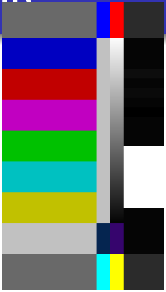
Dynamical Algebraic Combinatorics and the Homomesy Phenomenon: Toggling, whirling, and Bulgarian solitaire



Tom Roby (UConn)

Describing joint research with
Michael Joseph & James Propp

Okayama University
Okayama-shi, JAPAN

Slides for this talk are available online (or will be soon) at
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Abstract: Given a group acting on a finite set of combinatorial objects, one can often find natural statistics on these objects which are *homomesic*, i.e., over each orbit of the action, the average value of the statistic is the same. Since the notion was codified a few years ago, homomesic statistics have been uncovered in a wide variety of situations within dynamical algebraic combinatorics. We discuss a couple of interesting examples in depth, including "Coxeter toggling" the independent sets in a path graph (joint work with Michael Joseph), "whirling" functions between finite sets (joint work with Michael Joseph and James Propp), and (if time permits) "Bulgarian Solitaire".

Acknowledgments

This seminar talk discusses joint work with Michael Joseph and James Propp.

Please feel free to interrupt with questions or comments.

ご質問コメント等があれば、ご遠慮なくおっしゃって下さい。

Outline

- Cyclic rotation of bitstrings (or k-subsets);
- Actions, orbits, and homomesy;
- Toggling Independent Sets of a Path Graph;
- Whirling injections and surjections;
- 時間あれば "Bulgarian solitaire"

- Let $\binom{[n]}{k}$ be the set of length n binary strings with k 1s.
- Let $C_R : \binom{[n]}{k} \to \binom{[n]}{k}$ be rightward cyclic rotation.

Example

$n = 6,\ k = 2$

$$101000 \underset{C_R}{\longmapsto} 010100$$

$$000011 \underset{C_R}{\longmapsto} 100001$$

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

$n = 6,\ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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100010	5	000011	0		
010001	3	100001	4		
Average	4	**Average**	4	**Average**	4

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R.[PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in \binom{[n]}{k}$.

Then the function $\mathrm{inv} : \binom{[n]}{k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Theorem (Propp & R.[PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in \binom{[n]}{k}$.

Then the function $\mathrm{inv} : \binom{[n]}{k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ■

Example

$n = 6, \ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
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100010	5	000011	0	010010	4
010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of SSYT; Rowmotion of "nice" (e.g., minuscule heap) posets [PrRo15, StWi11, Had14, RuWa15+] ;

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- Liftings of homomesy from combinatorial actions to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].

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- Liftings of homomesy from combinatorial actions to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].
- There are many others, including the next two examples.

Toggling Independent Sets of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

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Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the subsequence 11.

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex *i*** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$. まず、危害を加えない。

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

Toggles

Proposition

Each toggle τ_i is an involution, i.e., τ_i^2 is the identity. Also, τ_i and τ_j commute if and only if $|i-j| \neq 1$.

Definition

The **toggle group** is the group generated by the n toggles.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

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	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	2	3	4	5	6	7
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph-R.[JoRo18])

Define $\chi_i : \mathcal{I}_n \to \{0,1\}$ *to be the indicator function of vertex* i*.*

For $1 \leq i \leq n$, $\chi_i - \chi_{n+1-i}$ *is 0-mesic on* φ*-orbits of* $\mathcal{I}_n$*.*

Also $2\chi_1 + \chi_2$ *and* $\chi_{n-1} + 2\chi_n$ *are 1-mesic on* φ*-orbits of* $\mathcal{I}_n$*.*

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
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Total:	6	3	4	4	4	4	4	4	3	6

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
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$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
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Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board" (not in the rightmost column), there is also a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
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$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e., the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1)$ mod 4.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Whirling injections and surjections

Whirling action on injections/surjections

We write functions $f \in [k]^{[n]}$ in one-line notation $f(1)f(2)\cdots f(n)$.

Definition

Let $\mathcal{S}$ denote either $\mathrm{Inj}(n,k)$ or $\mathrm{Sur}(n,k)$ for a given $n,k \in \mathbb{P}$. Define a map $\mathrm{wh}_i : \mathcal{S} \to \mathcal{S}$, called **whirling at index** i in the following way. Given $f \in \mathcal{S}$, repeatedly add 1 (mod k) to the value of $f(i)$ until we get a function in $\mathcal{S}$. The new function is $\mathrm{wh}_i(f)$.

EG: $f = 124 \in \mathrm{Inj}(3,6) \implies \mathrm{wh}_1(124) = 324$, $\mathrm{wh}_2(124) = 134$, and $\mathrm{wh}_3(124) = 125$.

These generalize *toggle operations*, which are involutions. The composition $\mathrm{wh} := \mathrm{wh}_n \, \mathrm{wh}_{n-1} \cdots \mathrm{wh}_2 \, \mathrm{wh}_1$ is called **whirling**.

EG: $\mathrm{wh}(124) = (324 \cdots 354 \cdots 356) = 356$. 124 generates the whirling orbit

$$124 \mapsto 356 \mapsto 412 \mapsto 534 \mapsto 651 \mapsto 263 \mapsto 415 \mapsto 621 \mapsto 342 \mapsto 563$$

EG: Let $v = 21444323 \in \mathrm{Sur}(8,4)$. Then $\mathrm{wh}_1(v) = 31444323$, while $\mathrm{wh}_2(v) = v$. The orbit generated by v is:

$21444323 \mapsto 31114424 \mapsto 32211134 \mapsto 43222141 \mapsto 13332242 \mapsto 14433312 \,\Lsh$

$\eta_1(32211134) = 3$; $\eta_2(32211134) = 2$; $\eta_3(32211134) = 2$; $\eta_4(32211134) = 1$;

Theorem (Joseph-R.)

Fix $\mathcal{S}$ to be either $\mathrm{Inj}(n,k)$ or $\mathrm{Sur}(n,k)$ for a given $n,k \in \mathbb{P}$. For $i \in [k]$, define $\eta_i(f) = \#f^{-1}(\{i\})$ to be the number of times i appears as an output of the function f. Then η_i is $\frac{n}{k}$-mesic for any $i \in [k]$.

Equivalently, $\eta_i - \eta_j$ is 0-mesic for any $i,j \in [k]$, i.e., i and j appear as outputs of functions the same number of times across any orbit.

Proving homomesy for injections

Key Idea: Partition the orbit into $[k]$-chunks. If a value j appears in ith spot, then $j+1 \bmod k$ must occur directly below, unless it was already in the row when w_i was applied. Thus, the next $j+1$ occurs no later than the nth letter after j. Color these the same.

124	It's easy to see this relation goes backwards as well as forwards, so partitions the orbit into chunks each containing all of $[k]$. (The chunks can wrap around from bottom to top.)
356	
412	
534	
651	
263	
415	
621	
342	
563	

Proving homomesy for surjections

This uses a somewhat different partitioning argument. Since v_i and $\mathrm{wh}(v)_i$ either agree or differ by one, we could just partition into vertical chunks, except when $v_i = \mathrm{wh}(v)_i$ (i.e., same values on top of one another). So it suffices to show that the number of pairs of a j directly below another j is the same for all $j \in [k]$. The tops of such pairs are circled below in red.

Finally, one shows that every circled j is followed within the next n slots by a circled $j+1$, allowing these to be partitioned as well.

2	1	4	4	4	3	2	3
3	1	1	1	4	4	2	4
3	2	2	1	1	1	3	4
4	3	2	2	2	1	4	1
1	3	3	3	2	2	4	2
1	4	4	3	3	3	1	2

A consequence of homomesy for orbits

From this homomesy we can deduce information about orbit sizes (that we currently don't know by any other means).

Let $\ell(\mathcal{O})$ be the length of the orbit $\mathcal{O}$.

If we consider surjective functions from [7] to [4], then across every orbit , the numbers 1, 2, 3, 4 all appear as outputs the same number of times; hence, $4 \mid 7\ell(\mathcal{O}) \implies 4 \mid \ell(\mathcal{O})$.

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On the other hand, if we consider surjective functions from [8] to [4], then across every orbit, $4 \mid 8\ell(\mathcal{O})$, which gives no new information.

A function $f : S \to T$ between two sets S and T is m-**injective** if $\#f^{-1}(t) \leq m$ for every $t \in T$ and m-**surjective** if $\#f^{-1}(t) \geq m$ for every $t \in T$. Let $\mathrm{Inj}_m(n,k)$ and $\mathrm{Sur}_m(n,k)$ denote the set of m-injective (resp. m-surjective) functions from $[n]$ to $[k]$.

Conjecture (Joseph)

Fix $\mathcal{S}$ to be either $\mathrm{Inj}_m(n,k)$ or $\mathrm{Sur}_m(n,k)$ for fixed $n, k, m \in \mathbb{P}$. For $i \in [k]$, define $\eta_i(f) = \#f^{-1}(\{i\})$ to be the number of times i appears as an output of the function f. Then η_i is $\frac{n}{k}$-mesic for any $i \in [k]$.

Bulgarian solitaire

Alternative Definition: Homomesy

For situations where our map is *not* invertible, we have an alternative definition of homomesy as "Same limiting average from any starting point". Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite, and
- a statistic $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if the ergodic average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i(x)) = c$$

is **independent** of the starting point $x \in S$.

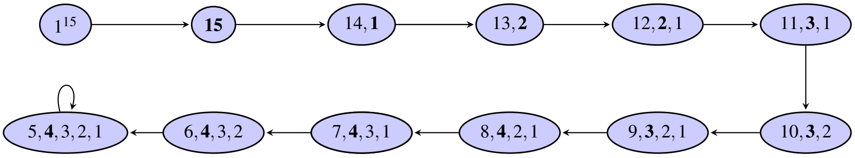
In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

This agrees with our earlier definition when τ is an invertible action, but allows it to generalize to other situations, e.g., the one upcoming.

Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define $\text{б}(\lambda)$ as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

- First surfaced as a puzzle in Russia around 1980, and a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardner column; see survey of Brian Hopkins [Hop12].
- Initial puzzle: starting from any of 176 partitions of 15, one ends at $(5, 4, 3, 2, 1)$.



Bulgarian solitaire: homomesies

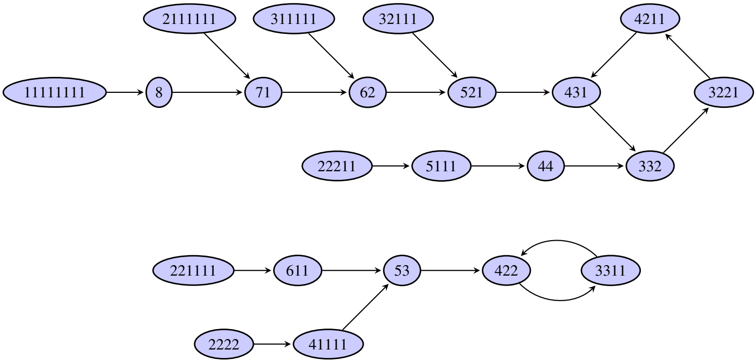
E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

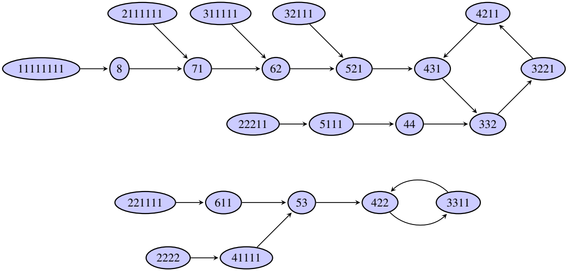
and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).



Bulgarian solitaire: homomesies



Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5,3)$, the average value of φ is $(4+3)/2 = 7/2$; while for $\lambda = (6,2)$, the average value of φ is $(3+4+4+3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic б; similarly for "size of (kth) largest part".

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

So an equivalent way of stating our main definition in this case is, φ is homomesic with respect to (S, τ) iff the average of φ over each periodic τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$.

In the rest of this talk, we restricted attention to maps τ that are invertible on S, so transience was not an issue.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
(Call f c-mesic for short.)

The final slide of this talk (before the references)

SUMMARY: We defined the notion of homomesy, and gave some examples:

- Rotation of bit strings with the inversion statistic;
- Toggling independent sets of a path graph once at each vertex, with statistic the indicator function at a vertex;
- Whirling injections and surjections, with statistic indicating number of times a value appears within an orbit;
- Bulgarian solitaire map on partitions, with indicator statistic the number of parts (non-invertible!).

We're happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

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Thanks very much for coming to this talk!

どうも有り難う御座いました。

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