Paths to Understanding Birational Rowmotion on a Product of Two Chains

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Outline

1. Classical Rowmotion
2. Piecewise-linear (PL) and Birational Rowmotion
3. Formula in terms of Lattice Paths
4. Sketch of Proof
5. Applications (Periodicity and Homomesy)

We are grateful for the 2015 AIM workshop on *Dynamical Algebraic Combinatorics* and for Darij Grinberg’s implementation of birational rowmotion in SageMath.

http://math.umn.edu/~musiker/Birational18.pdf
Main Ideas

- The combinatorial rowmotion map has liftings (via a decomposition into involutions called *toggles*) to the piecewise-linear (order polytope) and then birational settings. Proving results at the birational level implies them at the other levels.

- For rectangular posets $P = [0, r] \times [0, s]$, we give a formula in terms of NILPs that allows us to compute $\rho_B^k$, the $k$th iteration of birational rowmotion.

- The key lemma is a Plücker-like relation satisfied by certain polynomials we define, proven by a colorful combinatorial bijection on pairs of NILPs (along the lines of Fulmek-Kleber).

- Using our formula, we obtain more direct proofs of the periodicity and “antipodal-reciprocity” of this system, as well as the first proof of “homomesy along files”.
Classical rowmotion is the rowmotion studied by Striker-Williams (2012), who coined the term. It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or “nonnesting partitions”, with relations to Lie theory).
- Propp-Roby (2015), as one of several actions that displays the homomesy phenomenon on the product of two chains.
Classical rowmotion

Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

Let $S$ be the following order ideal (indicated by the ●’s):

![Diagram](image-url)
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

Mark the complement in red.
Let $P$ be a finite poset. **Classical rowmotion** is the map

$r : J(P) \rightarrow J(P)$

sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

Mark $M$ (the minimal elements of the complement) in blue.
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal.

Remove the old order ideal:
Classical rowmotion

Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \to J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

$r(S)$ is the order ideal generated by $M$ (“everything below $M$”):

```
    M
   /|
  /  |  \
 M  M
   |
    M
```
Examples of Orbits of this Dynamic on Order Ideals.

$$\frac{(0 + 1 + 3 + 5 + 7 + 8)}{6} = 4$$

\[
(2+4+6+6+4+2) / 6 = 4
\]

$$\frac{(3+5+4+3+5+4)}{6} = 4$$
Definition ([PR15])

Given an (invertible) action $\tau$ on a finite set of objects $S$, call a statistic $f : S \to \mathbb{C}$ homomesic with respect to $(S, \tau)$ if the average of $f$ over each $\tau$-orbit $O$ is the same constant $c$ for all $O$, i.e.,

$$\frac{1}{\#O} \sum_{s \in O} f(s) = c$$

does not depend on the choice of $O$.

(Call $f$ c-mesic for short.) Greek for “same-middle”

Theorem ([PR15])

For the action of rowmotion on order ideals $J(P)$ of rectangular posets $P = [p] \times [q]$, the cardinality statistic is homomesic (with average $pq/2$).
Motivations and Connections

• Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].

• Birational rowmotion can be related to $Y$-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].

• The orbits of these actions all have natural *homomesic* statistics [PR15, EiPr13, EiPr14].

• Periodicity of these systems is generally nontrivial to prove.
Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.
Classical rowmotion: Periodicity

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

However, for some types of $P$, the order can be explicitly computed or bounded from above. See Striker-Williams [StWi11] (and the very recent Thomas-Williams [TW17]) for an exposition of known results.

- If $P$ is a $p \times q$-rectangle:

\[
\begin{array}{c}
(1,2) \\
(1,1) \quad (0,2) \\
(1,0) \quad (0,1) \\
(0,0)
\end{array}
\]

(shown here for $p = 2$ and $q = 3$), then $\text{ord}(r) = p + q$. 

Classical rowmotion: Periodicity (Example)

\[ S = (1, 2), \quad r(S) = (1, 2), \]
\[ (1, 1) \quad (0, 2) \]
\[ (1, 0) \quad (0, 1) \]
\[ (0, 0) \]

\[ r^2(S) = (1, 2), \quad r^3(S) = (1, 2), \]
\[ (1, 1) \quad (0, 2) \]
\[ (1, 0) \quad (0, 1) \]
\[ (0, 0) \]

\[ r^4(S) = (1, 2), \quad r^5(S) = (1, 2), \]
\[ (1, 1) \quad (0, 2) \]
\[ (1, 0) \quad (0, 1) \]
\[ (0, 0) \]
The **average value** along **antipodal (N-S, E-W) pairs** is 1 for both orbits, and is also **constant**, as

\[
\begin{array}{c}
\frac{1}{2} (1, 0) & 1 & (0, 1) \frac{1}{2} \\
(0, 0) & \end{array}
\]

on files.

We will generalize this to birational rowmotion.
There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $t_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - $S$ otherwise.
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- $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
- $S$ otherwise.

(“Try to add or remove $v$ from $S$, as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave $S$ fixed.”)
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  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - $S$ otherwise.

  (“Try to add or remove $v$ from $S$, as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave $S$ fixed.”)

- More formally, if $P$ is a poset and $v \in P$, then the $v$-toggle is the map $t_v : J(P) \to J(P)$ which takes every order ideal $S$ to:
  - $S \cup \{v\}$, if $v$ is not in $S$ but all elements of $P$ covered by $v$ are in $S$ already;
  - $S \setminus \{v\}$, if $v$ is in $S$ but none of the elements of $P$ covering $v$ is in $S$;
  - $S$ otherwise.

- Note that $t_v^2 = \text{id}$. 
Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

**Example:**

Start with this order ideal \(S\):

\[
\begin{array}{c}
(1, 1) \\
/ \\
(1, 0) \\
/ \\
(0, 0) \\
\end{array}
\quad \quad
\begin{array}{c}
(0, 1) \\
/ \\
(1, 0) \\
/ \\
(0, 0) \\
\end{array}
\]
Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

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\[
\mathbf{r} = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

**Example:**

First apply \(t_{(1,1)}\), which changes nothing:

\[
\begin{align*}
(1, 1) \\
(1, 0) & \quad (0, 1) \\
(0, 0)
\end{align*}
\]
Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[ r = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}. \]

**Example:**

Then apply \(t_{(1,0)}\), which removes \((1,0)\) from the order ideal:

```
     (1,1)
    /   \
(1,0)   (0,1)
  /     \
(0,0)
```
Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of $P$; this means a list of all elements of $P$ (each only once) such that $i < j$ whenever $v_i < v_j$.

Cameron and Fon-der-Flaass showed that

\[ r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}. \]

**Example:**

Then apply $t_{(0,1)}$, which adds $(0,1)$ to the order ideal:

```
(1,1)
(1,0)  (0,1)
(0,0)
```
Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
\mathbf{r} = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}.
\]

**Example:**

Finally apply \(t_{(0,0)}\), which changes nothing:
Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
r = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

**Example:**

So this is \(S \rightarrow r(S)\):

\[
(1, 1) \quad \quad \rightarrow \quad \quad (1, 1)
\]

\[
(1, 0) \quad \quad (0, 1) \quad \quad (1, 0) \quad \quad (0, 1)
\]

\[
(0, 0) \quad \quad (0, 0)
\]
The decomposition of classical rowmotion into toggles allows us to define a piecewise-linear (PL) version of rowmotion acting on functions on a poset.

Let $P$ be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.
Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let $P$ be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$. ( $\mathcal{J}(P) = \{f : P \to \{0,1\} : f$ is monotone$\}$)
The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let $P$ be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \succ x$ means $z$ covers $x$ and $w \prec x$ means $x$ covers $w$. 


Generalizing to the piecewise-linear setting

For each \( x \in P \), define the flip-map \( \sigma_x : \mathcal{O}(P) \to \mathcal{O}(P) \) sending \( f \) to the unique \( f' \) satisfying

\[
f'(y) = \begin{cases} 
  f(y) & \text{if } y \neq x, \\
  \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x) & \text{if } y = x,
\end{cases}
\]

where \( z \succ x \) means \( z \) covers \( x \) and \( w \prec x \) means \( x \) covers \( w \).

Note that the interval \([\min_{z \succ x} f(z), \max_{w \prec x} f(w)]\) is precisely the set of values that \( f'(x) \) could have so as to satisfy the order-preserving condition.

If \( f'(y) = f(y) \) for all \( y \neq x \), the map that sends

\[ f(x) \text{ to } \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x) \]

is just the affine involution that swaps the endpoints.
Example of flipping at a node

\[
\min_{z \succ x} f(z) + \max_{w \prec x} f(w) = .7 + .2 = .9
\]

\[
f(x) + f'(x) = .4 + .5 = .9
\]
Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get \textit{piecewise-linear rowmotion}: 

\[
\begin{array}{c}
\sigma_N \\
\sigma_W \\
\sigma_E \\
\sigma_S \\
\end{array}
\]

\[
\begin{array}{c}
.8 \\
.4 \\
.6 \\
.6 \\
\end{array}
\rightarrow
\begin{array}{c}
.6 \\
.4 \\
.3 \\
.3 \\
\end{array}
\rightarrow
\begin{array}{c}
.6 \\
.3 \\
.3 \\
.3 \\
\end{array}
\rightarrow
\begin{array}{c}
.6 \\
.3 \\
.4 \\
.4 \\
\end{array}
\rightarrow
\begin{array}{c}
.6 \\
.3 \\
.4 \\
.2 \\
\end{array}
\]

(We successively flip at \(N = (1, 1), \ W = (1, 0), \ E = (0, 1), \) and \(S = (0, 0)\) in order to get \(\rho_{PL}(f)\).)
How PL rowmotion generalizes classical rowmotion

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means $z$ covers $x$ and $w < \cdot x$ means $x$ covers $w$.

Example:

Start with this order ideal $S$:

```
(1, 1)
   / \  /
  /   /
(1, 0)  (0, 1)
   \  /
  /   /
(0, 0)
```
For each \( x \in P \), define the flip-map \( \sigma_x : O(P) \to O(P) \) sending \( f \) to the unique \( f' \) satisfying

\[
f'(y) = \begin{cases} 
  f(y) & \text{if } y \neq x, \\
  \min_{z > x} f(z) + \max_{w < x} f(w) - f(x) & \text{if } y = x,
\end{cases}
\]

where \( z \cdot > x \) means \( z \) covers \( x \) and \( w < \cdot x \) means \( x \) covers \( w \).

**Example:**

Translated to the PL setting:

![Diagram](image-url)
How PL rowmotion generalizes classical rowmotion

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where $z \cdot > x$ means $z$ covers $x$ and $w < \cdot x$ means $x$ covers $w$.

Example:

First apply $t_{(1,1)}$, which changes nothing:

```
1
/ \
0   1
/ \
0
```
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means $z$ covers $x$ and $w < \cdot x$ means $x$ covers $w$.

**Example:**

Then apply $t_{(1,0)}$, which removes $(1,0)$ from the order ideal:

```
          1
         / \  \
        1  1  1
       / \  / \  \\
      1  1 0
```
For each \( x \in P \), define the flip-map \( \sigma_x : \mathcal{O}(P) \to \mathcal{O}(P) \) sending \( f \) to the unique \( f' \) satisfying

\[
f'(y) = \begin{cases} 
  f(y) & \text{if } y \neq x, \\
  \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x,
\end{cases}
\]

where \( z \cdot > x \) means \( z \) covers \( x \) and \( w < \cdot x \) means \( x \) covers \( w \).

**Example:**

Then apply \( t_{(0,1)} \), which adds \((0, 1)\) to the order ideal:
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \rightarrow \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means $z$ covers $x$ and $w < \cdot x$ means $x$ covers $w$.

**Example:**

Finally apply $t_{(0,0)}$, which changes nothing:
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} 
  f(y) & \text{if } y \neq x, \\
  \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x) & \text{if } y = x,
\end{cases}$$

where $z \succ x$ means $z$ covers $x$ and $w \prec x$ means $x$ covers $w$.

Example:

So this is $S \rightarrow r(S)$:
In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at $x$ replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with $f'$, where

$$f'(x) := \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x)$$

We can “detropicalize” this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$$\min(z_i) = -\max(-z_i),$$

to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w \prec x} f(w)}{f(x) \sum_{z \succ x} \frac{1}{f(z)}}$$
Let $P$ be a finite poset. We define $\hat{P}$ to be the poset obtained by adjoining two new elements $\hat{0}$ and $\hat{1}$ to $P$ and forcing
- $\hat{0}$ to be less than every other element, and
- $\hat{1}$ to be greater than every other element.

Let $K$ be a field.

A $K$-labelling of $P$ will mean a function $\hat{P} \rightarrow K$.

The values of such a function will be called the labels of the labelling.

We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\hat{P}$.

For any $v \in P$, define the birational $v$-toggle as the rational map

$$T_v : K^{\hat{P}} \dasharrow K^{\hat{P}}$$

defined by

$$(T_v f)(w) = \frac{\sum_{\hat{P} \ni u < v} f(u)}{f(v) \sum_{\hat{P} \ni u > v} \frac{1}{f(u)}}$$

for $w = v$.

(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)
Birational rowmotion: definition

- For any \( v \in P \), define the **birational** \( v \)-toggle as the rational map
  \[
  T_v : \mathbb{K} \hat{P} \to \mathbb{K} \hat{P}
  \]
  defined by
  \[
  (T_v f)(w) = \frac{\sum_{u < v} f(u)}{f(v) \sum_{u > v} \frac{1}{f(u)}}
  \]
  for \( w = v \).
- Notice that this is a **local change** only to the label at \( v \).
- We have \( T_v^2 = id \) (on the range of \( T_v \)), and \( T_v \) is a birational map.
For any \( v \in P \), define the **birational v-toggle** as the rational map

\[
T_v : \mathbb{K}^\hat{P} \to \mathbb{K}^\hat{P} \text{ defined by } (T_v f)(w) = \frac{\sum_{u < v} f(u)}{f(v) \sum_{u > v} \frac{1}{f(u)}} \text{ for } w = v.
\]

Notice that this is a **local change** only to the label at \( v \).

We have \( T_v^2 = id \) (on the range of \( T_v \)), and \( T_v \) is a birational map.

We define **birational rowmotion** as the rational map

\[
\rho_B := T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_n} : \mathbb{K}^\hat{P} \to \mathbb{K}^\hat{P},
\]

where \( (v_1, v_2, ..., v_n) \) is a linear extension of \( P \).

This is indeed independent of the linear extension, because

- \( T_v \) and \( T_w \) commute whenever \( v \) and \( w \) are incomparable (even whenever they are not adjacent in the Hasse diagram of \( P \));
- we can get from any linear extension to any other by switching incomparable adjacent elements.

This is originally due to Einstein and Propp [EiPr13, EiPr14], following the lead of Kirillov-Berenstein [KiBe95].
Example:

Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

\[
\begin{array}{c|c}
\text{poset} & \text{labelling} \\
\hline
\hat{1} & 1 \\
(1, 1) & \downarrow \\
(1, 0) & \downarrow \\
(0, 1) & \downarrow \\
(0, 0) & \downarrow \\
\hat{0} & 1 \\
\end{array}
\]

That is, toggle in the order “top, left, right, bottom”.
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>poset</th>
<th>labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{1}$</td>
<td>1</td>
</tr>
<tr>
<td>$(1,1)$</td>
<td>$z$</td>
</tr>
<tr>
<td>$(1,0)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$(0,1)$</td>
<td>$y$</td>
</tr>
<tr>
<td>$(0,0)$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$ using the linear extension $((1,1), (1,0), (0,1), (0,0))$.

That is, toggle in the order “top, left, right, bottom”.
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>original labelling $f$</th>
<th>labelling $T_{(1,1)}f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( z )</td>
<td>( z )</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>( w )</td>
<td>( w )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$. 
Example:

Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

\[
\begin{array}{c|c}
\text{original labelling } f & \text{labelling } T_{(1,0)} T_{(1,1)} f \\
1 & 1 \\
\downarrow & \downarrow \\
z & \frac{(x+y)}{z} \\
\begin{array}{c}
x \\
y \\
w \\
1
\end{array} & \begin{array}{c}
w(x+y) \\
\frac{xz}{y} \\
w \\
1
\end{array}
\end{array}
\]

We are using \( \rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}. \)
Example:

Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

<table>
<thead>
<tr>
<th>original labelling ( f )</th>
<th>labelling ( T_{(0,1)} T_{(1,0)} T_{(1,1)} f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( z )</td>
<td>( \frac{x+y}{z} )</td>
</tr>
<tr>
<td>( w )</td>
<td>( \frac{w(x+y)}{xz} )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

We are using \( \rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)} \).
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>original labelling $f$</th>
<th>labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$z$</td>
<td>$(x+y)$</td>
</tr>
<tr>
<td>$x$</td>
<td>$w(x+y)$</td>
</tr>
<tr>
<td>$y$</td>
<td>$w(x+y)$</td>
</tr>
<tr>
<td>$w$</td>
<td>$\frac{1}{z}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$. 
Example: Iterating this procedure we get

\[ \rho_B f = \frac{(x+y)w}{xz} \]

\[ \rho_B^2 f = \frac{(x+y)w}{yz} \]

\[ \rho_B^3 f = \frac{(x+y)w}{xz} \]

\[ \rho_B^4 f = \frac{xz}{(x+y)w} \]

Notice that \( \rho_B^4 f = f \), which generalizes to \( \rho_B^{r+s+2} f = f \) for \( P = [0, r] \times [0, s] \) [Grinberg-R 2015]. Notice also "antipodal reciprocity".
Example: Iterating this procedure we get

\[
\rho_B f = \frac{(x+y)w}{xz}, \quad \frac{(x+y)w}{yz}, \quad \frac{1}{z}, \quad \frac{1}{w}, \quad \frac{xy}{(x+y)w}, \quad \frac{xz}{(x+y)w}, \quad \frac{1}{x}, \quad \frac{z}{x+y}, \quad \frac{y}{w}.
\]

Notice that \( \rho_4^4 f = f \), which generalizes to \( \rho_B^{r+s+2} f = f \) for \( P = [0, r] \times [0, s] \) [Grinberg-R 2015]. Notice also “antipodal reciprocity”. 
Birational homomesy on files

The poset $[0, 1] \times [0, 1]$ has three files, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$
\prod_{k=1}^{4} \rho_B^k(f)(1, 0) = \frac{(x + y)w}{xz} \frac{1}{y} \frac{yz}{(x + y)w} (x) = 1,
$$

Each of these products equalling one is the manifestation, for the poset of a product of two chains, of homomesy along files at the birational level.
The poset $[0, 1] \times [0, 1]$ has three files, \{(1, 0)\}, \{(0, 0), (1, 1)\}, and \{(0, 1)\}.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_{B}^{k}(f)(1, 0) = \frac{(x + y)w}{xz} \frac{1}{y} \frac{yz}{(x + y)w} \quad (x) = 1,$$

$$\prod_{k=1}^{4} \rho_{B}^{k}(f)(0, 0)\rho_{B}^{k}(f)(1, 1) =$$

$$\frac{1}{z} \frac{x + y}{z} \frac{z}{x + y} \frac{(x + y)w}{xy} \frac{xy}{(x + y)w} \frac{1}{w} \quad (w) (z) = 1,$$
The poset $[0, 1] \times [0, 1]$ has three files, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_B^k(f)(1, 0) = \frac{(x + y)w}{xz} \frac{1}{y} \frac{yz}{(x + y)w} \quad (x) = 1,$$

$$\prod_{k=1}^{4} \rho_B^k(f)(0, 0)\rho_B^k(f)(1, 1) = \frac{1}{z} \frac{x + y}{z} \frac{z}{x + y} \frac{(x + y)w}{xy} \frac{xy}{(x + y)w} \frac{1}{w} \quad (w) (z) = 1,$$

$$\prod_{k=1}^{4} \rho_B^k(f)(0, 1) = \frac{(x + y)w}{yz} \frac{1}{x} \frac{xz}{(x + y)w} \quad (y) = 1.$$
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$$\prod_{k=1}^{4} \rho_{B}^{k}(f)(1, 0) = \frac{(x + y)w}{xz} \frac{1}{y} \frac{yz}{(x + y)w} (x) = 1,$$

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$$\frac{1}{z} \frac{x + y}{z} \frac{z}{x + y} \frac{(x + y)w}{xy} \frac{xy}{(x + y)w} \frac{1}{w} (w) (z) = 1,$$

$$\prod_{k=1}^{4} \rho_{B}^{k}(f)(0, 1) = \frac{(x + y)w}{yz} \frac{1}{x} \frac{xz}{(x + y)w} (y) = 1.$$

Each of these products equalling one is the manifestation, for the poset of a product of two chains, of homomesy along files at the birational level.
We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r + s + 1]$. 
We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i, j)$ for $(i, j) \in [0, r] \times [0, s]$ and $k \in [0, r + s + 1]$.

1) Let $\bigwedge_{(m,n)} := \{(u, v) : (u, v) \geq (m, n)\}$ be the principal order filter at $(m, n)$, $\bigcap^k_{(m,n)}$ be the rank-selected subposet, of elements in $\bigwedge_{(m,n)}$ whose rank (within $\bigwedge_{(m,n)}$) is at least $k - 1$ and whose corank is at most $k - 1$. 

\[
\begin{array}{c}
(2, 2) \\
(2, 1) & (1, 2) \\
(2, 0) & (1, 1) & (0, 2) \\
(1, 0) & (0, 1) \\
(0, 0)
\end{array}
\]
2) Let \( s_1, s_2, \ldots, s_k \) be the \( k \) minimal elements and let \( t_1, t_2, \ldots, t_k \) be the \( k \) maximal elements of \( \Box^k_{(m,n)} \). (For \( k \leq \min\{r - m, s - n\} + 1 \).)
2) Let $s_1, s_2, \ldots, s_k$ be the $k$ minimal elements and let $t_1, t_2, \ldots, t_k$ be the $k$ maximal elements of $\square^k_{(m,n)}$. (For $k \leq \min\{r - m, s - n\} + 1$.)

Let $A_{ij} := \frac{\sum_{z < (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i, j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\hat{P}$).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_k(m, n)$ in terms of the $A_{ij}$'s as follows.
We define a lattice path of length \( k \) within \( P = [0, r] \times [0, s] \) to be a sequence \( v_1, v_2, \ldots, v_k \) of elements of \( P \) such that each difference of successive elements \( v_i - v_{i-1} \) is either \((1, 0)\) or \((0, 1)\) for each \( i \in [k] \). We call a collection of lattice paths non-intersecting if no two of them share a common vertex.
3) Let $S_k(m, n)$ be the set of non-intersecting lattice paths in $\bigcirc^k_{(m,n)}$, from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. Let $\mathcal{L} = \{L_1, L_2, \ldots L_k\} \in S_k^k(m, n)$ denote a $k$-collection of such lattice paths.

4) Define $\varphi_k(m, n) := \sum_{\mathcal{L} \in S_k^k(m, n)} \prod_{(i,j) \in \bigcirc^k_{(m,n)}} A_{ij}$.

**Theorem(*)**: 

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$$

**EG:** $\rho_B^2(1,1) = \frac{\varphi_1(0,0)}{\varphi_2(0,0)}$.

= sum of 6 quartic terms in $A_{ij}$

= \frac{A_{20} + A_{11} + A_{02}}{\varphi_1(0,0)}$ 

\begin{itemize}
  \item \[\rho_B^2(1,1) = \frac{\varphi_1(0,0)}{\varphi_2(0,0)}.\]
\end{itemize}

(*) Caveats explained and general statement given in the next few slides.
Fix $k \in [0, r + s + 1]$, and let $\rho^{k+1}_B(i,j)$ denote the rational function associated to the poset element $(i,j)$ after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0, r] \times [0, s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k - i]_+ + [k - j]_+$. 
Main Theorem (M-Roby 2018)

Fix \( k \in [0, r + s + 1] \), and let \( \rho_{B}^{k+1}(i, j) \) denote the rational function associated to the poset element \((i, j)\) after \((k + 1)\) applications of the birational rowmotion map to the generic initial labeling of \( P = [0, r] \times [0, s] \). Set \([\alpha]_+ := \max\{\alpha, 0\}\) and \( M = [k - i]_+ + [k - j]_+ \).

(a1) When \( M = 0 \), i.e., \((i - k, j - k)\) still lies in the poset \([0, r] \times [0, s] \):

\[
\rho_{B}^{k+1}(i, j) = \frac{\varphi_k(i - k, j - k)}{\varphi_{k+1}(i - k, j - k)}
\]

where \( \varphi_t(v, w) \) is defined in 4) above.
Fix $k \in [0, r + s + 1]$, and let $\rho^{k+1}_B(i, j)$ denote the rational function associated to the poset element $(i, j)$ after $(k + 1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0, r] \times [0, s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k - i]_+ + [k - j]_+.$

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$$\rho^{k+1}_B(i, j) = \frac{\varphi_k(i - k, j - k)}{\varphi_{k+1}(i - k, j - k)}$$

where $\varphi_t(v, w)$ is defined in 4) above.

(a2) When $0 < M \leq k$:

$$\rho^{k+1}_B(i, j) = \mu([k-j]_+,[k-i]_+) \left( \frac{\varphi_{k-M}(i - k + M, j - k + M)}{\varphi_{k-M+1}(i - k + M, j - k + M)} \right)$$

where $\mu^{(a, b)}$ is the operator that takes a rational function in $\{A(u,v)\}$ and simply shifts each index in each factor of each term: $A(u,v) \mapsto A(u-a,v-b)$. 
Main Theorem (M-Roby 2018)

Fix $k \in [0, r + s + 1]$ and set $M = [k - i]_+ + [k - j]_+$. After $(k + 1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0, r] \times [0, s]$ we get:

(a) When $0 \leq M \leq k$:

$$\rho_B^{k+1}(i, j) = \mu^{([k-j]_+, [k-i]_+)} \left( \frac{\varphi_{k-M}(i - k + M, j - k + M)}{\varphi_{k-M+1}(i - k + M, j - k + M)} \right)$$

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where $\varphi_t(v, w)$ and $\mu^{(a, b)}$ are as defined above.

(b) When $M \geq k$: $\rho_B^{k+1}(i, j) = 1/\rho_B^{k-i-j}(r - i, s - j)$, which is well-defined by part (a).

Remark: We prove that our formulae in (a) and (b) agree when $M = k$, allowing us to give a new proof of periodicity:

$$\rho_B^{r+s+2+d} = \rho_B^d$$; thus we get a formula for all iterations of the birational rowmotion map.
Corollaries of the Main Theorem

**Corollary**

For \( k \leq \min\{i, j\} \), \( \rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}. \)

**Corollary ([GrRo15, Thm. 30])**

The birational rowmotion map \( \rho_B \) on the product of two chains \( P = [0, r] \times [0, s] \) is periodic, with period \( r + s + 2 \).

**Corollary ([GrRo15, Thm. 32])**

The birational rowmotion map \( \rho_B \) on the product of two chains \( P = [0, r] \times [0, s] \) satisfies the following reciprocity:

\[
\rho_B^{i+j+1} = 1 / \rho_B^0(r - i, s - j) = \frac{1}{x_{r-i,s-j}}.
\]
Corollaries of the Main Theorem

**Theorem**

Given a file $F$ in $[0, r] \times [0, s]$, \[ \prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i,j) = 1. \]

The poset $[0, 1] \times [0, 1]$ has three files, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

\[
\prod_{k=1}^{4} \rho_B^k(f)(1, 0) = \frac{(x + y)w}{xz} \cdot \frac{1}{y} \cdot \frac{yz}{(x + y)w} \cdot (x) = 1,
\]
Corollaries of the Main Theorem

**Theorem**

Given a file $F$ in $[0, r] \times [0, s]$, \[ \prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1. \]

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Multiplying over all iterates of birational rowmotion in a given file:

\[ \prod_{k=1}^{4} \rho^k_B(f)(1, 0) = \frac{(x + y)w}{xz} \frac{1}{y} \frac{yz}{(x + y)w} (x) = 1, \]

\[ \prod_{k=1}^{4} \rho^k_B(f)(0, 0)\rho^k_B(f)(1, 1) = \]

\[ \frac{1}{z} \frac{x + y}{z} \frac{z}{x + y} \frac{(x + y)w}{xy} \frac{xy}{(x + y)w} \frac{1}{w} (w) (z) = 1, \]

\[ \prod_{k=1}^{4} \rho^k_B(f)(0, 1) = \frac{(x + y)w}{yz} \frac{1}{x} \frac{xz}{(x + y)w} (y) = 1. \]
Example

We use our main theorem to compute $\rho_{B}^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2,$ and $j = 1$ throughout.
We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2,$ and $j = 1$ throughout.

Recall that in the case where shifting $(i, j) \mapsto (i - k, j - k)$ (straight down by $2k$ ranks) still gives a point in $P$, we get a simpler formula.

**Corollary:** For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$
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**Corollary:** For $k \leq \min\{i, j\}$, $\rho_{B}^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$

When $k = 0$, $M = 0$ and we get

$$\rho_{B}^{1}(2, 1) = \frac{\varphi_0(2, 1)}{\varphi_1(2, 1)} = \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22} + A_{31}}.$$
We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2,$ and $j = 1$ throughout.

Recall that in the case where shifting $(i, j) \mapsto (i - k, j - k)$ (straight down by $2k$ ranks) still gives a point in $P$, we get a simpler formula.

**Corollary:** For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$

When $k = 1$, we still have $M = 0$, and $\rho_B^2(2, 1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} = \frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}$.

For the numerator, $s_1 = (1, 0), t_1 = (3, 2)$, and there are six lattice paths from $s_1$ to $t_1$, each of which covers 5 elements and leaves 4 uncovered.

For the denominator, $s_1 = (2, 0), s_2 = (1, 1), t_1 = (3, 1), \text{ and } t_2 = (2, 2)$, and each pair of lattice paths leaves exactly one element uncovered.
When \( k = 1 \), we still have \( M = 0 \), and \( \rho^2_B(2, 1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} = \)

\[
\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.
\]

For the numerator, \( s_1 = (1, 0) \), \( t_1 = (3, 2) \), and there are six lattice paths from \( s_1 \) to \( t_1 \), each of which covers 5 elements and leaves 4 uncovered.
Example

When \( k = 1 \), we still have \( M = 0 \), and \( \rho_2^2(2, 1) = \frac{\varphi_1(1, 0)}{\varphi_2(1, 0)} = \frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}. \)

For the denominator, \( s_1 = (2, 0) \), \( s_2 = (1, 1) \), \( t_1 = (3, 1) \), and \( t_2 = (2, 2) \), and each pair of lattice paths leaves exactly one element uncovered.
Example

In the case where shifting \((i, j) \mapsto (i - k, j - k)\) (straight down by \(2k\) ranks) gives a point outside of \(P\), we must also apply a \(\mu\)-translation. We use our main theorem to compute \(\rho_B^{k+1}(2, 1)\) for \(P = [0, 3] \times [0, 2]\) for \(k = 0, 1, 2, 3, 4, 5, 6\). Here \(r = 3, s = 2, i = 2,\) and \(j = 1\) throughout.

When \(k = 2\), we get \(M = [2 - 2]_+ + [2 - 1]_+ = 1 \leq 2 = k\). So by part (a) of the main theorem we have

\[
\rho_B^3(2, 1) = \mu^{(1,0)} \left[ \begin{array}{c} \varphi_1(1, 0) \\ \varphi_2(1, 0) \end{array} \right] = (\text{just shifting indices in the } k = 1 \text{ formula})
\]

\[
\frac{A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}}{A_{02} + A_{11} + A_{20}}.
\]
Example

In the case where shifting \((i, j) \mapsto (i - k, j - k)\) (straight down by \(2k\) ranks) gives a point outside of \(P\), we must also apply a \(\mu\)-translation.

We use our main theorem to compute \(\rho_B^k(2, 1)\) for \(P = [0, 3] \times [0, 2]\) for \(k = 0, 1, 2, 3, 4, 5, 6\). Here \(r = 3, s = 2, i = 2,\) and \(j = 1\) throughout.

When \(k = 3\), we get \(M = [3 - 2]_+ + [3 - 1]_+ = 3 = k\). Therefore,

\[
\rho_B^4(2, 1) = \mu^{(2,1)} \left[ \frac{\varphi_0(2, 1)}{\varphi_1(2, 1)} \right] = \mu^{(2,1)} \left[ \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22} + A_{31}} \right] = \frac{A_{00}A_{01}A_{10}A_{11}}{A_{01} + A_{10}}.
\]

In this situation, we can also use part (b) of the main theorem to get

\[
\rho_B^4(2, 1) = 1/\rho_B^{3-2-1}(3 - 2, 2 - 1) = 1/\rho_B^0(1, 1) = \frac{1}{x_{11}}.
\]

The equality between these two expressions is easily checked.
In the case where $\mu$-translation would lead to negative subscripts for the $\phi$'s, i.e. $M > k$, part (a) of the Theorem does not apply.

We use our main theorem to compute $\rho_{B}^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3$, $s = 2$, $i = 2$, and $j = 1$ throughout.

When $k = 4$, we get $M = [4 - 2]_{+} + [4 - 1]_{+} = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_{B}^{5}(2,1) = 1/\rho_{B}^{4-2-1}(3-2, 2-1) = 1/\rho_{B}^{1}(1,1) = \frac{\phi_{1}(1,1)}{\phi_{0}(1,1)} = \frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}}.$$ 

Each term in the numerator is associated with one of the three lattice paths from $(1, 1)$ to $(3, 2)$ in $P$, while the denominator is just the product of all $A$-variables in the principal order filter $\bigvee (1,1)$. 


In the case where \( \mu \)-translation would lead to negative subscripts for the \( \varphi \)'s, i.e. \( M > k \), part (a) of the Theorem does not apply.

We use our main theorem to compute \( \rho_{k+1}^B(2,1) \) for \( P = [0,3] \times [0,2] \) for \( k = 0, 1, 2, 3, 4, 5, 6 \). Here \( r = 3, s = 2, i = 2, \) and \( j = 1 \) throughout.

When \( k = 5 \), we get \( M = [5 - 2]_+ + [5 - 1]_+ = 7 > k \). Therefore, by part (b) of the main theorem, then part (a),

\[
\rho_B^6(2,1) = 1/\rho_B^{5-2-1}(3 - 2, 2 - 1) = 1/\rho_B^2(1,1) = \frac{\varphi_2(0,0)}{\varphi_1(0,0)} =
\]

\[
(A_{02}A_{12} + A_{02}A_{21} + A_{11}A_{21} + A_{30}A_{02} + A_{30}A_{11} + A_{30}A_{20})/(A_{01}A_{11}A_{21}A_{12}A_{22} + A_{01}A_{11}A_{02}A_{30}A_{12}A_{22} + A_{01}A_{11}A_{02}A_{30}A_{12}A_{31} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{31} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{10}A_{20}A_{11}A_{30}A_{20} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{10}A_{20}A_{11}A_{30}A_{20})
\]
In the case where $\mu$-translation would lead to negative subscripts for the $\varphi$’s, i.e. $M > k$, part (a) of the Theorem does not apply.

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $k = 6$, we get $M = [6 - 2]_+ + [6 - 1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2, 1) = 1/\rho_B^{6-2-1}(3 - 2, 2 - 1) = 1/\rho_B^3(1, 1) = \mu^{(1,1)} \left[ \frac{\varphi_1(1, 1)}{\varphi_0(1, 1)} \right]$$

$$= \mu^{(1,1)} \left[ \frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}} \right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x$$
When \( k = 6 \), we get \( M = [6 - 2]_+ + [6 - 1]_+ = 9 > k \). Therefore, by part (b) of the main theorem, then part (a),

\[
\rho_B(2, 1) = \frac{1}{\rho_B(6 - 2 - 1)(3 - 2, 2 - 1)} = \frac{1}{\rho_B(1, 1)} = \mu^{(1,1)} \left[ \frac{\phi_1(1, 1)}{\phi_0(1, 1)} \right]
\]

\[
= \mu^{(1,1)} \left[ \frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}} \right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x_21
\]

The lattice paths involved here are the same as for the \( k = 4 \) computation.

We can deduce this by \( A_{00} = 1/x_{00}, A_{10} = x_{00}/x_{10}, A_{01} = x_{00}/x_{01}, A_{11} = (x_{10} + x_{01})/x_{11}, A_{20} = x_{10}/x_{20}, \) and \( A_{21} = (x_{20} + x_{11})/x_{21}. \)

Periodicity also kicks in: \( \rho_B(2, 1) = \rho_B(0, 1) = x_{21} \) using \((r + s + 2) = 7.\)
Sketch of Proof

By definition of birational rowmotion,

\[
\rho_B^{k+1}(i, j) = \frac{\left( \rho_B^k(i, j - 1) + \rho_B^k(i - 1, j) \right) \cdot \left( \rho_B^{k+1}(i + 1, j) \parallel \rho_B^{k+1}(i, j + 1) \right)}{\rho_B^k(i, j)}
\]

where

\[
A \parallel B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.
\]
Sketch of Proof

By definition of birational rowmotion,

\[ \rho_B^{k+1}(i,j) = \frac{\left( \rho_B^k(i,j - 1) + \rho_B^k(i, j - 1) \right) \cdot \left( \rho_B^{k+1}(i + 1, j) \parallel \rho_B^{k+1}(i, j + 1) \right)}{\rho_B^k(i,j)} \]

where

\[ A \parallel B = \frac{1}{A^{+1} + B^{-1}}. \]

By induction on \( k \), and the fact that we apply birational rowmotion from top to bottom, we can apply algebraic manipulations to reduce our result to proving the following Plücker-like identity:

\[ \varphi_k(i-k, j-k) \varphi_{k-1}(i-k+1, j-k+1) = \]

\[ \varphi_k(i - k, j - k + 1) \varphi_{k-1}(i - k + 1, j - k) + \varphi_k(i - k + 1, j - k) \varphi_{k-1}(i - k, j - k + 1). \]
It is sufficient to verify the following Plücker-like identity

\[ \varphi_k(i-k, j-k) \varphi_{k-1}(i-k+1, j-k+1) = \varphi_k(i-k, j-k+1) \varphi_{k-1}(i-k+1, j-k) + \varphi_k(i-k+1, j-k) \varphi_{k-1}(i-k, j-k+1). \]

**Example (k=5):**
We build **bounce paths** and **twigs** (paths of length one from \( \circ \) to \( \times \)) starting from the bottom row of \( \circ \)'s.

**Example (k=5):**

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Sketch of Proof

We then reverse the colors along the \((k - 2)\) twigs and the one bounce path from \(\circ\) to \(\times\) (rather than \(\circ\) to \(\circ\)).

Example \((k=5)\):

\[
\begin{array}{ccccccc}
\times & \times & \times & \times & \times \\
\circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
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\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
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\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]
Swap in the new colors and shift the \( \circ \)'s and \( \times \)'s in the bottom two rows.

Example (k=5):
Sketch of Proof

\[ \varphi_k(i - k, j - k) \varphi_{k-1}(i - k + 1, j - k + 1) = \]
\[ \varphi_k(i - k, j - k + 1) \varphi_{k-1}(i - k + 1, j - k) \]
\[ + \varphi_k(i - k + 1, j - k) \varphi_{k-1}(i - k, j - k + 1). \]

Example (k=5):

[Diagram showing a pattern with crosses and circles]
**Theorem**

Given a file \( F \) in \([0, r] \times [0, s]\),

\[
\prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i,j) = 1.
\]

**Sketch of Proof:**
Double-counting argument, followed by color-coded cancellations and several entries immediately equal to 1, as in ensuing table.
Further Application: Birational File Homomesy

Theorem

Given a file $F$ in $[0, r] \times [0, s]$, \[ \prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i,j) = 1. \]

Sketch of Proof: Double-counting argument, followed by color-coded cancellations and several entries immediately equal to 1, as in ensuing table.
Further Application: Birational File Homomesy

Let \((r, s) = (4, 3), \ d = 2, \) and consider the file \(F = \{(4, 2), (3, 1), (2, 0)\}\). The following table displays the values of \(\rho_B^k(i, j)\) for \(0 \leq k \leq 8, (i, j) \in F\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>((4, 2))</th>
<th>((3, 1))</th>
<th>((2, 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k = 0)</td>
<td>(\varphi_0(4, 2))</td>
<td>(\varphi_0(3, 1))</td>
<td>(\varphi_0(2, 0))</td>
</tr>
<tr>
<td>(\varphi_1(4, 2) = 1)</td>
<td>(\varphi_1(3, 1))</td>
<td>(\varphi_1(2, 0))</td>
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</tr>
<tr>
<td>(\varphi_2(3, 1) = 1)</td>
<td>(\varphi_2(2, 0))</td>
<td>(\varphi_2(2, 0))</td>
<td></td>
</tr>
<tr>
<td>(\varphi_3(2, 0) = 1)</td>
<td>(\varphi_2(2, 0))</td>
<td>(\varphi_2(2, 0))</td>
<td></td>
</tr>
<tr>
<td>(k = 1)</td>
<td>(\mu^{(1, 0)})</td>
<td>(\mu^{(2, 0)})</td>
<td>(\mu^{(1, 0)})</td>
</tr>
<tr>
<td>(\varphi_2(2, 0))</td>
<td>(\varphi_1(2, 0))</td>
<td>(\varphi_0(2, 0))</td>
<td></td>
</tr>
<tr>
<td>(\varphi_3(2, 0) = 1)</td>
<td>(\varphi_2(2, 0))</td>
<td>(\varphi_2(2, 0))</td>
<td></td>
</tr>
<tr>
<td>(k = 2)</td>
<td>(\mu^{(3, 1)})</td>
<td>(\mu^{(3, 1)})</td>
<td>(\mu^{(2, 3)})</td>
</tr>
<tr>
<td>(\varphi_0(3, 1))</td>
<td>(\varphi_1(3, 1))</td>
<td>(\varphi_0(2, 3) = 1)</td>
<td></td>
</tr>
<tr>
<td>(\varphi_1(2, 0))</td>
<td>(\varphi_1(2, 0))</td>
<td>(\varphi_3(0, 1) = 1)</td>
<td></td>
</tr>
<tr>
<td>(\varphi_2(3, 1) = 1)</td>
<td>(\varphi_1(3, 1))</td>
<td>(\varphi_2(0, 1))</td>
<td></td>
</tr>
<tr>
<td>(k = 3)</td>
<td>(\mu^{(4, 2)})</td>
<td>(\mu^{(4, 2)})</td>
<td>(\mu^{(0, 1)})</td>
</tr>
<tr>
<td>(\varphi_1(4, 2) = 1)</td>
<td>(\varphi_2(0, 1))</td>
<td>(\varphi_3(0, 1) = 1)</td>
<td></td>
</tr>
<tr>
<td>(\varphi_0(0, 1))</td>
<td>(\varphi_1(0, 1))</td>
<td>(\varphi_2(0, 1))</td>
<td></td>
</tr>
<tr>
<td>(k = 7)</td>
<td>(\mu^{(0, 1)})</td>
<td>(\mu^{(0, 1)})</td>
<td>(\mu^{(1, 2)})</td>
</tr>
<tr>
<td>(\varphi_1(0, 1))</td>
<td>(\varphi_2(0, 1))</td>
<td>(\varphi_2(1, 2) = 1)</td>
<td></td>
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<tr>
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<td>(\varphi_1(0, 1))</td>
<td>(\varphi_1(1, 2))</td>
<td></td>
</tr>
<tr>
<td>(k = 8)</td>
<td>(\mu^{(0, 1)})</td>
<td>(\mu^{(1, 2)})</td>
<td>(\mu^{(2, 3)})</td>
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<tr>
<td>(\varphi_1(0, 1))</td>
<td>(\varphi_1(1, 2))</td>
<td>(\varphi_2(2, 3) = 1)</td>
<td></td>
</tr>
<tr>
<td>(\varphi_0(0, 1))</td>
<td>(\varphi_0(1, 2))</td>
<td>(\varphi_0(2, 3))</td>
<td></td>
</tr>
</tbody>
</table>


