A Bijection that Counts: The Robinson-Schensted-Knuth Correspondence

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Outline

- Increasing subsequences in permutations (Boarding, Sorting)
- Schensted's original counting problem
- Partitions and tableaux
- RSK Correspondence
- Application: Erdős-Szekeres

Increasing subsequences in permutations

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DEF by EG: 269 is an **increasing subsequence** of $w = 527361948$. Are there any longer ones? Yes, e.g., 527361948, and several others of size 4. Note that **not** every inc. subseq. of size 3 can be completed to one of size 4.

$$
\begin{aligned}
\operatorname{lis}(w) &:= \text{length of the longest inc. subseq. of } w && = 4 \\
\operatorname{lds}(w) &:= \text{length of the longest dec. subseq. of } w && = 3
\end{aligned}
$$

Airplane Boarding

Q: Why should we care about increasing subsequences?

A1: In a naive model of airplane boarding, there's one seat per row, and passengers board in some permuted order. It takes one unit of time for those with higher numbers to wait behind those with lower numbers.

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For example, if the boarding order is $w = 527361948$, then boarding proceeds:

Seats	9	8	7	6	5	4	3	2	1
Time 1					5			2736	1948
Time 2			7				36948		
Time 3				69		48			
Time 4	9	8							

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Easy: Total boarding time $= \text{lis}(w)$.

Patience Sorting

A2: **Patience Sorting:** Shuffle a deck of cards labeled $1, 2, \ldots n$, turn up cards one-by-one, and sort them into piles according to the rule:

- A low card may be placed on a higher card (e.g., a 3 on a 6), **or** may be placed on a new pile to the right of existing piles.

EG: If the order of the cards is $w = 527361948$, then we get the following sequence of piles:

$$5\,,\quad \begin{array}{c}2\\5\end{array},\quad \begin{array}{cc}2&\\5&7\end{array},\quad \begin{array}{cc}2&3\\5&7\end{array},\quad \begin{array}{ccc}2&3&\\5&7&6\end{array},\quad \begin{array}{ccc}1&&\\2&3&\\5&7&6\end{array},\quad \begin{array}{cccc}1&&&\\2&3&&\\5&7&6&9\end{array},\quad \begin{array}{cccc}1&&&\\2&3&4&\\5&7&6&9\end{array},\quad \begin{array}{cccc}1&&&\\2&3&4&8\\5&7&6&9\end{array}$$

Easy: Minimum number of piles $= \mathrm{lis}(w)$ (greedily achievable).
Ref: D. Aldous & P. Diaconis: "Longest incr. subseq.: from patience sorting to the Baik-Deift-Johansson thm., *BAMS*, 1999.

Permutation Patterns

A3: We're combinatorialists—we'll count anything!
Including **permutation patterns** (cf. Pudlow's talk). The last two decades have seen a surge in interest in enumerating and understanding classes of permutations which *avoid* certain subpatterns. This turns out to have surprising connections to algebraic geometry, where avoidance of certain patterns turns up in interesting decompositions of certain varieties ("Schubert Calculus", Knutson's talk).

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Schensted was 30 years ahead of his time in studying the number of permutations that avoid the pattern $1, 2, \ldots k+1$.
Schensted's original question was to count the number of permutations on n whose long. incr. subseq. was a fixed value k:

$$\#\{w \in S_n : \mathrm{lis}(w) = k\}$$

To do this, he discovered a very *beautiful bijection* between permutations and pairs of "Standard Young Tableaux", which were previously *cleverly counted*.

Partitions & Tableaux

A **partition** λ **of** n is a sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \dots \lambda_l)$$

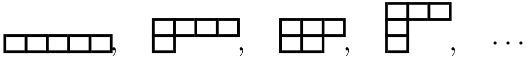
such that:

1. The terms are weakly decreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$
2. $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$

Suppressing commas and trailing zeroes, we have seven partitions of 5:

$$5, \quad 41, \quad 32, \quad 311, \quad 221, \quad 2111, \quad 11111$$

represented visually as left-justified shapes (whose squares are called **cells**):

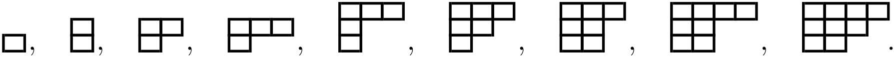


Standard Young Tableaux

A **standard Young tableau of shape** λ, is a labeling of the cells of a partition with the numbers $1, 2, \ldots, |\lambda|$ which increases along each row and column. For example,

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 8 \\ \hline 2 & 6 & 9 \\ \cline{1-3} 5 & 7 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 7 & 9 \\ \hline 4 & 8 & 10 & 13 \\ \cline{1-4} 6 & 11 & 12 \\ \cline{1-3}\end{array} \quad \text{but } \textcolor{red}{\text{not}} \quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 7 & 11 \\ \hline 4 & 8 & 12 & 13 \\ \cline{1-4} 6 & 9 & 10 \\ \cline{1-3}\end{array}$$

Such a labeling is in *easy* bijection with sequences ("chains") of shapes that grow one box in each step. For example, the first tableaux above corresponds to:



Hook-Length Formula

Let $\mathrm{SYT}(\lambda)$ = set of standard Young tableaux of a shape λ with n boxes. There's a surprisingly simple formula for counting these.

Frame-Robinson-Thrall Hook Length Formula:

$$f_\lambda = \#\,\mathrm{SYT}(\lambda) = \frac{n!}{\prod_{c\in\lambda} h_c}$$

where for each cell of λ, h_c represents the size of the **hook** centered at c (all cells to the right of or below c, including c).

EG: If $\lambda = (3,2) = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$, then we fill in the hook lengths:

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \quad\Longrightarrow\quad \#\,\mathrm{SYT}(3,2) = \frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5$$

corresponding to

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}\,.$$

Where are we?

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We are about at the point in the talk where it's easy to get lost. We have:

- Permutations and increasing sequences therein: 527361948; We'd like to count:

$$\#\{w \in S_n : \mathrm{lis}(w) = k\}$$

- We have partitions/shapes $\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$ and fillings of these $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$ called tableaux.
- We have a hook-length formula to count tableaux of a given shape:

$$f_\lambda = \#\mathrm{SYT}(\lambda) = \frac{n!}{\prod_{c\in\lambda} h_c!}$$

- How can we get from permutations to tableaux?

Schensted's Algorithm

Schensted's algorithm is based on inserting a number a into a partial SYT T as follows:

- Place a where it should go in increasing order in row 1 of T, possibly bumping an preexisting element b in that spot;
- Repeat with b in the next row, and continue iterating;
- Finish when some element comes to rest at the end of a row.

EG: Suppose we insert 4 into the tableau $\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&7\\ \cline{1-2} 6&9\\ \cline{1-2} 8\\ \cline{1-1}\end{array}$ We get (showing the intermediate steps)

$$\begin{array}{|c|c|c|}\hline 1&3&{}^{4}5\\ \hline 2&7\\ \cline{1-2} 6&9\\ \cline{1-2} 8\\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&{}^{5}7\\ \cline{1-2} 6&9\\ \cline{1-2} 8\\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2} 6&{}^{7}9\\ \cline{1-2} 8\\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2} 6&7\\ \cline{1-2} 8&9\\ \cline{1-2}\end{array}$$

The last tableau on the right is the result of this insertion.

RSK Correspondence

- Starting from an empty shape, insert each element of $w \in S_n$ in turn to get a SYT P of some shape.
- In order to make this **reversible**, we need to keep track of the order in which new cells are created. We record these in a second tableau Q, which has the same shape.
- Then the map $w \quad \longleftrightarrow \quad (P, Q)$ is a bijection $S_n \quad \longleftrightarrow \quad \prod_{|\lambda|=n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$

If we apply the above to the permutation $\begin{smallmatrix}1&2&3&4&5&6\\3&1&6&2&5&4\end{smallmatrix}$ We get the following sequence of tableaux:

$$P: \quad \emptyset \quad \begin{array}{|c|}\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&6\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&6\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

$$Q: \quad \emptyset \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

Why is it beautiful?

See basic Schensted bumping algorithm applet at:
`http://www.math.uconn.edu/~troby/Goggin/index.html`
Here are some beautiful properties and consequences of RSK:

- (Schensted) $\text{lis}(w)$ = length of the first row of P. Similarly, $\text{lds}(w)$ = length of the first col of P.
- (Schützenberger) If $w \overset{\text{R-S}}{\longleftrightarrow} (P, Q)$ then $w^{-1} \overset{\text{R-S}}{\longleftrightarrow} (Q, P)$.
- $\displaystyle\sum_{|\lambda|=n} f_\lambda^2 = n!$, where $f_\lambda = \#\text{SYT}(\lambda)$ = dimension of irreducible S_n rep. corr. to λ.

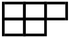
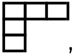
EG: When $n = 5$ one computes (via FRT or by hand) that $f_5 = 1, f_{41} = 4, f_{32} = 5, f_{311} = 6, f_{221} = 5, f_{2111} = 4, f_{11111} = 1$ so

$$1^2+4^2+5^2+6^2+5^2+4^2+1^2 = 1+16+25+36+25+16+1 = 120 = 5!$$

Counting permutations with fixed LIS

This bijection allowed Schensted to answer his original question of counting:

$$\#\{w \in S_n : \mathrm{lis}(w) = k\} = \#\{(P, Q) : \text{first row length} = k\}$$

by just counting the number of pairs of same shape tableau whose first row is a given length (using FRT hook-length formula).
EG: If $n = 6$ and $\mathrm{lis}(w) = k = 3$, then w corresponds to a pair of tableaux of shape  or , so the total number of such permutations is

$$f_{32}^2 + f_{311}^2 = 25 + 36 = 61\,,$$

the year in which Schensted published his paper.

The following well-known result of Erdős-Szekeres & Seidenberg is an immediate corollary of RSK:

Cor: Let $w \in S_{pq+1}$. Then either $\mathrm{lis}(w) > p$ or $\mathrm{lds}(w) > q$.

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Cor: Let $w \in S_{pq+1}$. Then either $\mathrm{lis}(w) > p$ or $\mathrm{lds}(w) > q$.

Proof: Let $w \quad \longleftrightarrow \quad (P, Q)$ of shape λ. If $\mathrm{lis}(w) \leq p$ and $\mathrm{lds}(w) \leq q$, then the shape must fit inside a $p \times q$ rectangle, which forces λ to have fewer than $pq + 1$ boxes, contradiction.

Bijections Rule! And they can lead to clever counting.

Summary & References

Bijections Rule! And they can lead to clever counting.

1. R.P. Stanley: Enumerative Combinatorics, Vol. 2, Sec. 7.11, and references therein.
2. Papers by C. Schensted, M-P. Schützenberger, S. Fomin.
3. Google "Schensted", "RSK", or "Tom Roby" for applets.