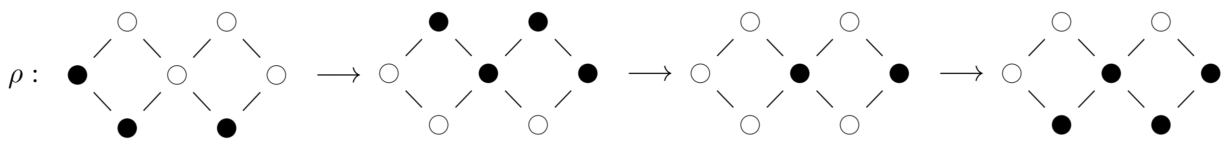
**Tom Roby Dynamical Algebraic Combinatorics in 10 Minutes 9 September 2020**

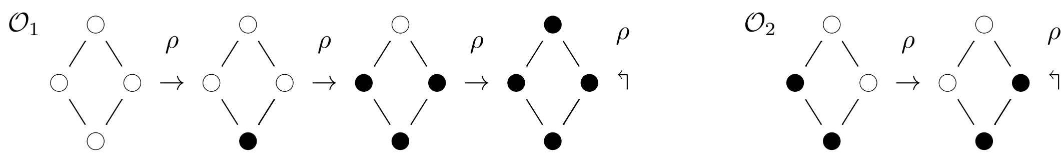
My broad interests are in enumerative and algebraic combinatorics, particularly bijective correspondences, partially-ordered sets, connections with representation theory, and discrete dynamics. My most recent focus has been on *dynamical algebraic combinatorics*, focussing on issues of *periodicity*, *orbit structure*, *homomesy*, and *equivariant bijections*.

A *poset* P is a set with an order relation $\leq$ which is reflexive, antisymmetric, and transitive. An *order ideal* $I \subseteq P$ satisfies $v \in P$ and $u \leq v \implies u \in P$. The set of all order ideals is denoted $J(P)$.

Define an operator $\rho$ on $J(P)$ by $\rho(I) =$ the order ideal $I'$ gen. by the minimal elements of $P - I$.




Here are the orbits of $\rho$ on the rectangular poset $[2] \times [2]$:




Given an action $\tau$ on a set $\mathcal{S}$, we call a statistic $g : \mathcal{S} \to \mathbb{C}$ *homomesic* (or *c-mesic*) if the average of $g$ over every $\tau$-orbit $\mathcal{O}$ is the same constant $c$, i.e., $\displaystyle\frac{1}{\#\mathcal{O}} \sum_{v \in \mathcal{O}} g(v) = c$ for every $\mathcal{O}$.

The cardinality statistic is 2-mesic for $\rho$ acting on $J(P)$.

One can also describe $\rho$ as a product of *toggling involutions*, one for each poset element, from top to bottom ("rowmotion in slowmotion"). More formally, for $I \in J(P)$ and $v \in P$, let $\mathbf{t}_v(I) = I \triangle \{v\}$ (symmetric difference) provided that $I \triangle \{v\}$ is an order ideal of $P$; otherwise, let $\mathbf{t}_v(I) = I$. The involutions $\mathbf{t}_x$ and $\mathbf{t}_y$ commute unless $x$ covers $y$ or $y$ covers $x$, and $\rho = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \cdots \circ \mathbf{t}_{v_n} : J(P) \to J(P)$ where $(v_1, \ldots, v_n)$ is any linear extension of $P$.

This setup generalizes nicely from order ideals (order-preserving 0-1 labelings) to order-preserving labelings $f : P \to [0,1]$ (the *order polytope* of Stanley). (No time today, but of independent interest.)

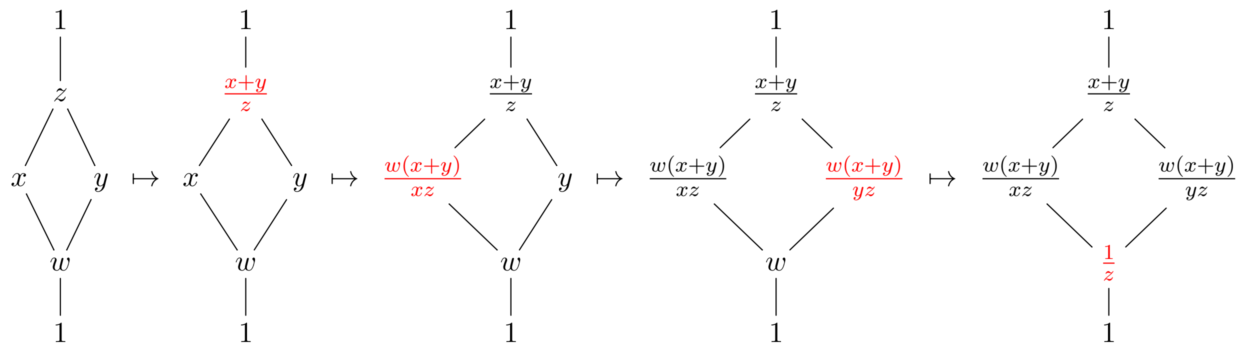
Detropicalizing these PL-toggles leads to an operator in the birational category: For any $v \in P$, define the **birational $v$-toggle** as the rational map $T_v : \mathbb{C}^{\widehat{P}} \dashrightarrow \mathbb{C}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

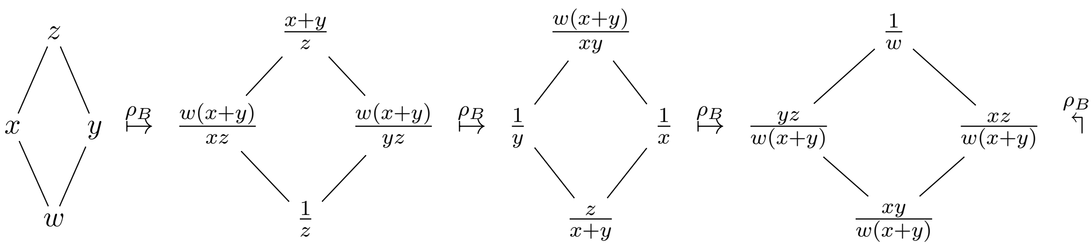
for all $w \in \widehat{P}$.

- Tropicalization $\cdot \mapsto +$ and $+ \mapsto \max$, recovers PL toggles.
- We can describe toggling at $v$ as: **(a) inverting** the label at $v$, **(b) multiplying** it with the **sum** of the labels at vertices **covered by** $v$, **(c) multiplying** it with the **harmonic sum** of the labels at vertices **covering** $v$.
- Note that $T_v$ changes *only* the label at $v$.
- These maps are involutions: $T_v^2 = \mathrm{id}$.

Here is one iteration, birationally toggling from top-to-bottom:




And here is a complete orbit:




**Surprises: (a)** This is still periodic on $[r] \times [s]$ with period $r + s$ [GrRo15, MR18]. This appears to generalize only to very special classes of posets [GrRo16, GrRo15].

**(b)** Homomesy generalizes to various products across an orbit being equal to 1 [GrRo15, MR18].

**(c)** There is a formula for iterating $\rho_B$ on a product of two chains in terms of families of NILPs [MR18].

**(d)** One can define a **noncommutative** version of this that still has periodicity [JR20].

For a 25-minute intro to my work, see the video https://www.youtube.com/watch?v=9TUajKFInwg of my talk at AlCove (http://www.math.uwaterloo.ca/~opecheni/alcove.htm).

J Propp, J Striker, N Williams, and I are running a workshop at BIRS, with virtual talks MWF of the last two weeks of October: http://www.birs.ca/events/2020/5-day-workshops/20w5164.

# References

[GrRo16] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. of Combin. **23**(1), #P1.33 (2016). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i1p33

[GrRo15] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. of Combin. **22**(3), #P3.40 (2015). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p40.

[JR20] Michael Joseph and Tom Roby, *Birational and noncommutative lifts of antichain toggling and rowmotion*, *to appear in* Algebraic Combin., arXiv:1909.09658.

[MR18] Gregg Musiker and Tom Roby "Paths to Understanding Birational Rowmotion on Products of Two Chains," to appear in *Algebraic Combinatorics*, https://arxiv.org/abs/1801.03877.

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22(3)** (2015), #P3.4, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4.

[Rob16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in Andrew Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652. Available at http://www.math.uconn.edu/~troby/research.php.