My broad interests are in enumerative and algebraic combinatorics, particularly bijective correspondences, partially-ordered sets, connections with representation theory, and discrete dynamics. My most recent focus has been on dynamical algebraic combinatorics, focussing on issues of periodicity, orbit structure, homomesy, and equivariant bijections.

A poset $P$ is a set with an order relation $\leq$ which is reflexive, antisymmetric, and transitive. An order ideal $I \subseteq P$ satisfies $v \in P$ and $u \leq v \implies u \in P$. The set of all order ideals is denoted $J(P)$.

Define an operator $\rho$ on $J(P)$ by $\rho(I) =$ the order ideal $I'$ gen. by the minimal elements of $P - I$.

Here are the orbits of $\rho$ on the rectangular poset $[2] \times [2]$:

\[ \begin{array}{ccc}
O_1 & \rho \to & O_2 \\
\bullet & \rho & \bullet \\
\circ & \rho & \circ \\
\circ & \rho & \circ
\end{array} \]

Given an action $\tau$ on a set $S$, we call a statistic $g : S \to \mathbb{C}$ homomesic (or $c$-mesic) if the average of $g$ over every $\tau$-orbit $O$ is the same constant $c$, i.e., $\frac{1}{\#O} \sum_{v \in O} g(v) = c$ for every $O$.

The cardinality statistic is 2-mesic for $\rho$ acting on $J(P)$.

One can also describe $\rho$ as a product of toggling involutions, one for each poset element, from top to bottom (“rowmotion in slowmotion”). More formally, for $I \in J(P)$ and $v \in P$, let $t_v(I) = I \Delta \{v\}$ (symmetric difference) provided that $I \Delta \{v\}$ is an order ideal of $P$; otherwise, let $t_v(I) = I$. The involutions $t_x$ and $t_y$ commute unless $x$ covers $y$ or $y$ covers $x$, and $\rho = t_{v_1} \circ t_{v_2} \circ \cdots \circ t_{v_n} : J(P) \to J(P)$ where $(v_1, \ldots, v_n)$ is any linear extension of $P$.

This setup generalizes nicely from order ideals (order-preserving 0-1 labelings) to order-preserving labelings $f : P \to [0, 1]$ (the order polytope of Stanley). (No time today, but of independent interest.)

Detropicalizing these PL-toggles leads to an operator in the birational category: For any $v \in P$, define the birational $v$-toggle as the rational map $T_v : \mathbb{C}^\hat{P} \to \mathbb{C}^\hat{P}$ defined by

\[
(T_v f)(w) = \begin{cases} 
  f(w), & \text{if } w \neq v; \\
  \frac{1}{f(v)} \sum_{u \in \hat{P}, u \geq v} f(u), & \text{if } w = v
\end{cases}
\]

for all $w \in \hat{P}$.

- Tropicalization $\cdot \mapsto +$ and $+ \mapsto \max$, recovers PL toggles.
- We can describe toggling at $v$ as: (a) inverting the label at $v$, (b) multiplying it with the sum of the labels at vertices covered by $v$, (c) multiplying it with the harmonic sum of the labels at vertices covering $v$.
- Note that $T_v$ changes only the label at $v$.
- These maps are involutions: $T_v^2 = \text{id}$. 

Here is one iteration, birationally toggling from top-to-bottom:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & \frac{x+y}{x} & \frac{x+y}{y} & \frac{x+y}{y} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\frac{x+y}{x} & \frac{w(x+y)}{x} & \frac{w(x+y)}{y} & \frac{w(x+y)}{y} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\frac{w(x+y)}{x} & \frac{w(x+y)}{y} & \frac{w(x+y)}{y} & \frac{w(x+y)}{y} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\frac{w(x+y)}{x} & \frac{w(x+y)}{y} & \frac{w(x+y)}{y} & \frac{w(x+y)}{y} \\
\end{array}
\]

And here is a complete orbit:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & \frac{x+y}{x} & \frac{w(x+y)}{y} & \frac{1}{y} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\frac{w(x+y)}{x} & \frac{w(x+y)}{y} & \frac{1}{y} & \frac{1}{y} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\frac{w(x+y)}{x} & \frac{w(x+y)}{y} & \frac{1}{y} & \frac{1}{y} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\frac{w(x+y)}{x} & \frac{w(x+y)}{y} & \frac{1}{y} & \frac{1}{y} \\
\end{array}
\]

**Surprises:**

(a) This is still periodic on \([r] \times [s]\) with period \(r + s\) [GrRo15, MR18]. This appears to generalize only to very special classes of posets [GrRo16, GrRo15].

(b) Homomesy generalizes to various products across an orbit being equal to 1 [GrRo15, MR18].

(c) There is a formula for iterating \(\rho_B\) on a product of two chains in terms of families of NILPs [MR18].

(d) One can define a **noncommutative** version of this that still has periodicity [JR20].

For a 25-minute intro to my work, see the video [https://www.youtube.com/watch?v=9TUajKFInwg](https://www.youtube.com/watch?v=9TUajKFInwg) of my talk at AlCove ([http://www.math.uwaterloo.ca/~opecheni/alcove.htm](http://www.math.uwaterloo.ca/~opecheni/alcove.htm)).

J Propp, J Striker, N Williams, and I are running a workshop at BIRS, with virtual talks MWF of the last two weeks of October: [http://www.birs.ca/events/2020/5-day-workshops/20w5164](http://www.birs.ca/events/2020/5-day-workshops/20w5164).

**References**


