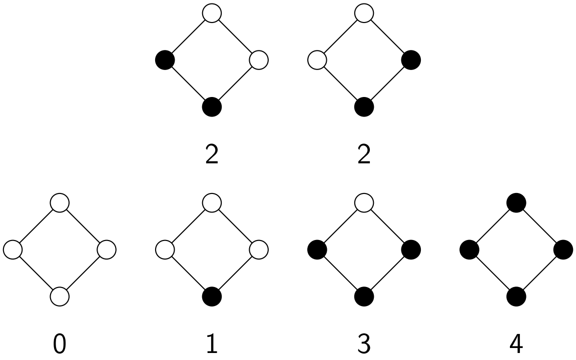
Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan)

Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign an order ideal $I \in \mathcal{J}(P)$ weight $f(I) = \sum_{x \in I} \mathrm{wt}(x)$, then f is homomesic under rowmotion and promotion, with average $n/2$.

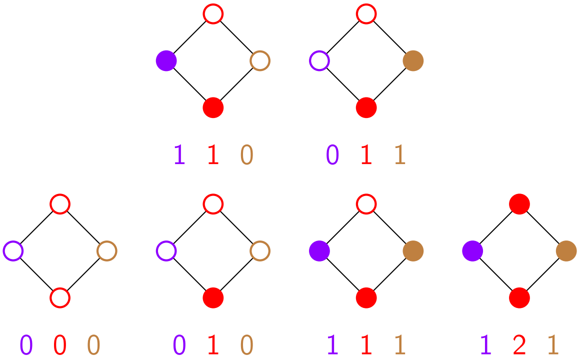
Ideals in $[a] \times [b]$: the case $a = b = 2$

We have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ of a violet element, 1 red element, and $1/2$ of a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

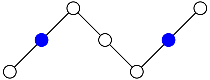
Theorem (Propp, R.)

For every ρ-orbit $\mathcal{O}$ in $J([a] \times [b])$:

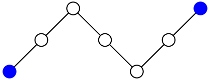
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Some homomesies for (order-ideal) rowmotion on fence posets

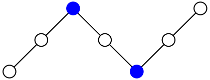
Periodicity, orbit structure, and homomesy for rowmotion on *fence posets* was explored in work of Elizalde–Plante–Roby–Sagan [EPRS23].



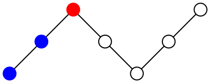
Average:1



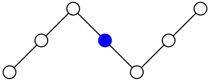
Average:1



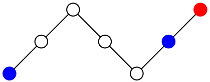
Average:1



Average:1



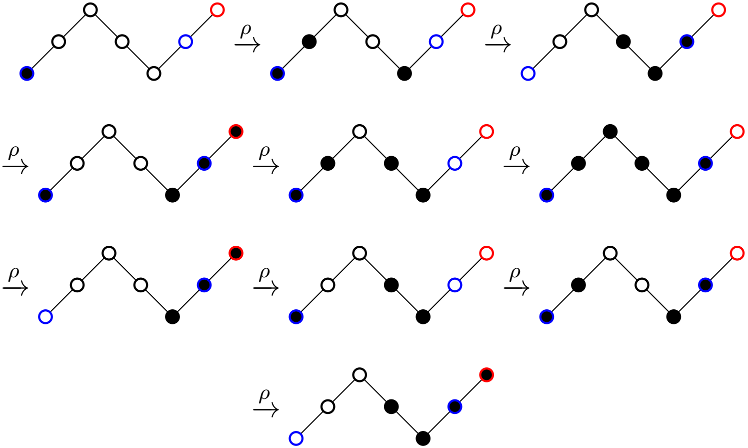
Average:$\frac{1}{2}$



Average:1

Jamie Kimble presented a poster last night for rowmotion on *rooted trees*, based on joint work with Pranjal Dangwal, Jinting Liang, Jianzhi Lou, Bruce Sagan, and Zach Stewart.

One orbit of rowmotion on a fence poset, highlighting a homomesy

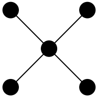


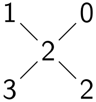
Checking the statistic we see $\frac{\color{blue}{13}\color{red}{-3}}{10} = 1$

Whirling on posets

Definition of whirling on posets

Let $\mathcal{F}_k$ be the set of order-reversing functions from P to $\{0, 1, 2, \ldots, k\}$.

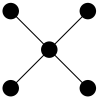
$P =$ 

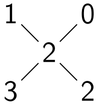


$\in \mathcal{F}_3(P)$

Definition of whirling on posets

Let $\mathcal{F}_k$ be the set of order-reversing functions from P to $\{0, 1, 2, \ldots, k\}$.

$P =$ 

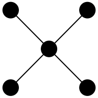


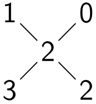
$\in \mathcal{F}_3(P)$

Definition ([JPR18])

Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$ define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at* x, as follows: repeatedly add 1 $(\mathrm{mod}\ k+1)$ to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

Let $\mathcal{F}_k$ be the set of order-reversing functions from P to $\{0, 1, 2, \ldots, k\}$.

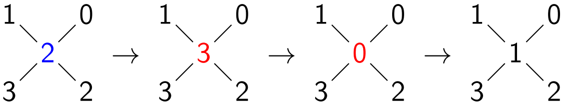
$P =$ 



$\in \mathcal{F}_3(P)$

Definition ([JPR18])

Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$ define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at* x, as follows: repeatedly add 1 $(\mathrm{mod}\ k+1)$ to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.



$\in \mathcal{F}_3(P)$

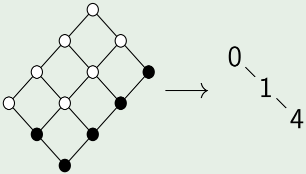
Equivariant bijection between whirling and rowmotion

Now let $\{x_1, x_2, \ldots, x_n\}$ be any linear extension of P (with $\#P = n$.) It is easy to show that w_x and w_y commute when $x, y \in P$ are *incómparable*. Thus the *whirling operator* $w := w_{x_1} w_{x_2} \cdots w_{x_n}$ is well-defined (whirling each poset element from top to bottom).

Theorem (Plante)

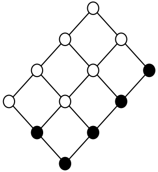
There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w to $\rho_{\mathcal{J}}$.

Example ($\mathcal{J}([3] \times [4])$ to $\mathcal{F}_4([3])$)

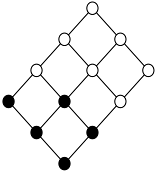


The number of order ideal elements in each fiber is recorded as an order-reversing function on [3].

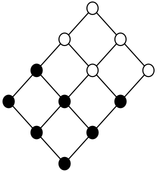
Product of two chains orbit bijection example



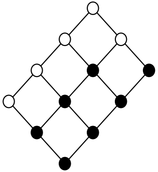
0 \ 1 \ 4



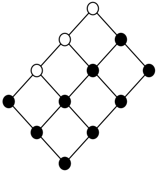
1 \ 2 \ 2



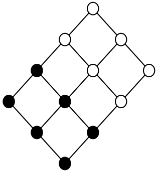
2 \ 2 \ 3



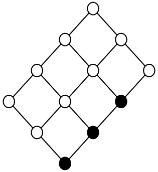
0 \ 3 \ 4



1 \ 4 \ 4



2 \ 2 \ 2



0 \ 0 \ 3

0	1	2	0	1	2	0
1	2	2	3	4	2	0
4	2	3	4	4	2	3

Theorem (Plante)

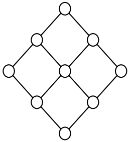
Let w denote the whirling operator on order-reversing functions $\mathcal{F}_k([m])$. Consider a superorbit board of w with length $k+m$.

1. *The board can be partitioned into m snakes of length $k+m$ under the following rules:*
   1. *Start at zero in the top row.*
   2. *Stay in a row until the value does not increase then move down.*
   3. *End once the snake contains k in the bottom row.*
2. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the segments of a snake α, that is, α_i is the number of blocks of the snake in row i. Each snake in the board has segments which are a cyclic rotation of $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.*
3. *The average sum of values along a snake is $k(m+k)/2$.*

An orbit board of $(0,1,4) \in \mathcal{F}_4([3])$:

0	1	2	0	1	2	0
1	2	2	3	4	2	0
4	2	3	4	4	2	3

Orbits of a product of two chains

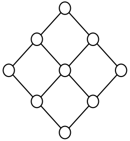
$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

$$\begin{matrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 & 3 & 3 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 & 2 \\ 3 & 1 & 2 & 3 & 3 & 2 \end{matrix}$$

$$\begin{matrix} 0 & 1 & 0 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 & 2 & 2 \\ 3 & 1 & 2 & 3 & 2 & 3 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 2 & 1 \\ 2 & 3 \end{matrix}$$

Orbits of a product of two chains

$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

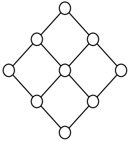
0	0	0	1	2	3
0	0	1	2	3	3
0	1	2	3	3	3

0	0	1	0	1	2
0	1	1	2	3	2
3	1	2	3	3	2

0	1	0	0	1	2
3	1	0	1	2	2
3	1	2	3	2	3

0	1
2	1
2	3

Orbits of a product of two chains

$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

0	0	0	1	2	3
0	0	1	2	3	3
0	1	2	3	3	3

0	0	1	0	1	2
0	1	1	2	3	2
3	1	2	3	3	2

0	1	0	0	1	2
3	1	0	1	2	2
3	1	2	3	2	3

0	1	0	1	0	1
2	1	2	1	2	1
2	3	2	3	2	3

(expanding the last orbit to a superorbit).

The $\vee \times [k]$ poset

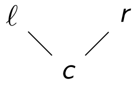
The poset $\mathsf{V} \times [k]$

- Let V be the three-element partially ordered set with Hasse diagram

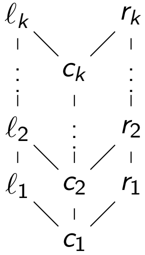
$$\begin{tikzcd}[column sep=small] \ell \arrow[dr, dash] & & r \arrow[dl, dash] \\ & c & \end{tikzcd}$$

The poset V $\times$ $[k]$

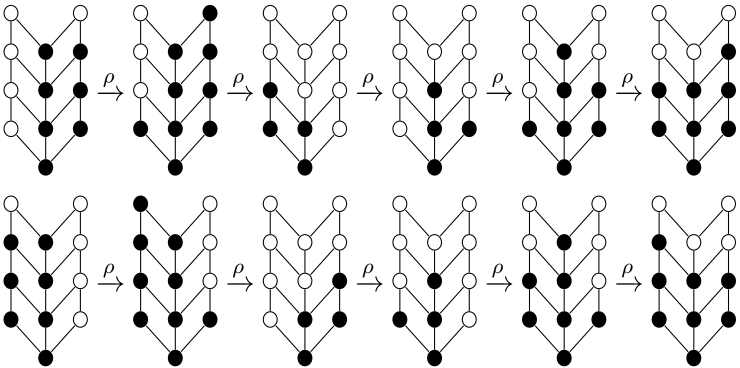
- Let V be the three-element partially ordered set with Hasse diagram



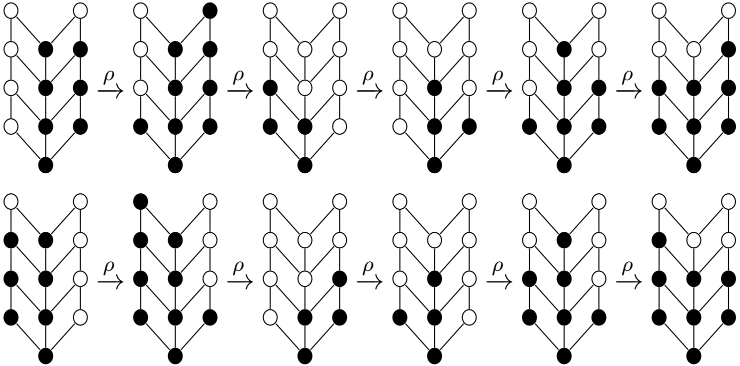
- The poset of interest is $V(k) := \mathrm{V} \times [k]$



Order-ideal rowmotion on $\mathsf{V} \times [k]$



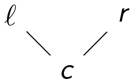
Order-ideal rowmotion on $\mathsf{V} \times [k]$



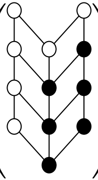
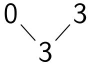
Theorem (Plante)

Order ideals of $V(k)$ are reflected about the center chain after $k+2$ iterations of ρ, and furthermore, the order of ρ on order ideals of $V(k)$ is $2(k+2)$.

Map to order-reversing functions on V



1. Define $\mathcal{F}_k(\mathsf{V}) = \{(\ell, c, r) \in \{0, \ldots, k\}^3 : \ell, r \leq c\}$.
2. Define $\phi : \mathcal{J}(\mathsf{V}(k)) \to \mathcal{F}_k(\mathsf{V})$ by $\phi(I) = \left(\sum \chi_{\ell_i}, \sum \chi_{c_i}, \sum \chi_{r_i}\right)$.

$\phi\Bigg($  $\Bigg) = (0, 3, 3) \leftrightarrow$ 

We whirl the example $\begin{array}{ccc} \ell & & r \\ & c & \end{array}$ first at ℓ, r, then c.

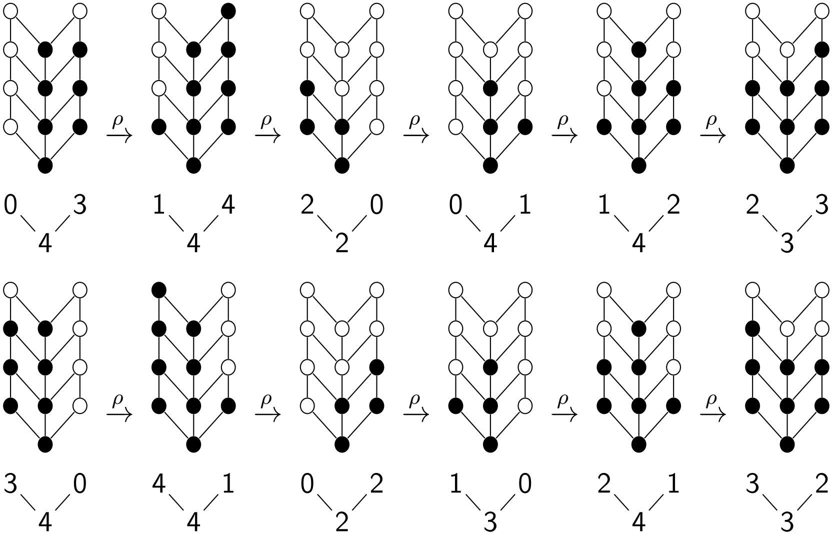
Start with $(0, 2, 2) \in \mathcal{F}_2(\mathrm{V})$.

$$\begin{array}{ccc} {\color{blue}0} & & 2 \\ & 2 & \end{array} \xrightarrow{w_\ell} \begin{array}{ccc} 1 & & 2 \\ & 2 & \end{array}$$

$$\begin{array}{ccc} 1 & & {\color{blue}2} \\ & 2 & \end{array} \xrightarrow{w_r} \begin{array}{ccc} 1 & & 0 \\ & 2 & \end{array}$$

$$\begin{array}{ccc} 1 & & 0 \\ & {\color{blue}2} & \end{array} \xrightarrow{w_c} \begin{array}{ccc} 1 & & 0 \\ & {\color{red}0} & \end{array} \rightarrow \begin{array}{ccc} 1 & & 0 \\ & 1 & \end{array}$$

Example of rowmotion orbit with triples



Equivariant bijection example

Alternatively we may define w on $(\ell, c, r) \in \mathcal{F}_k(\mathrm{V})$ as the process:

1. $\ell \to \ell + 1$ unless $\ell = c$, then $\ell \to 0$.

Equivariant bijection example

Alternatively we may define w on $(\ell, c, r) \in \mathcal{F}_k(\mathrm{V})$ as the process:

1. $\ell \to \ell + 1$ unless $\ell = c$, then $\ell \to 0$.
2. Repeat step 1 with r instead of ℓ.

Equivariant bijection example

Alternatively we may define w on $(\ell, c, r) \in \mathcal{F}_k(\mathsf{V})$ as the process:

1. $\ell \to \ell + 1$ unless $\ell = c$, then $\ell \to 0$.
2. Repeat step 1 with r instead of ℓ.
3. $c \to c + 1$ unless $c = k$, then $c \to \max(\ell, r)$.

Corollary

The map ϕ is an equivariant bijection that sends ρ to w.

$$
\begin{tikzcd}
\mathcal{J}(\mathsf{V}(k)) \arrow[r, "\rho"] & \mathcal{J}(\mathsf{V}(k)) \\
\mathcal{F}_k(\mathsf{V}) \arrow[r, "w"] \arrow[u, leftrightarrow, "\phi"] & \mathcal{F}_k(\mathsf{V}) \arrow[u, leftrightarrow, "\phi"']
\end{tikzcd}
$$

Theorem (Plante)

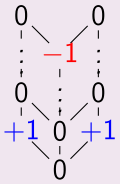
Order ideals of $V(k)$ are reflected about the center chain after $k+2$ iterations of ρ, and furthermore, the order of ρ on order ideals of $V(k)$ is $2(k+2)$.

Direct inspection of order-reversing functions on V as tuples gives a straightforward proof of periodicity.

Theorem (Plante)

For the action of rowmotion on order ideals of $\mathsf{V}(k)$*:*

1. *The statistic* $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ *is* $\frac{2(k-1)}{k+2}$*-mesic.*



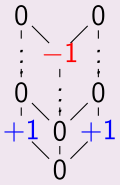
.

Homomesy

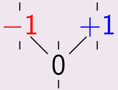
Theorem (Plante)

For the action of rowmotion on order ideals of $\mathsf{V}(k)$*:*

1. *The statistic* $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ *is* $\frac{2(k-1)}{k+2}$*-mesic.*



.

2. *The statistic* $\chi_{r_i} - \chi_{\ell_i}$ *is 0-mesic* 

for each $i = 1, \ldots, k$*, where* χ_x *is the indicator function.*

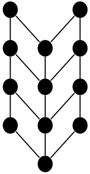
Center Seeking Snakes

We decompose the orbit board into 6 snakes of length $k+2$. Or 2 two-tailed snakes if the order-reversing functions are symmetric. Recall that snakes start at the top of a poset and move down. Since the least element of V is in the center, we call these snakes, *center-seeking snakes*.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Sketch of Proof of Homomesy



$$\sum \chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$$

$$= (2(k+2)-3)+(2(k+2)-3)-6$$

Thus we see

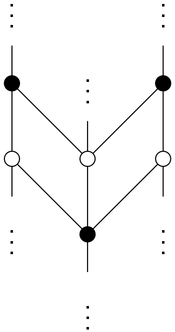
$$\frac{4(k+2)-12}{2(k+2)} = \frac{2k-2}{k+2}.$$

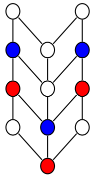
1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

$2(k+2)$

Back to the Future!

There is another nice looking homomesy for the "chain of V's poset." Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the following flux-capacitor shape in $\mathrm{V}(k)$.



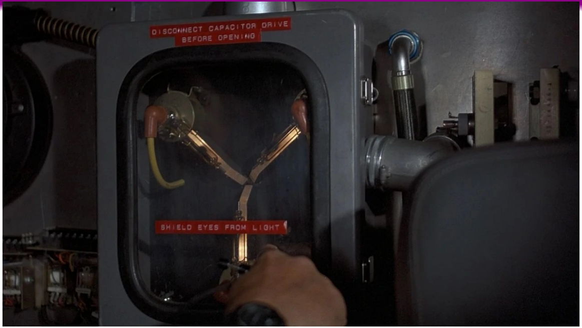


$F_3 - F_2$

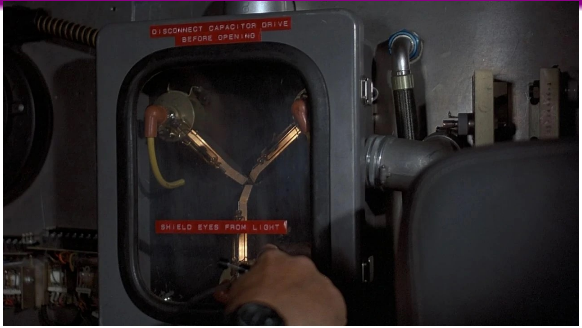
Theorem (Plante)

The difference $F_i - F_j$ is $\frac{3(i-j)}{(k+2)}$-mesic with respect to rowmotion on $\mathrm{V}(k)$.

Flux Capacitor??



Flux Capacitor??



https://www.youtube.com/watch?v=VcZe8_RZO8c

References

[AST11] Drew Armstrong, Christian Stump, and Hugh Thomas, *A uniform bijection between nonnesting and noncrossing partitions*, Trans. Amer. Math. Soc. **365** (2013), no. 8, 4121–4151.

[BlPeSa13] Jonathan Bloom, Oliver Pechenik, and Dan Saracino, *Proofs and generalizations of a homomesy conjecture of Propp and Roby*, Discrete Math., **339** (2016), 194–206.

[Def21+] Colin Defant, *Pop-stack-sorting for Coxeter Groups*, `arXiv:2104.02675`.

[EFGJMPR16] David Einstein, Miriam Farber, Emily Gunawan, Michael Joseph, Matthew Macauley, James Propp, and Simon Rubinstein-salzedo, *Noncrossing partitions, toggles, and homomesies*, Electron. J. of Combin. **23**(3 (2016).

[EiPr21] David Einstein and James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains*, Alg. Combin. **4** #2 (2021) `arXiv:1310.5294`.

[EPRS23] Sergi Elizalde, Matthew Plante, Tom Roby, and Bruce Sagan, *Rowmotion on fences*, Algebraic Combinatorics, **6**#1, (2023), pp. 17–36.

[GR16] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. of Combin. **23**(1), #P1.33 (2016). `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i1p33`

[GR15] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Elec. J. Combin. **22**(3), #P3.40, 2015. `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p40`

[GR23] Darij Grinberg and Tom Roby, *Birational rowmotion on a rectangle over a noncommutative ring*, Combin. Thy. **3**(3) (2023) #7, `https://doi.org/10.5070/C63362790`.

[Had14] Shahrzad Haddadan, *Some Instances of Homomesy Among Ideals of Posets*, 2014, `arXiv:1410.4819v2`.

[HS22] Erika Hanaoka and Taizo Sadahiro, *A generalization of rotation of binary sequences and its applications to toggle dynamical systems*, Theoretical Comp. Sci. **910** #2, (2022), `arXiv:1909.01125`.

[Hop12] Brian Hopkins, *30 years of Bulgarian solitaire*, College Math J., **43**, #2, March 2012, 135–140.

[J19] Michael Joseph, *Antichain toggling and rowmotion*, Electronic Journal of Combin., **26(1)**, 2019, #P1.29.

[JPR17+] James Propp, Michael Joseph, and Tom Roby, *Whirling injections, surjections, and other functions between finite sets*, 2017, `arXiv:1711.02411`.

[JR18] Michael Joseph and Tom Roby, *Toggling Independent Sets of a Path Graph*, Electronic Journal of Combin., **25(1)**, 2018, #P1.18.

[JR20] Michael Joseph and Tom Roby, *Birational and noncommutative lifts of antichain toggling and rowmotion*, Alg. Combin. **3** #4 (2020) `arXiv:1909.09658`.

[JR21] M. Joseph and T. Roby, *A birational lifting of the Stanley–Thomas word on products of two chains*, 2019, DM&TCS **23**(1) #17 (2021), `arXiv:/2001.03811`.

[MR19] Gregg Musiker, Tom Roby, *Paths to understanding birational rowmotion on products of two chains*, Algebraic Comin. **2(2)** (2019), pp. 275–304. `arXiv:1801.03877`. `https://alco.centre-mersenne.org/item/ALCO_2019__2_2_275_0/`.

[Pan09] Dmitri I. Panyushev, *On orbits of antichains of positive roots*, Europ. J. Combin. **30(2)** (2009), 586–594.

[PrRo15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electronic J. Combin. **22(3)** (2015), #P3.4, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4`.

[RSW04] V. Reiner, D. Stanton, and D. White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), 17–50.

[Rob16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in A. Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

[RuWa15+] David B. Rush and Kelvin Wang, *On orbits of order ideals of minuscule posets II: Homomesy*, `arXiv:1509.08047`.

[Stan11] Richard P. Stanley, *Enumerative Combinatorics, volume 1, 2nd edition*, no. 49 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2011.

[Sta09] Richard P. Stanley, *Promotion and Evacuation*, Electron. J. Combin. **16(2)** (2009), #R9.

[Sta86] R. Stanley, *Two Poset Polytopes*, Disc. & Comp. Geom. **1** (1986), 9–23.

[Str18] Jessica Striker, *Rowmotion and generalized toggle groups*, Discrete Math & Theoretical Comp. Sci. **20**, no. 1. `arXiv:1601.03710`.

[StWi11] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942,

[ThWi17] H. Thomas and N. Williams, *Rowmotion in slow motion*, `arXiv:1712.10123v1`.

[Yil17] Emine Yıldırım, *Coxeter transformation on Cominuscule Posets*, `arXiv:1710.10632`.

[Volk06] Alexandre Yu. Volkov, *On the Periodicity Conjecture for Y-systems*, 2007. (Old version available at http://arxiv.org/abs/hep-th/0606094)

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to questions of periodicity/order, orbit structure, homomesy, and equivariant bijections.
- Actions that can be built out of smaller, simpler actions (toggles and whirls) often have interesting and unexpected properties.
- Much more remains to be explored, perhaps for combinatorial objects or actions that you work with for other reasons.

Slides for this talk will be available online at

Google "Tom Roby".

Thanks very much for coming to this talk!