Paths to Understanding Rowmotion on a Product of Two Chains

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Abstract: Birational rowmotion is an action on the space of assignments of rational functions to the elements of a finite partially-ordered set (poset). It is lifted from the well-studied rowmotion map on order ideals (equivalently on antichains) of a poset $P$, which when iterated on special posets, has unexpectedly nice properties in terms of periodicity, cyclic sieving, and homomesy (statistics whose averages over each orbit are constant) In this context, rowmotion appears to be related to Auslander-Reiten translation on certain quivers [Yil17], and birational rowmotion to $Y$-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity.

We give a formula in terms of families of non-intersecting lattice paths for iterated actions of the birational rowmotion map on a product of two chains. This allows us to give a much simpler direct proof of the key fact that the period of this map on a product of chains of lengths $r$ and $s$ is $r + s + 2$ (first proved by D. Grinberg and the author), as well as the first proof of the birational analogue of homomesy along files for such posets. This work is joint with Gregg Musiker.
Outline & Acknowledgements

1. Classical Rowmotion
2. Piecewise-linear (PL) and Birational Rowmotion
3. Formula in terms of Lattice Paths
4. Sketch of Proof
5. Applications (Periodicity and Homomesy)

We are grateful for the 2015 AIM workshop on *Dynamical Algebraic Combinatorics* and for Darij Grinberg’s implementation of birational rowmotion in SageMath.
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Please feel free to interrupt with questions or comments.
The combinatorial rowmotion map has liftings (via a decomposition into involutions called *toggles*) to the piecewise-linear (order polytope) and then birational settings. Proving results at the birational level implies them at the other levels.

For rectangular posets $P = [0, r] \times [0, s]$, we give a formula in terms of NILPs that allows us to compute $\rho_B^k$, the $k$th iteration of birational rowmotion.

The key lemma is a Plücker-like relation satisfied by certain polynomials we define, proven by a colorful combinatorial bijection on pairs of NILPs (along the lines of Fulmek-Kleber).

Using our formula, we obtain more direct proofs of the periodicity and “antipodal-reciprocity” of this system, as well as the first proof of “homomesy along files”.

Classical rowmotion is the rowmotion studied by Striker-Williams (2012), who coined the term. It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or “nonnesting partitions”, with relations to Lie theory).
- Propp-Roby (2015), as one of several actions that displays the homomesy phenomenon on the product of two chains.
Let $P$ be a finite poset. **Classical rowmotion** is the map 

$r : J(P) \rightarrow J(P)$

sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

Let $S$ be the following order ideal (indicated by the ●’s)
Classical rowmotion

Let $P$ be a finite poset. Classical rowmotion is the map $r : J(P) \to J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let $S$ be the following order ideal
Mark the complement in red.
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \to J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

Mark $M$ (the minimal elements of the complement) in blue.
Let $P$ be a finite poset. **Classical rowmotion** is the map $r : J(P) \to J(P)$ sending every order ideal $S$ to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal

Remove the old order ideal:
Let $P$ be a finite poset. **Classical rowmotion** is the map
\[ r : J(P) \rightarrow J(P) \]
sending every order ideal $S$ to a new order ideal $r(S)$ generated by
the minimal elements of $P \setminus S$.

**Example:** Let $S$ be the following order ideal
\[ r(S) \]

is the order ideal generated by $M$ ("everything below $M"): 

![Diagram](image-url)
Examples of Orbits of this Dynamic on Order Ideals.
Classical rowmotion: Homomesy

**Definition ([PR15])**

Given an (invertible) action $\tau$ on a finite set of objects $S$, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to $(S, \tau)$ if the average of $f$ over each $\tau$-orbit $O$ is the same constant $c$ for all $O$, i.e., $\frac{1}{\#O} \sum_{s \in O} f(s) = c$ does not depend on the choice of $O$.

(Call $f$ $c$-mesic for short.) Greek for “same-middle”

**Theorem ([PR15])**

*For the action of rowmotion on order ideals $J(P)$ of rectangular posets $P = [p] \times [q]$, the cardinality statistic is homomesic (with average $pq/2$).*
Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- Birational rowmotion can be related to $Y$-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PR15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.
- Galashin & Pylyavskyy have a very interesting extension of birational rowmotion to directed graphs and "$R$-systems." [GaPy17].
Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.
Classical rowmotion: Periodicity

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

However, **for some types of** $P$, the order can be explicitly computed or bounded from above. See Striker-Williams [StWi11] (and the very recent Thomas-Williams [TW17]) for an exposition of known results.

- If $P$ is a $p \times q$-rectangle:

  \[(1, 2)\]
  \[(1, 1)\]
  \[(0, 2)\]
  \[(1, 0)\]
  \[(0, 1)\]
  \[(0, 0)\]

  (shown here for $p = 2$ and $q = 3$), then $\text{ord}(r) = p + q$. 
Classical rowmotion: Periodicity (Example)

\[ S = \begin{array}{c}
(1, 2) \\
(1, 1) \\
(1, 0) \\
(0, 0)
\end{array} , \quad r(S) = \begin{array}{c}
(1, 2) \\
(1, 1) \\
(1, 0) \\
(0, 0)
\end{array} , \]

\[ r^2(S) = \begin{array}{c}
(1, 2) \\
(1, 1) \\
(1, 0) \\
(0, 0)
\end{array} , \quad r^3(S) = \begin{array}{c}
(1, 2) \\
(1, 1) \\
(1, 0) \\
(0, 0)
\end{array} , \]

\[ r^4(S) = \begin{array}{c}
(1, 2) \\
(1, 1) \\
(1, 0) \\
(0, 0)
\end{array} , \quad r^5(S) = \begin{array}{c}
(1, 2) \\
(1, 1) \\
(1, 0) \\
(0, 0)
\end{array} , \]

which is precisely the \( S \) we started with.

\[ \text{ord}(r) = p + q = 2 + 3 = 5. \]
Example in lattice cell form

Viewing the elements of the poset as squares below, we would map:

Area = 8

Area = 10