Paths to Understanding Birational Rowmotion on a Product of Two Chains

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Describing joint research with Gregg Musiker[1] (UMN)

Algebraic and Enumerative Combinatorics in Okayama
Okayama University, Japan

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Outline

1. Classical Rowmotion
2. Piecewise-linear (PL) and Birational Rowmotion
3. Formula in terms of Lattice Paths
4. Sketch of Proof
5. Applications (Periodicity and Homomesy)

We are grateful for the 2015 AIM workshop on *Dynamical Algebraic Combinatorics* and for Darij Grinberg's implementation of birational rowmotion in SageMath.

`http://math.umn.edu/~musiker/Birational18.pdf`

Main Ideas

- The combinatorial rowmotion map has liftings (via a decomposition into involutions called *toggles*) to the piecewise-linear (order polytope) and then birational settings. Proving results at the birational level implies them at the other levels.
- For rectangular posets $P = [0, r] \times [0, s]$, we give a formula in terms of NILPs that allows us to compute ρ_B^k, the kth iteration of birational rowmotion.
- The key lemma is a Plücker-like relation satisfied by certain polynomials we define, proven by a colorful combinatorial bijection on pairs of NILPs (along the lines of Fulmek-Kleber).
- Using our formula, we obtain more direct proofs of the periodicity and "antipodal-reciprocity" of this system, as well as the first proof of "homomesy along files".

Classical rowmotion is the rowmotion studied by Striker-Williams (2012), who coined the term. It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).
- Propp-Roby (2015), as one of several actions that displays the homomesy phenomenon on the product of two chains.

Let P be a finite poset. **Classical rowmotion** is the map
$\mathbf{r} : J(P) \longrightarrow J(P)$

sending every order ideal S to a new order ideal $\mathbf{r}(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let S be the following order ideal

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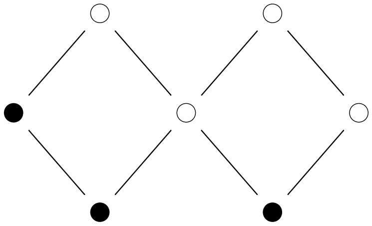

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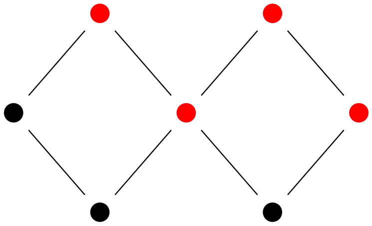
Mark the complement in red.



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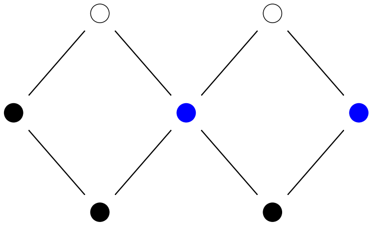
Mark M (the minimal elements of the complement) in blue.



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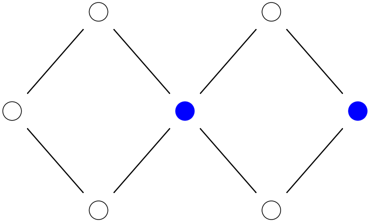
Remove the old order ideal:



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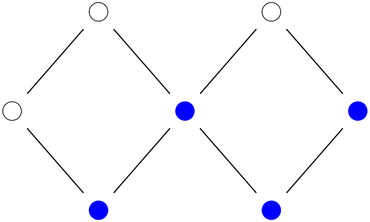
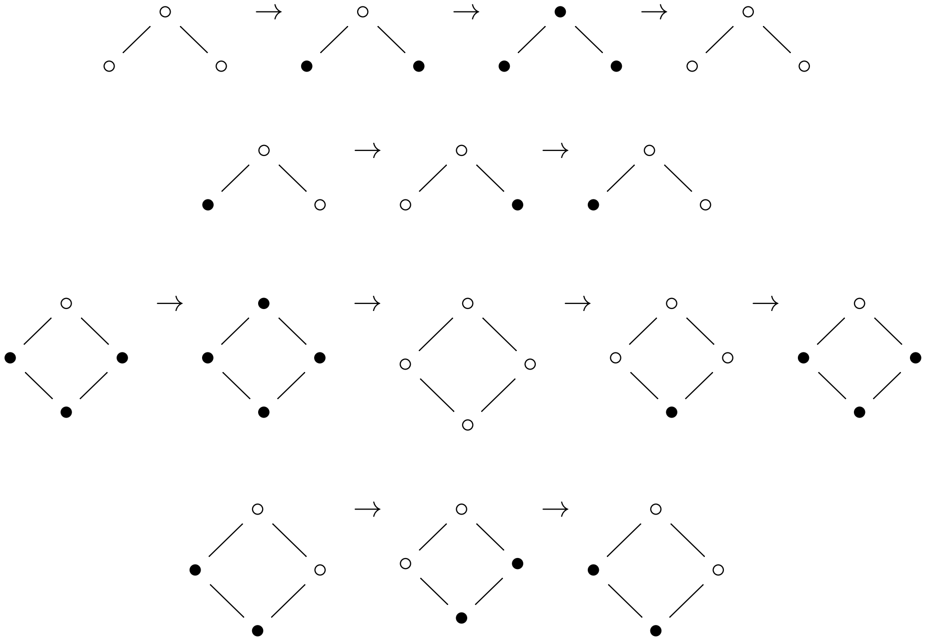
Example: Let S be the following order ideal

$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



Examples of Orbits of this Dynamic on Order Ideals.



Rowmotion on $[4] \times [2]$ A

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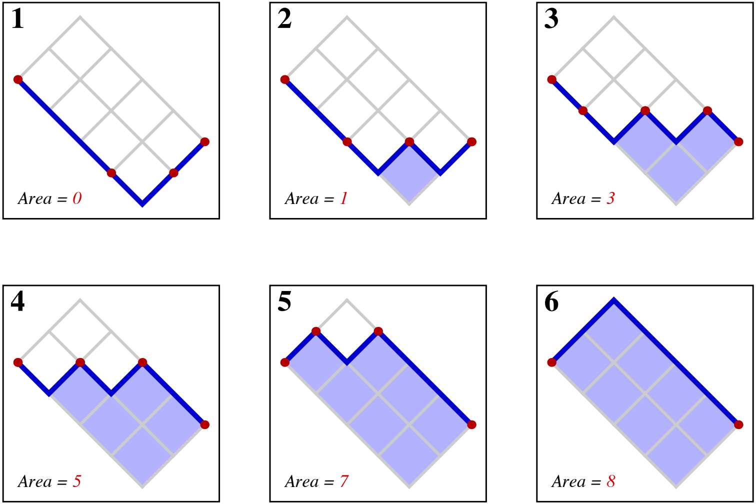

$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ B

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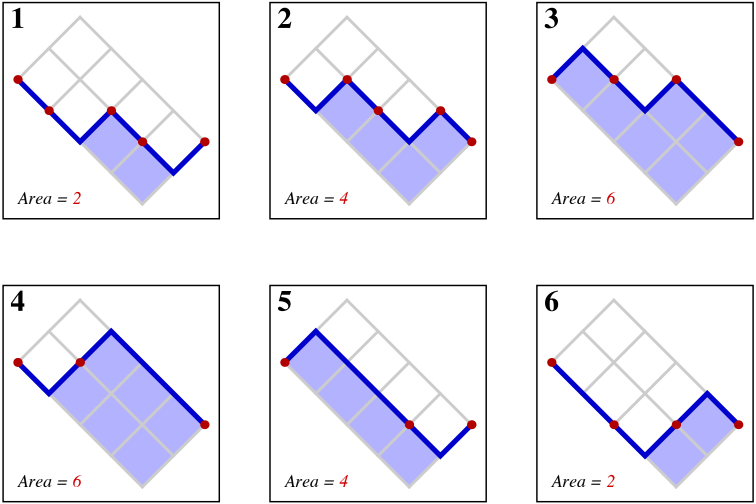

$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ C

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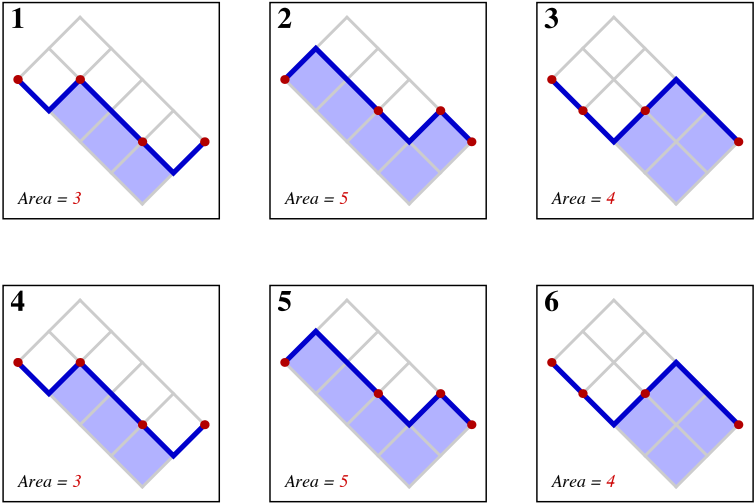

$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Definition ([PR15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
(Call f c-mesic for short.) Greek for "same-middle"

Theorem ([PR15])

For the action of rowmotion on order ideals $J(P)$ of rectangular posets $P = [p] \times [q]$, the cardinality statistic is homomesic (with average $pq/2$).

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PR15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.

Classical rowmotion: Periodicity

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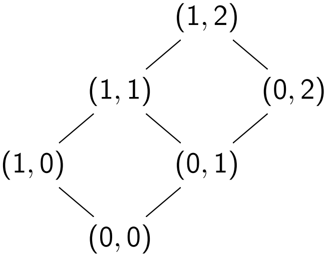
However, **for some types of** P, the order can be explicitly computed or bounded from above. See Striker-Williams [StWi11] (and the **very recent** Thomas-Williams [TW17]) for an exposition of known results.

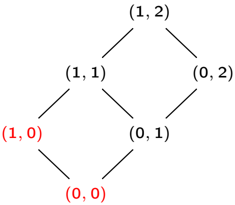
- If P is a $p \times q$-rectangle:



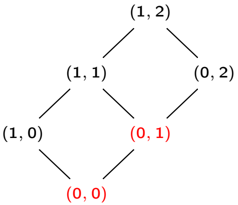
(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

Classical rowmotion: Periodicity (Example)

$S =$

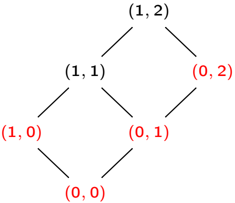


, $\mathbf{r}(S) =$

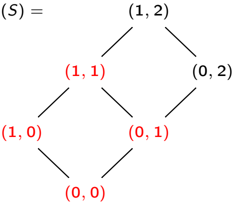


,

$\mathbf{r}^2(S) =$



, $\mathbf{r}^3(S) =$



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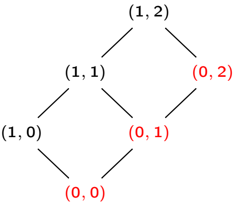
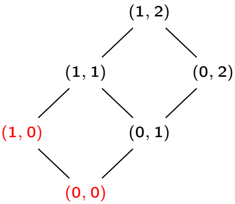


, $\mathbf{r}^5(S) =$



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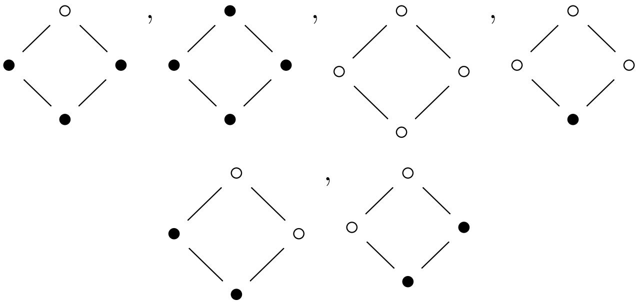
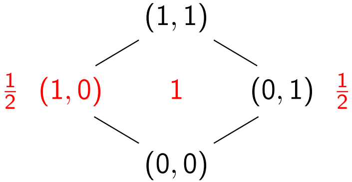


The average value along antipodal (N-S, E-W) pairs is 1 for both orbits,

and is also **constant**, as



, on files.

We will generalize this to birational rowmotion.

Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.

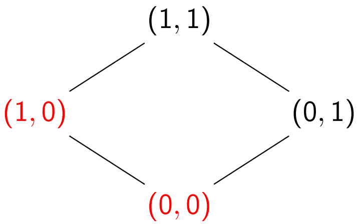
Classical rowmotion via toggling ("rowmotion in slowmotion")

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Example:

Start with this order ideal S:



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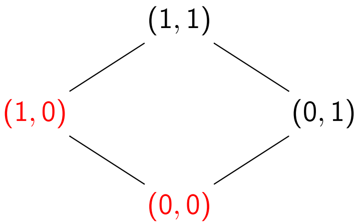

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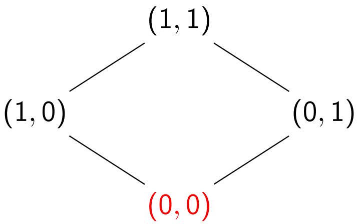

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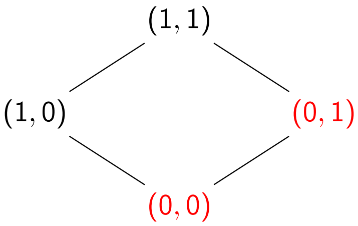
Example:

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0,1)$ to the order ideal:



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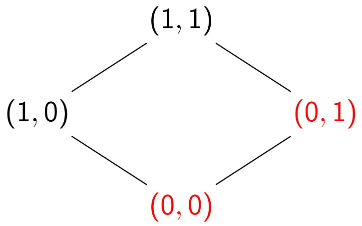

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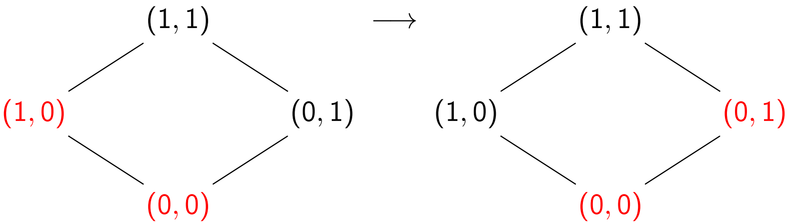
Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$. ($J(P) = \{f : P \to \{0,1\} : f \text{ is monotone}\}$)

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

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where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

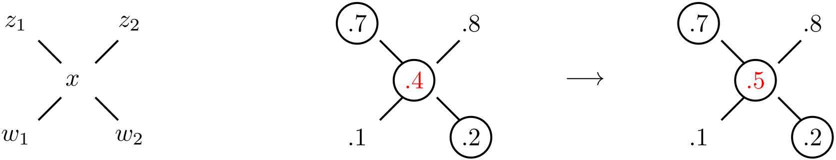
Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot\, x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

Example of flipping at a node

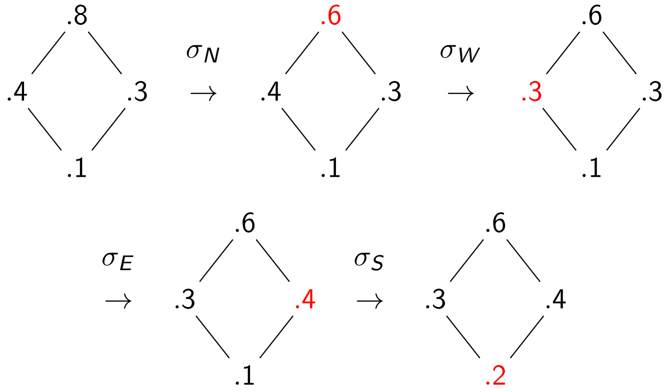


$$\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order to get $\rho_{PL}(f)$.)

How PL rowmotion generalizes classical rowmotion

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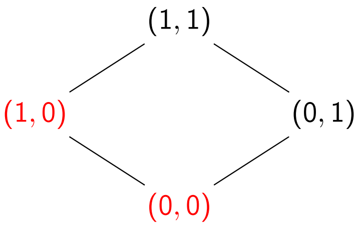

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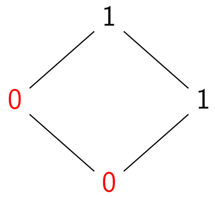
Example:

Translated to the PL setting:



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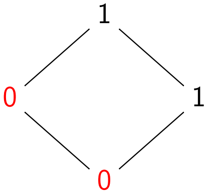

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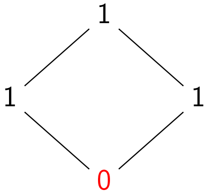

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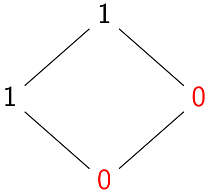
$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example:

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0, 1)$ to the order ideal:



How PL rowmotion generalizes classical rowmotion

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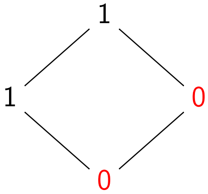
where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

Example:

Finally apply $\mathbf{t}_{(0,0)}$, which changes nothing:



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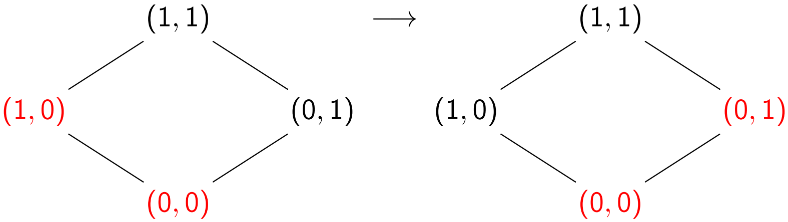
Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x)$$

We can"detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w < \cdot\, x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements $\widehat{0}$ and $\widehat{1}$ to P and forcing
  - $\widehat{0}$ to be less than every other element, and
  - $\widehat{1}$ to be greater than every other element.
- Let $\mathbb{K}$ be a field.
- A **$\mathbb{K}$-labelling of P** will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.
- For any $v \in P$, define the **birational v-toggle** as the rational map

  $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $(T_v f)(w) = \dfrac{\sum_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.

  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.

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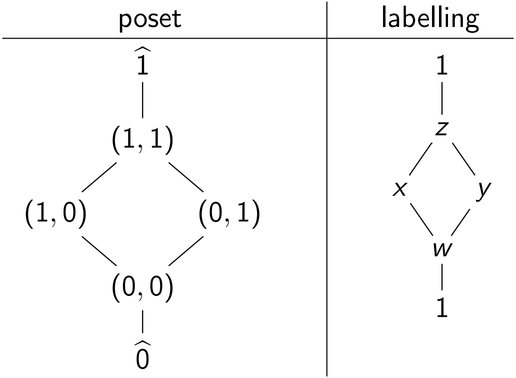
- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.
- We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- This is originally due to Einstein and Propp [EiPr13, EiPr14], following the lead of Kirillov-Berenstein [KiBe95].

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



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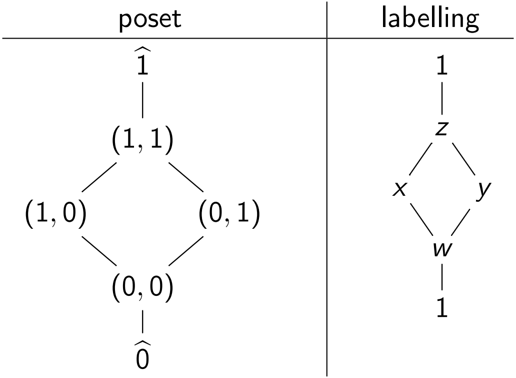

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$ using the linear extension $((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

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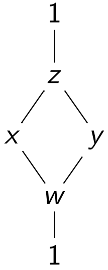
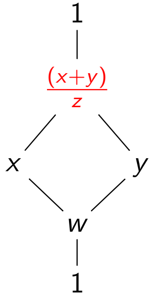
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

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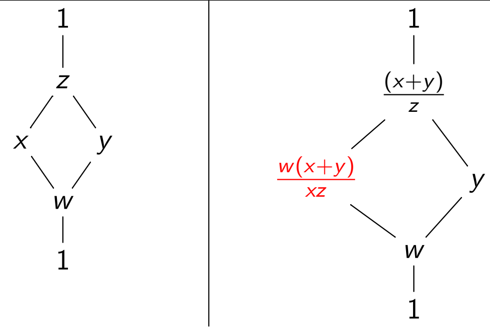

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Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,1)} T_{(1,0)} T_{(1,1)} f$
$\begin{array}{ccc} & 1 & \\ & z & \\ x & & y \\ & w & \\ & 1 & \end{array}$	$\begin{array}{ccc} & 1 & \\ & \frac{(x+y)}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & w & \\ & 1 & \end{array}$

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$
$\begin{tikzcd} & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$	$\begin{tikzcd} & 1 \arrow[d, dash] & \\ & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{w(x+y)}{xz} \arrow[dr, dash] & & \frac{w(x+y)}{yz} \arrow[dl, dash] \\ & \frac{1}{z} \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion orbit on a product of chains

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd}[column sep=small] & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{(x+y)w}{xz} \arrow[dr, dash] & & \frac{(x+y)w}{yz} \arrow[dl, dash] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[column sep=small] & \frac{(x+y)w}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{1}{y} \arrow[dr, dash] & & \frac{1}{x} \arrow[dl, dash] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[column sep=small] & \frac{1}{w} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{yz}{(x+y)w} \arrow[dr, dash] & & \frac{xz}{(x+y)w} \arrow[dl, dash] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}.$$

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd}[arrows=dash] & \frac{(x+y)}{z} \arrow[dl] \arrow[dr] & \\ \frac{(x+y)w}{xz} \arrow[dr] & & \frac{(x+y)w}{yz} \arrow[dl] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[arrows=dash] & \frac{(x+y)w}{xy} \arrow[dl] \arrow[dr] & \\ \frac{1}{y} \arrow[dr] & & \frac{1}{x} \arrow[dl] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[arrows=dash] & \frac{1}{w} \arrow[dl] \arrow[dr] & \\ \frac{yz}{(x+y)w} \arrow[dr] & & \frac{xz}{(x+y)w} \arrow[dl] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[arrows=dash] & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & w & \end{tikzcd}.$$

Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Birational homomesy on files

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_B^k(f)(1,0) = \frac{(x+y)w}{xz} \quad \frac{1}{y} \quad \frac{yz}{(x+y)w} \quad (x) = 1,$$

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$$\prod_{k=1}^{4}\rho_B^k(f)(0,0)\rho_B^k(f)(1,1)=$$

$$\frac{1}{z}\;\;\frac{x+y}{z}\;\;\frac{z}{x+y}\;\;\frac{(x+y)w}{xy}\;\;\frac{xy}{(x+y)w}\;\;\frac{1}{w}\;\;(w)\;\;(z)=1,$$

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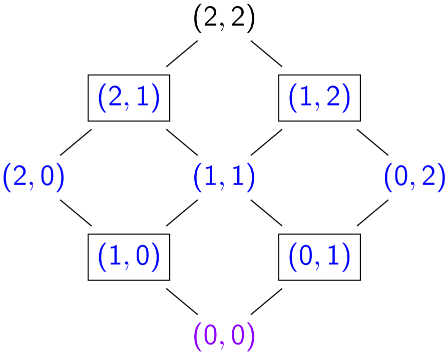
Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

Birational Rowmotion on the Rectangular Poset

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

1) Let $\bigvee_{(m,n)} := \{(u,v) : (u,v) \geq (m,n)\}$ be the *principal order filter at* (m,n), $\hexagon^k_{(m,n)}$ be the *rank-selected subposet*, of elements in $\bigvee_{(m,n)}$ whose rank (within $\bigvee_{(m,n)}$) is at least $k-1$ and whose corank is at most $k-1$.



2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$. (For $k \leq \min\{r-m, s-n\}+1$.)

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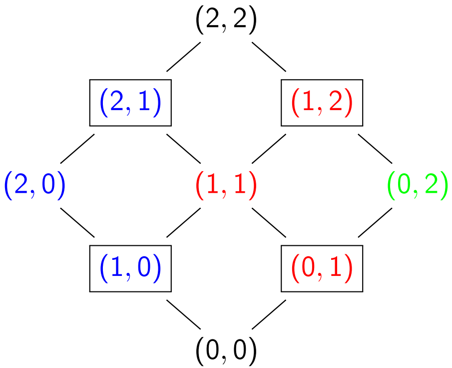
Let $A_{ij} := \frac{\sum_{z \lessdot (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1}+x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i,j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\widehat{P}$).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_{\mathbf{k}}(\mathbf{m}, \mathbf{n})$ in terms of the A_{ij}'s as follows.

We define a **lattice path of length k** within $P = [0, r] \times [0, s]$ to be a sequence $v_1, v_2, \ldots, v_k$ of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1, 0)$ or $(0, 1)$ for each $i \in [k]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex.



3) Let $S_k(m,n)$ be the set of non-intersecting lattice paths in $\hexagon^k_{(m,n)}$, from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. Let $\mathcal{L} = \{L_1, L_2, \ldots L_k\} \in S_k^k(m,n)$ denote a k-collection of such lattice paths.

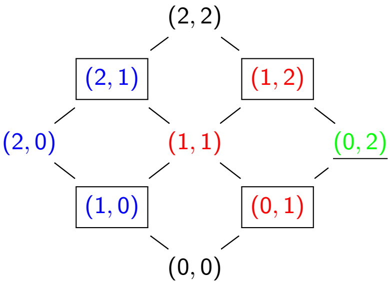
4) Define $\varphi_k(m,n) :=$
$$\sum_{\mathcal{L} \in S_k^k(m,n)} \prod_{\substack{(i,j) \in \hexagon^k_{(m,n)} \\ (i,j) \notin L_1 \cup L_2 \cup \cdots \cup L_k}} A_{ij}.$$

Theorem(*):

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

EG: $\rho_B^2(1,1) = \dfrac{\varphi_1(0,0)}{\varphi_2(0,0)}$.

$$= \frac{\text{sum of 6 quartic terms in } A_{ij}}{A_{20} + A_{11} + A_{02}}$$



(*) Caveats explained and general statement given in the next few slides.

Main Theorem (M-Roby 2018)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k-i]_+ + [k-j]_+$.

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(a1) When $M = 0$, i.e., $(i-k, j-k)$ still lies in the poset $[0,r] \times [0,s]$:

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

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(a2) When $0 < M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+,[k-i]_+)}\left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)}\right)$$

where $\mu^{(a,b)}$ is the operator that takes a rational function in $\{A_{(u,v)}\}$ and simply shifts each index in each factor of each term: $A_{(u,v)} \mapsto A_{(u-a,v-b)}$.

Main Theorem (M-Roby 2018)

Fix $k \in [0, r+s+1]$ and set $M = [k-i]_+ + [k-j]_+$. After $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$ we get:

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(a) When $0 \le M \le k$:

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where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined above.

(b) When $M \ge k$: $\rho_B^{k+1}(i,j) = 1/\rho_B^{k-i-j}(r-i, s-j)$, which is well-defined by part (a).

Remark: We prove that our formulae in (a) and (b) agree when $M = k$, allowing us to give a new proof of periodicity: $\rho_B^{r+s+2+d} = \rho_B^d$; thus we get a formula for **all** iterations of the birational rowmotion map.

Corollaries of the Main Theorem

Corollary

For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$.

Corollary ([GrRo15, Thm. 30])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *is periodic, with period* $r+s+2$.

Corollary ([GrRo15, Thm. 32])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *satisfies the following reciprocity:*
$\rho_B^{i+j+1} = 1/\rho_B^0(r-i,s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem

Given a file F in $[0, r] \times [0, s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1.$

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_B^k(f)(1,0) = \frac{(x+y)w}{xz} \quad \frac{1}{y} \quad \frac{yz}{(x+y)w} \quad (x) = 1,$$

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$$\prod_{k=1}^{4}\rho_B^k(f)(0,0)\rho_B^k(f)(1,1)=$$

$$\frac{1}{z}\ \ \frac{x+y}{z}\ \ \frac{z}{x+y}\ \ \frac{(x+y)w}{xy}\ \ \frac{xy}{(x+y)w}\ \ \frac{1}{w}\ \ (w)\ \ (z)=1,$$

$$\prod_{k=1}^{4}\rho_B^k(f)(0,1)=\frac{(x+y)w}{yz}\ \ \frac{1}{x}\ \ \frac{xz}{(x+y)w}\ \ (y)=1.$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

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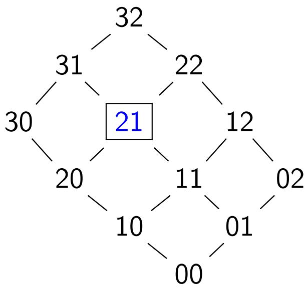
Recall that in the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get a simpler formula.

Corollary: For $k \le \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$



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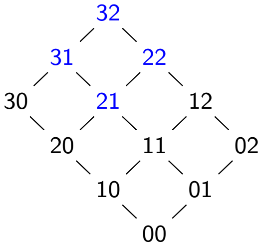
Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

When k = 0, $M = 0$ and we get



$$\rho_B^1(2,1) = \frac{\varphi_0(2,1)}{\varphi_1(2,1)} = \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}.$$

Example

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Corollary: For $k \le \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

For the numerator, $s_1 = (1,0)$, $t_1 = (3,2)$, and there are six lattice paths from s_1 to t_1, each of which covers 5 elements and leaves 4 uncovered.

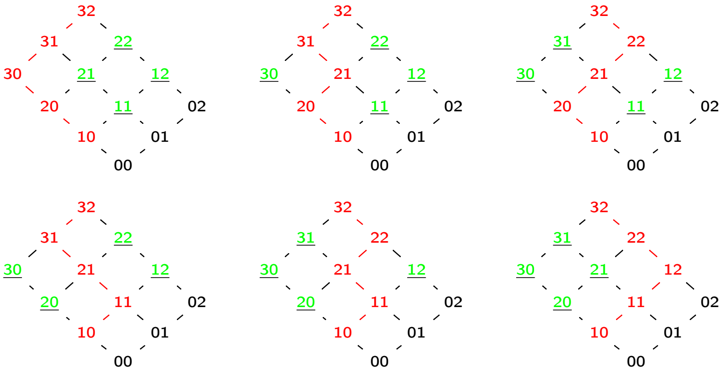
For the denominator, $s_1 = (2,0)$, $s_2 = (1,1)$, $t_1 = (3,1)$, and $t_2 = (2,2)$, and each pair of lattice paths leaves exactly one element uncovered.

Example

When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

For the numerator, $s_1 = (1,0)$, $t_1 = (3,2)$, and there are six lattice paths from s_1 to t_1, each of which covers 5 elements and leaves 4 uncovered.



Example

When k $= \mathbf{1}$, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1{,}0)}{\varphi_2(1{,}0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

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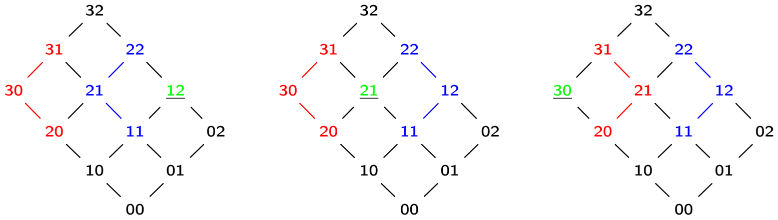

Example

In the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) gives a point outside of P, we must also apply a μ-translation.

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 2, we get $M = [2-2]_+ + [2-1]_+ = 1 \leq 2 = k$. So by part (a) of the main theorem we have

$$\rho_B^3(2,1) = \mu^{(1,0)} \left[\frac{\varphi_1(1,0)}{\varphi_2(1,0)} \right] = \text{(just shifting indices in the } k=1 \text{ formula)}$$

$$\frac{A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}}{A_{02} + A_{11} + A_{20}}.$$

Example

In the case where shifting $(i, j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) gives a point outside of P, we must also apply a μ-translation.

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 3, we get $M = [3-2]_+ + [3-1]_+ = 3 = k$. Therefore,

$$\rho_B^4(2,1) = \mu^{(2,1)}\left[\frac{\varphi_0(2,1)}{\varphi_1(2,1)}\right] = \mu^{(2,1)}\left[\frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}\right] = \frac{A_{00}A_{01}A_{10}A_{11}}{A_{01}+A_{10}}.$$

In this situation, we can also use part (b) of the main theorem to get

$$\rho_B^4(2,1) = 1/\rho_B^{3-2-1}(3-2, 2-1) = 1/\rho_B^0(1,1) = \frac{1}{x_{11}}.$$

The equality between these two expressions is easily checked.

Example

In the case where μ-translation would lead to negative subscripts for the φ's, i.e. $M > k$, part (a) of the Theorem does not apply.

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 4, we get $M = [4-2]_+ + [4-1]_+ = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^5(2,1) = 1/\rho_B^{4-2-1}(3-2,2-1) = 1/\rho_B^1(1,1) = \frac{\varphi_1(1,1)}{\varphi_0(1,1)} = \frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}}.$$

Each term in the numerator is associated with one of the three lattice paths from $(1,1)$ to $(3,2)$ in P, while the denominator is just the product of all A-variables in the principal order filter $\bigvee(1,1)$.

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We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 5, we get $M = [5-2]_+ + [5-1]_+ = 7 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^6(2,1) = 1/\rho_B^{5-2-1}(3-2, 2-1) = 1/\rho_B^2(1,1) = \frac{\varphi_2(0,0)}{\varphi_1(0,0)} =$$

$$(A_{02}A_{12} + A_{02}A_{21} + A_{11}A_{21} + A_{30}A_{02} + A_{30}A_{11} + A_{30}A_{20})\Big/$$

$$(A_{01}A_{11}A_{02}A_{21}A_{12}A_{22} + A_{01}A_{11}A_{02}A_{30}A_{12}A_{22} + A_{01}A_{11}A_{02}A_{30}A_{12}A_{31} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{01}A_{20}A_{02}A_{30}$$

$$A_{01}A_{20}A_{02}A_{30}A_{21}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{12}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{21}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{10}A_{20}A_{11}A_{30}$$

In the case where μ-translation would lead to negative subscripts for the φ's, i.e. $M > k$, part (a) of the Theorem does not apply.

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2,$ and $j = 1$ throughout.

When k = 6, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right]$$

$$= \mu^{(1,1)}\left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} =$$

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The lattice paths involved here are the same as for the $k = 4$ computation.

We can deduce this by $A_{00} = 1/x_{00}, A_{10} = x_{00}/x_{10}, A_{01} = x_{00}/x_{01}$,

$A_{11} = (x_{10} + x_{01})/x_{11}, A_{20} = x_{10}/x_{20}$, and $A_{21} = (x_{20} + x_{11})/x_{21}$.

Periodicity also kicks in: $\rho_B^7(2,1) = \rho_B^0(2,1) = x_{21}$ using $(r+s+2) = 7$.

Sketch of Proof

By definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \mid\mid \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

$$A \mid\mid B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

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where

$$A \parallel B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

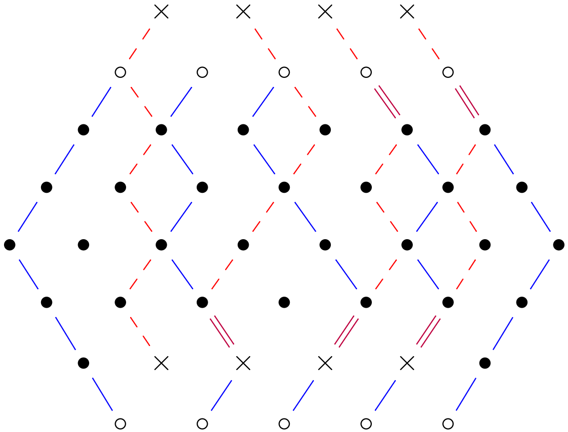
By induction on k, and the fact that we apply birational rowmotion from top to bottom, we can apply algebraic manipulations to reduce our result to proving the following **Plücker-like identity**:

$$\begin{aligned}\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \\ \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) \\ +\varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).\end{aligned}$$

It is sufficient to verify the following Plücker-like identity

$$\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k)+\varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).$$

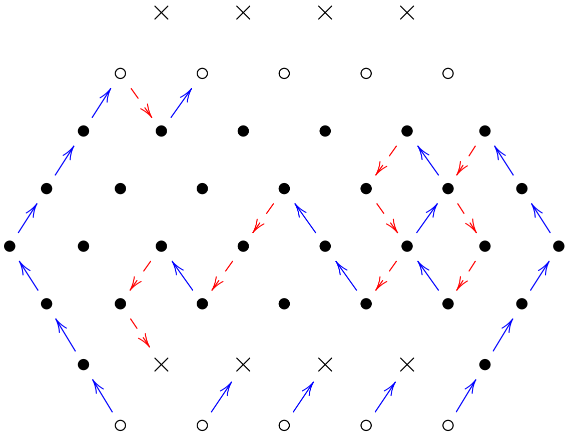
Example (k=5):



Sketch of Proof

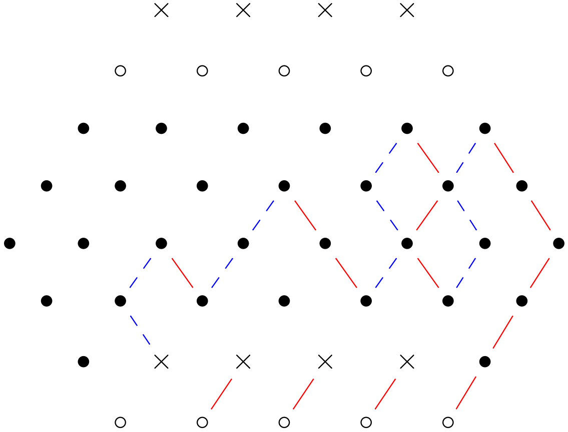
We build **bounce paths** and **twigs** (paths of length one from $\circ$ to $\times$) starting from the bottom row of $\circ$'s.

Example (k=5):



We then reverse the colors along the $(k-2)$ **twigs** and the **one bounce path from** $\circ$ **to** $\times$ (rather than $\circ$ to $\circ$).

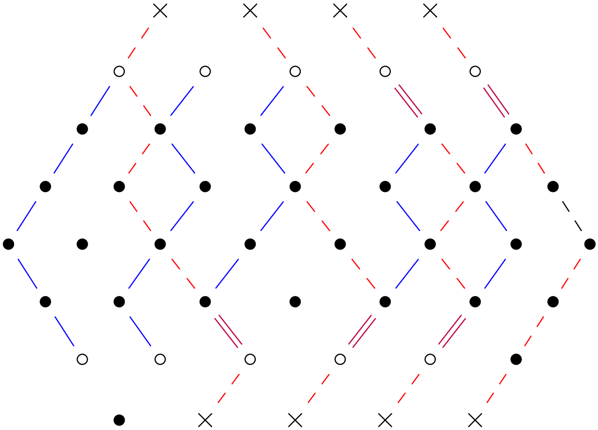
Example (k=5):



Sketch of Proof

Swap in the new colors and shift the ∘'s and ×'s in the bottom two rows.

Example (k=5):



Sketch of Proof

$$\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) + \varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).$$

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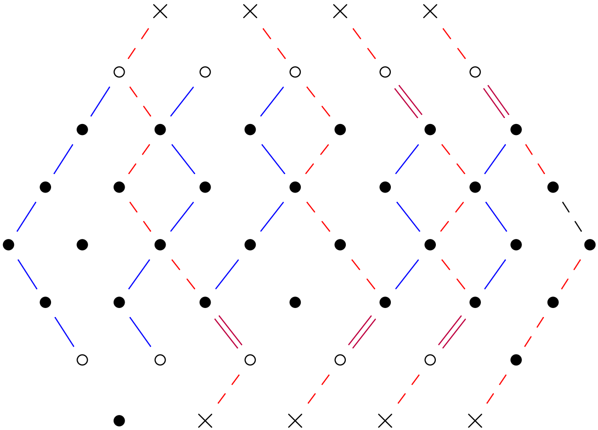

Further Application: Birational File Homomesy

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Given a file F in $[0, r] \times [0, s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i, j) = 1.$

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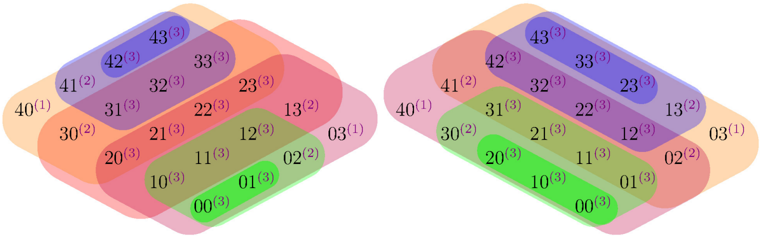
Sketch of Proof: Double-counting argument, followed by color-coded cancellations and several entries immediately equal to 1, as in ensuing table.



Further Application: Birational File Homomesy

Let $(r,s)=(4,3)$, $d=2$, and consider the file $F=\{(4,2),(3,1),(2,0)\}$. The following table displays the values of $\rho_B^k(i,j)$ for $0\le k\le 8$, $(i,j)\in F$.

	$(4,2)$	$(3,1)$	$(2,0)$
$k=0$	$\frac{\varphi_0(4,2)}{\varphi_1(4,2)=1}$	$\frac{\varphi_0(3,1)}{\color{red}\varphi_1(3,1)}$	$\frac{\varphi_0(2,0)}{\color{blue}\varphi_1(2,0)}$
$k=1$	$\frac{\color{red}\varphi_1(3,1)}{\varphi_2(3,1)=1}$	$\frac{\color{blue}\varphi_1(2,0)}{\color{red}\varphi_2(2,0)}$	$\mu^{(1,0)}\left[\frac{\varphi_0(2,0)}{\color{blue}\varphi_1(2,0)}\right]$
$k=2$	$\frac{\color{red}\varphi_2(2,0)}{\varphi_3(2,0)=1}$	$\mu^{(1,0)}\left[\frac{\color{blue}\varphi_1(2,0)}{\color{red}\varphi_2(2,0)}\right]$	$\mu^{(2,0)}\left[\frac{\varphi_0(2,0)}{\color{blue}\varphi_1(2,0)}\right]=\frac{1}{x_{23}}$
$k=3$	$\mu^{(1,0)}\left[\frac{\color{red}\varphi_2(2,0)}{\varphi_3(2,0)=1}\right]$	$\mu^{(2,0)}\left[\frac{\color{blue}\varphi_1(2,0)}{\color{red}\varphi_2(2,0)}\right]$	$\frac{\varphi_1(2,3)=1}{\varphi_0(2,3)}$
$k=4$	$\mu^{(2,0)}\left[\frac{\color{red}\varphi_2(2,0)}{\varphi_3(2,0)=1}\right]$	$\mu^{(3,1)}\left[\frac{\varphi_0(3,1)}{\color{red}\varphi_1(3,1)}\right]=\frac{1}{x_{12}}$	$\frac{\varphi_2(1,2)=1}{\color{green}\varphi_1(1,2)}$
$k=5$	$\mu^{(3,1)}\left[\frac{\color{red}\varphi_1(3,1)}{\varphi_2(3,1)=1}\right]$	$\frac{\color{green}\varphi_1(1,2)}{\varphi_0(1,2)}$	$\frac{\varphi_3(0,1)=1}{\color{green}\varphi_2(0,1)}$
$k=6$	$\mu^{(4,2)}\left[\frac{\varphi_0(4,2)}{\varphi_1(4,2)=1}\right]=\frac{1}{x_{01}}$	$\frac{\color{green}\varphi_2(0,1)}{\color{blue}\varphi_1(0,1)}$	$\mu^{(0,1)}\left[\frac{\varphi_3(0,1)=1}{\color{green}\varphi_2(0,1)}\right]$
$k=7$	$\frac{\color{blue}\varphi_1(0,1)}{\varphi_0(0,1)}$	$\mu^{(0,1)}\left[\frac{\color{green}\varphi_2(0,1)}{\color{blue}\varphi_1(0,1)}\right]$	$\mu^{(1,2)}\left[\frac{\varphi_2(1,2)=1}{\color{green}\varphi_1(1,2)}\right]$
$k=8$	$\mu^{(0,1)}\left[\frac{\color{blue}\varphi_1(0,1)}{\varphi_0(0,1)}\right]=x_{42}$	$\mu^{(1,2)}\left[\frac{\color{green}\varphi_1(1,2)}{\varphi_0(1,2)}\right]=x_{31}$	$\mu^{(2,3)}\left[\frac{\varphi_1(2,3)=1}{\varphi_0(2,3)}\right]=x_{20}$

[BrS74] Andries E. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math. Centr. report ZW24, Amsterdam (Jun. 1974), http://www.win.tue.nl/~aeb/preprints/zw24.pdf.

[CaFl95] Peter J. Cameron and Dmitry G. Fon-der-Flaass, *Orbits of Antichains Revisited*, Europ. J. of Combin., **16(6)**, (1995), 545–554, http://www.sciencedirect.com/science/article/pii/0195669895900365.

[EiPr13] David Einstein and James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains* (2013), arXiv:1310.5294.

[EiPr14] David Einstein and James Propp, *Piecewise-linear and birational toggling (Extended abstract)*, DMTCS proc. FPSAC 2014, http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAT0145/4518. Also available at arXiv:1404.3455v1.

[Flaa93] Dmitry G. Fon-der-Flaass, *Orbits of Antichains in Ranked Posets*, Europ. J. Combin., **14(1)**, (1993), 17–22, http://www.sciencedirect.com/science/article/pii/S0195669883710036.

[GrRo14] Darij Grinberg and Tom Roby, *The order of birational rowmotion (Extended abstract)*, DMTCS proc. FPSAC 2014, http://www.dmtcs.org/pdfpapers/dmAT0165.pdf.

[GrRo15] Darij Grinberg, Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. of Combin. **22**(3), #P3.40 (2015). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p40.

[KiBe95] Anatol N. Kirillov and Arkadiy D. Berenstein, *Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz, **7**, #1 (1995), 92–152.

[MR18] Gregg Musiker, Tom Roby, *Paths to understanding birational rowmotion on products of two chains* (2018), arXiv:1801.03877.

[AIM15] James Propp, Tom Roby, Jessica Striker, and Nathan Williams (organizers), Sam Hopkins (notetaker), *Notes from the AIM workshop on dynamical algebraic combinatorics*, American Institute of Math., San Jose, CA, 23–27 March 2015, http://aimath.org/pastworkshops/dynalgcomb.html, http://mit.edu/~shopkins/docs/aim_dyn_alg_comb_notes.pdf.

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22(3)** (2015), #P3.4, [Rob16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in Andrew Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

[Sage17] William A. Stein et. al., *Sage Mathematics Software (Version 8.1)*, The Sage Development Team (2017), http://www.sagemath.org.

[StWi11] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942, http://www.sciencedirect.com/science/article/pii/S0195669812000972. Also available at arXiv:1108.1172v3.

[TW17] HughThomas and Nathan Williams, *Rowmotion in Slow Motion*, arXiv:1712.10123.

[Yil17] Emine Yıldırım, *Coxeter transformation on Cominuscule Posets*, arXiv:1710.10632.

Gregg Musiker and Tom Roby, *Paths to understanding binational rowmotion on products of two chains* (2018), arXiv:1801.03877.