Dynamical algebraic combinatorics: Actions, orbits, and averages

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Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

Abstract: Dynamical algebraic combinatorics explores actions on sets of discrete combinatorial objects, many of which can be built up by small local changes, e.g., Schützenberger's promotion and evacuation, or the rowmotion map on order ideals. There are strong connections to the combinatorics of representation theory and with Coxeter groups. Birational liftings of these actions are related to the Y-systems of statistical mechanics, thereby to cluster algebras, in ways that are still relatively unexplored.

The term "homomesy" (fka "combinatorial ergodicity") describes the following widespread phenomenon: Given a group action on a set of combinatorial objects, a statistic on these objects is called "homomesic" if its average value is the same over all orbits. Along with its intrinsic interest as a kind of "hidden invariant", homomesy can be used to prove certain properties of the action, e.g., facts about the orbit sizes. Homomesy can often be found among the same dynamics that afford cyclic sieving. Proofs of homomesy often involve developing tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection.

This talk will be a introduction to these ideas, giving a number of examples of such actions and pointing out connections to other areas.

Acknowledgments

This talk discusses joint work, mostly with (chronologically) Jim Propp, Mike Joseph, and Darij Grinberg.

I'm grateful to Matthew, Mike, and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

Thanks also to Drew Armstrong, Arkady Berenstein, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, David Einstein, Darij Grinberg, Shahrzad Haddadan, Sam Hopkins, Mike La Croix, Svante Linusson, Gregg Musiker, Matthew Plante, Vic Reiner, Bruce Sagan, Richard Stanley, Jessica Striker, Ralf Schiffler, Hugh Thomas, Nathan Williams, and Ben Young.

And to Bernd Sturmfels and MPI MiS for gracious hospitality.

Please feel free to ask questions or make comments as we go along.

Some themes in dynamical algebraic combinatorics

1. Periodicity/order;
2. Orbit structure;
3. Homomesy;
4. Equivariant bijections; and
5. Lifting from combinatorial to piecewise-linear and birational settings.

Cyclic rotation of binary strings

"Immer mit den einfachsten Beispielen anfangen." — David Hilbert

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \underset{C_R}{\longmapsto} 010100$$

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- *Periodicity* is clear here. The map has order $n = 6$.
- *Orbit structure* is very nice; every orbit size must divide n.
- *Homomesy?* Need a statistic, first.
- *Equivariant bijection?* No need.

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6$, $k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case, we say that the function f is **homomesic** with average c (also called c-**mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Cyclic rotation of binary strings

Example

$n = 6, \ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
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Cyclic rotation of binary strings

Example

$n = 6,\ k = 2$

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010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well:
Let $\chi_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic?

Bulgarian Solitaire

Homomesy: A more general definition

There are some cases where we find a similar phenomenon, but where the map no longer has finite orbits. Here is a more general definition of homomesy that is useful for some purposes.

Definition

Let τ be an self-map on a discrete set of objects S, and f be a statistic on S. We say f is **homomesic** if the value of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = c$$

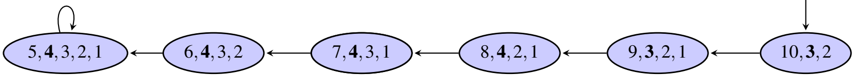
is **independent** of the starting point $x \in S$. (Also, f is c-mesic.)

This clearly reduces to the earlier definition in the case where we have an invertible action with finite orbits.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define б(λ) as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

- First surfaced as a puzzle in Russia around 1980, with a solution by Andrei Toom in *Kvant*; later popularized in a 1983 Martin Gardiner column; see survey of Brian Hopkins [Hop12].
- Initial puzzle: starting from any of 176 partitions of 15, one ends at $(5, 4, 3, 2, 1)$.



Bulgarian solitaire: “orbits” are now “trajectories”

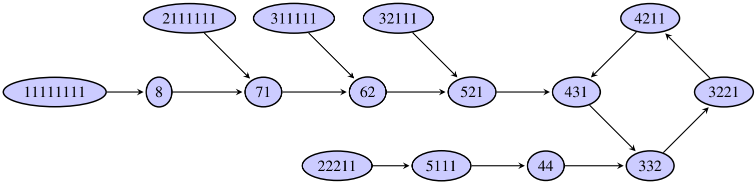
E.g., for $n = 8$, two trajectories are

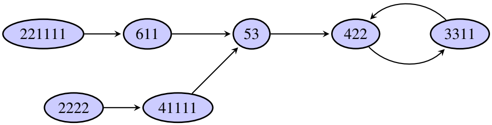
$$53 \to 42\underline{2} \to \underline{3}311 \to \underline{4}22 \to \dots$$

and

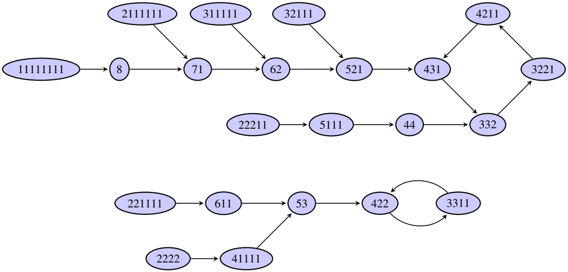
$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \underline{3}32 \to \underline{3}221 \to \underline{4}211 \to \underline{4}31 \to \dots$$

(the new heaps are underlined).





Bulgarian solitaire: homomesies



Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5, 3)$, the average value of φ is $(4 + 3)/2 = 7/2$; while for $\lambda = (6, 2)$, the average value of φ is $(3 + 4 + 4 + 3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic wrt/б; similarly for "size of (kth) largest part".

Further work on non-invertible maps and possible directions

Colin Defant has recently done some work on non-invertible maps that are variants of pop-stack sorting (including a dual version called pop-tsack torsing) [Def21+]. In his talk on this at the BIRS-DAC workshop at UBCO, he highlighted the following questions:

Let $f : X \to X$ be a noninvertible map on a finite set X. Define the *forward orbit* of $x \in X$ to be $O_f(X) := \{x, f(x), f^2(x), \dots\}$.

- What are the periodic points of f?
- What is the image of f?
- How can we compute the number of preimages of some $x_0 \in X$ under f?
- What is the maximum number of preimages an element of X can have?
- How many elements have exactly 1 preimage?
- What is $\max_{x \in X} \#O_f(X)$? For which x is the max attained?
- Which elements $x \in X$ maximize $O_f(X)$?
- How big is $O_f(X)$ on average?

Ignore transience (for the rest of this talk)

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

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This definition also works in situations where S is infinite. But for rest of this talk, we'll restrict attention to maps τ that are invertible on S, where S is finite, so our initial definition (below) makes sense.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
(Call f c-**mesic** for short.)

Coxeter Toggling

Independent Sets

of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

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Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the factor 11.

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex i** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

Toggles

Proposition

Each toggle τ_i is an involution, i.e., τ_i^2 is the identity. Also, τ_i and τ_j commute if and only if $|i-j| \neq 1$.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Order & Orbits

- The order of this action grows quite fast as n increases and is difficult to describe in general. It is the LCM of the orbit sizes, which are not all divisors of some small number (relative to n): $2, 3, 6, 15, 24, 231, 210, 1989, 240, 72105, 18018, 3354725, \ldots$
- For $n = 6$ the three orbits have sizes 3, 7, 11, giving order LCM(3,7,11)$=$ 231.
- The number of orbits appeared to be OEIS A000358 ("Number of binary necklaces of length n with no subsequence 00") , but we didn't understand why at first.
- This means that this action is unlikely to exhibit interesting Cyclic Sieving.
- But we can still find homomesy.

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	2	3	4	5	6	7
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

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	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph–R. [JR18])

Define $\chi_i : \mathcal{I}_n \to \{0, 1\}$ *to be the indicator function of vertex* i.

For $1 \leq i \leq n$, $\chi_i - \chi_{n+1-i}$ *is 0-mesic on* φ*-orbits of* $\mathcal{I}_n$.

Also $2\chi_1 + \chi_2$ *and* $\chi_{n-1} + 2\chi_n$ *are 1-mesic on* φ*-orbits of* $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
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$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1)$ mod 4.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using elementary Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Hanaoka & Sadahiro have generalized the "palindromic" homomesy to the case of "m-independent sets", leading them to an interesting variation of bitstring rotation [HS22]. Video lecture from BIRS-DAC (Kelowna) is available at `https://www.birs.ca/events/2021/5-day-workshops/21w5514/videos`

Rowmotion on Order Ideals of a Poset

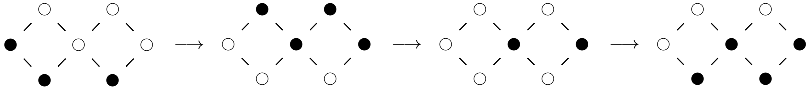
Rowmotion: an invertible operation on order ideals

We define the (cyclic) group action of **rowmotion** on the set of order ideals $\mathcal{J}(P)$ via the map $\text{Row} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the following three-step process.

Start with an order ideal, and

1. Θ: Take the complement (giving an order filter)
2. ∇: Take the minimal elements (giving an antichain)
3. Δ^{-1}: Saturate downward (giving a order ideal)

ρ_J :



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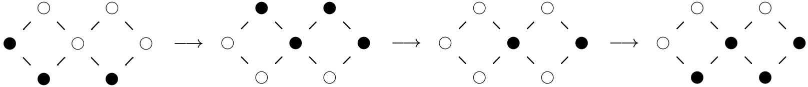


This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Dynamical properties of rowmotion: cardinality is homomesic

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a+b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a+b)/d$, some d dividing both a and b.

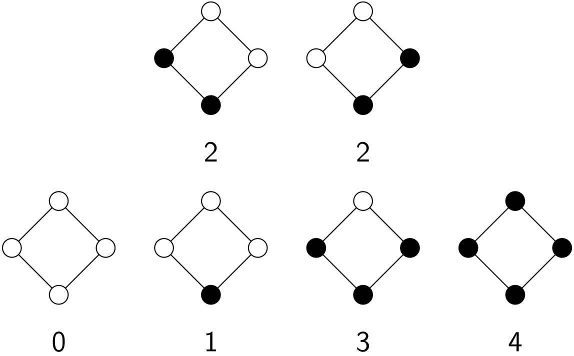
Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary rowmotion orbit in $\mathcal{J}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

We have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Toggling order ideals

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order ideals* (equivalently *order filters*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{J}(P)$ be the set of order ideals of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{J}(P) \to \mathcal{J}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{J}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{J}(P), \\ U & \text{otherwise.} \end{cases}$$

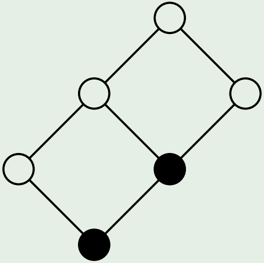
Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order ideals of P.

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example



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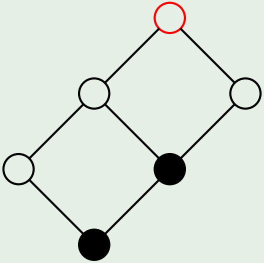

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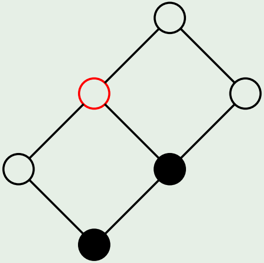

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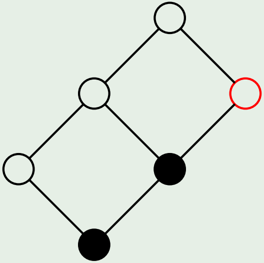

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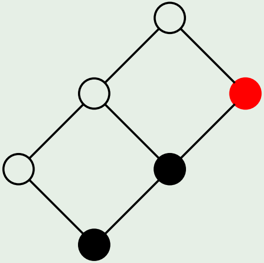

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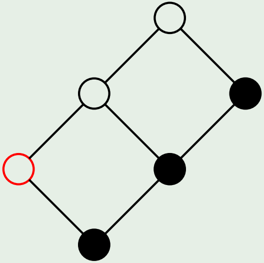

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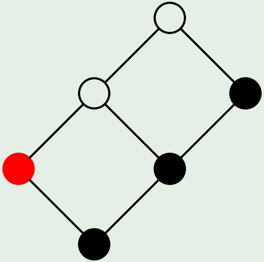

Toggling order ideals

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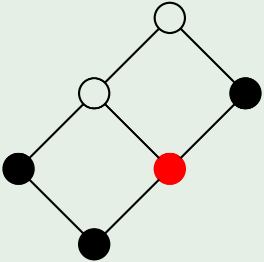
Example



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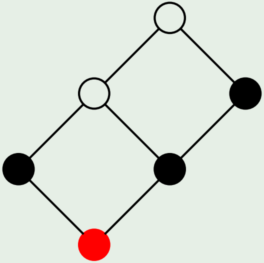

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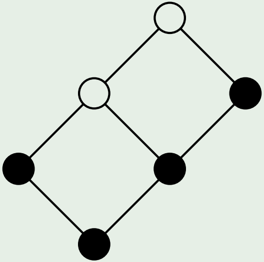
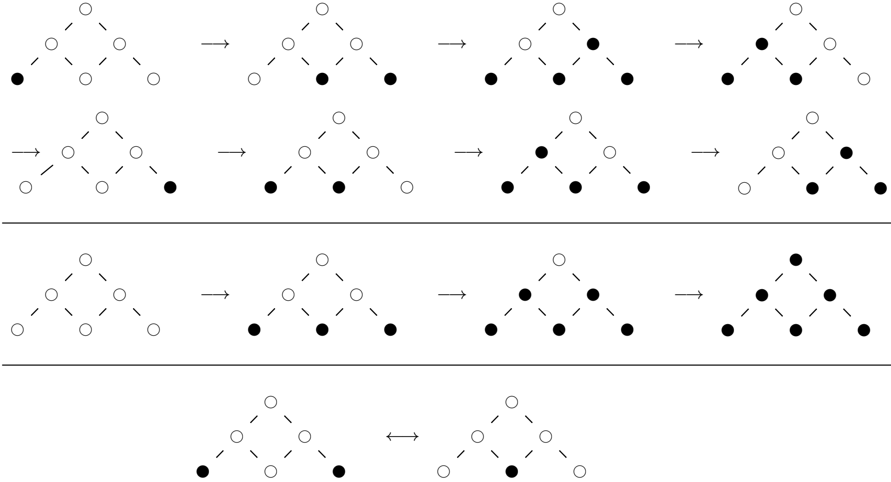


Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:



Checking the average cardinality for each orbit we find that

$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}; \quad \frac{0+3+5+6}{4} = \frac{7}{2}; \quad \frac{2+1}{2} = \frac{3}{2}.$$ Darn!

Example of order ideal rowmotion on A_3 root poset

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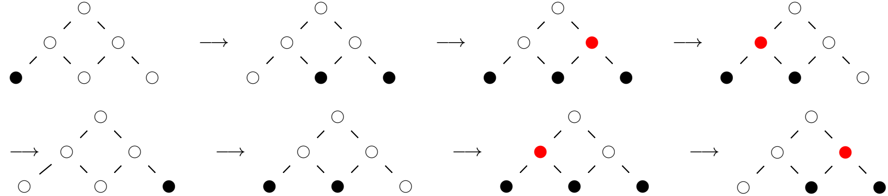
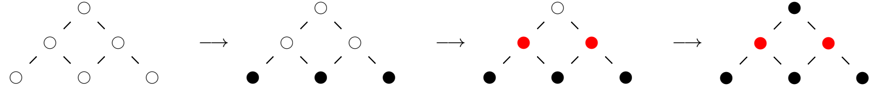
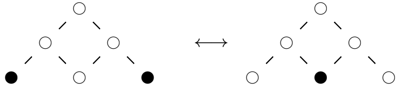






Checking the average rank-alternating cardinality for each orbit we find:

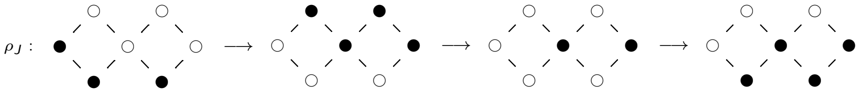
$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{1+2+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}$$ Yay!

Root posets of type A: rank-signed cardinality is homomesic

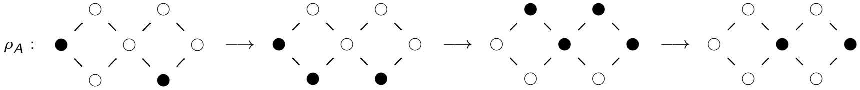
Theorem (Haddadan)

Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign an order ideal $I \in \mathcal{J}(P)$ weight $f(I) = \sum_{x \in I} \mathrm{wt}(x)$, then f is homomesic under rowmotion and promotion, with average $n/2$.

We've already seen examples of Rowmotion ρ on $J(P)$, the set of order ideals of a poset P.



We can also define it as an operator $\rho_{\mathcal{A}}$ on the set of **antichains** of P by shifting the waltz beat by 1:



Following Jessica Striker, Michael Joseph defined antichain toggles and showed that $\rho_{\mathcal{A}}$ can be written as a product of these toggles moving from bottom to top along any linear extension. These toggles commute less frequently than order-ideal toggles, giving a different, though related, theory.

Panyushev's conjecture (AST's theorem)

One of the earliest examples of homomesy appeared around the time Propp and Roby isolated the phenomenon.

Let Δ be a (reduced irreducible) root system in $\mathbb{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong–Stump–Thomas [AST11], Conj. [Pan09])

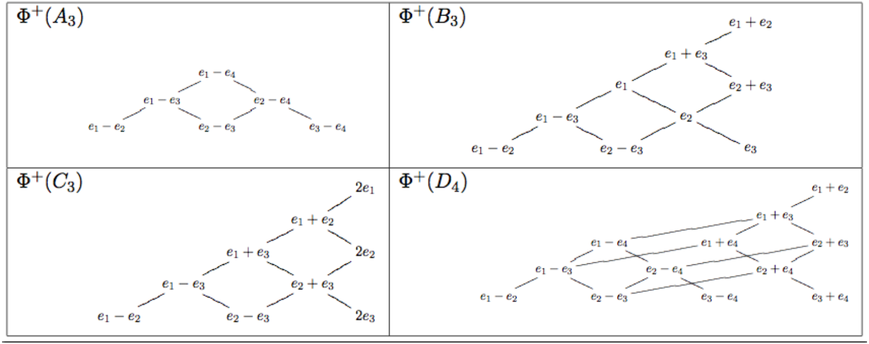
Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

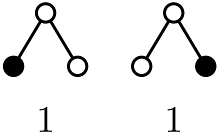
Picture of root posets

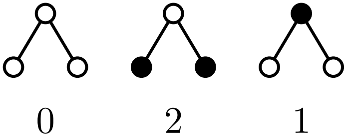
Here are the classes of posets included in Panyushev's conjecture.



(Graphic courtesy of Striker–Williams.)

Here we have just an orbit of size 2 and an orbit of size 3:

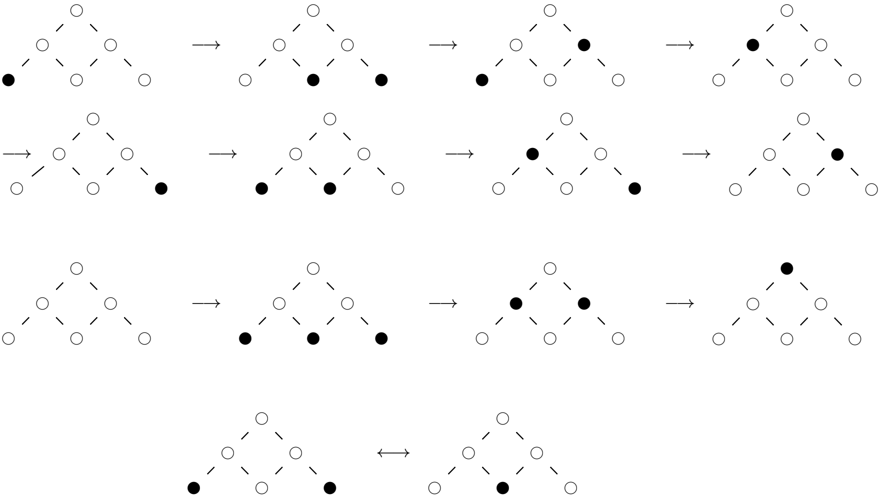




Within each orbit, the average antichain has cardinality $n/2 = 1$.

Example of antichain rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



Checking the average cardinality for each orbit we find that

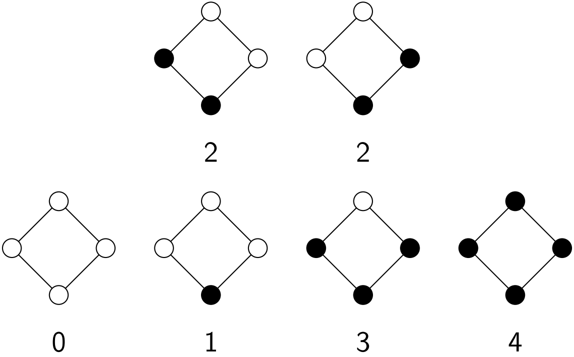
$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

Back to order-ideal rowmotion. . .

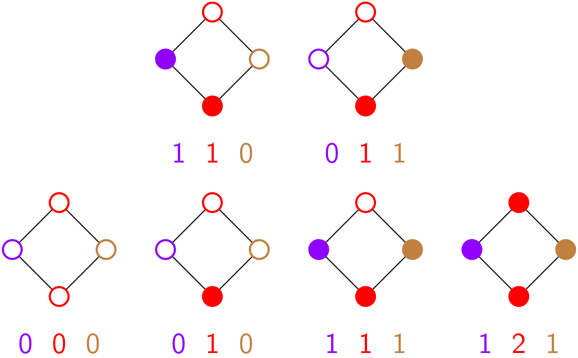
FIX

For the poset $P = [2] \times [2]$ have one orbit of size 2 and one orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ of a violet element, 1 red element, and $1/2$ of a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1-b \leq k \leq a-1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i-j=k\}.$$

For $1-b \leq k \leq a-1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every ρ-orbit $\mathcal{O}$ in $J([a] \times [b])$:

- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Piecewise-linear and birational liftings

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

For a finite poset P, let $\widehat{P}$ denote P with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : \widehat{P} \to [0,1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

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where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

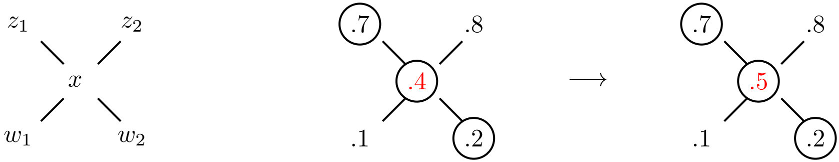
Note that the interval $[\min_{z \gtrdot x} f(z), \max_{w \lessdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

If $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \qquad \text{to} \qquad \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints of the interval.

Example of flipping at a node

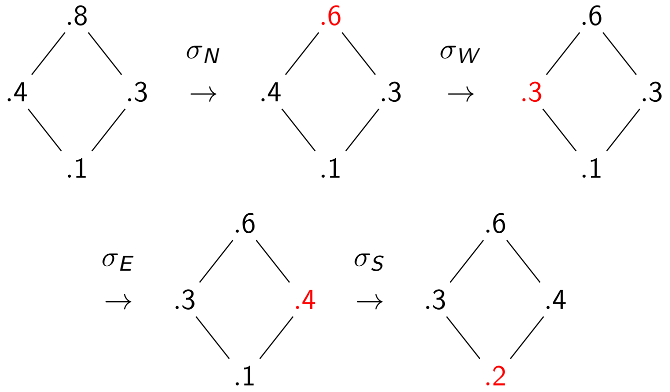


$$\min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

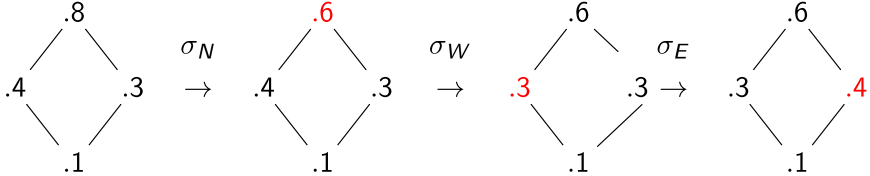
Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



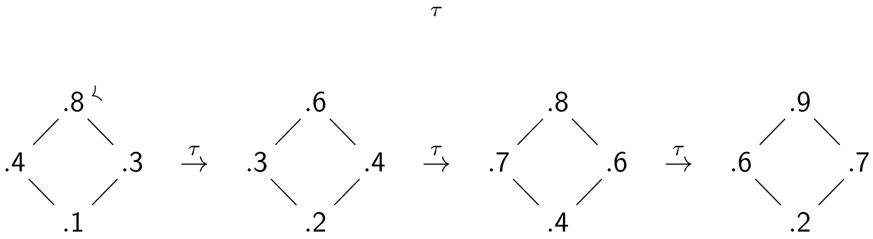
(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

Composing flips and example of PL rowmotion orbit

We can apply flip-maps from top to bottom (successively flipping at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.), to get *piecewise-linear rowmotion*:



Here's an orbit of this map ($\tau = \sigma_S \circ \sigma_E \circ \sigma_W \circ \sigma_N$), which again has period 4.



De-tropicalizing to birational maps

In the *tropical semiring*, one replaces the standard binary ring operations $(+,\cdot)$ with the tropical operations $(\max,+)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0,1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w \lessdot x} f(w)}{f(x) \sum_{z \gtrdot x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- For a field $\mathbb{K}$, a $\mathbb{K}$-**labelling of** P will mean a function $f : \widehat{P} \to \mathbb{K}$. We always set $f(\widehat{0}) = f(\widehat{1}) = 1$.
- For any $v \in P$, define the **birational** v**-toggle** as the rational map

  $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $(T_v f)(w) = \dfrac{\sum\limits_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum\limits_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.

  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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  $w = v$.

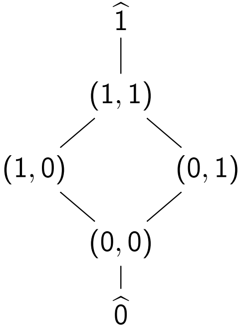
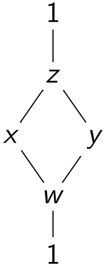
  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)
- This is a **local change** only to the label at v, and $T_v^2 = id$ (on the range of T_v).
- We define **birational rowmotion** as the rational map

  $$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset	labelling
	

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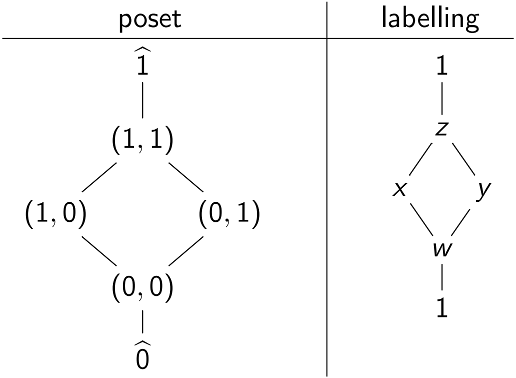


We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

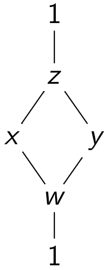
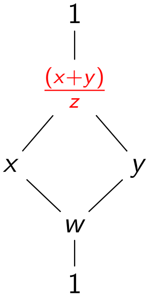
using the linear extension

$((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion: example

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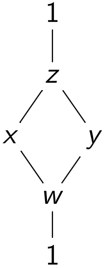	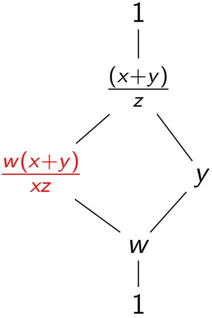

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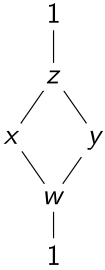	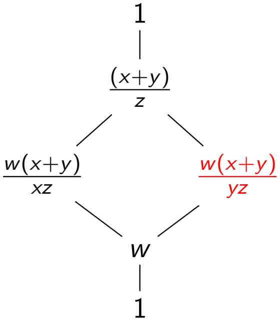

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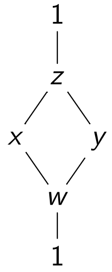
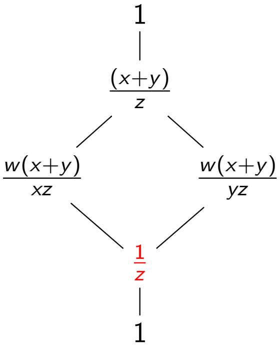
Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$
	

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example: Iterating this procedure we get

$$\rho_B f = \begin{array}{ccc} & \frac{(x+y)}{z} & \\ \frac{(x+y)w}{xz} & & \frac{(x+y)w}{yz} \\ & \frac{1}{z} & \end{array}, \qquad \rho_B^2 f = \begin{array}{ccc} & \frac{(x+y)w}{xy} & \\ \frac{1}{y} & & \frac{1}{x} \\ & \frac{z}{x+y} & \end{array},$$

$$\rho_B^3 f = \begin{array}{ccc} & \frac{1}{w} & \\ \frac{yz}{(x+y)w} & & \frac{xz}{(x+y)w} \\ & \frac{xy}{(x+y)w} & \end{array}, \qquad \rho_B^4 f = \begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array}.$$

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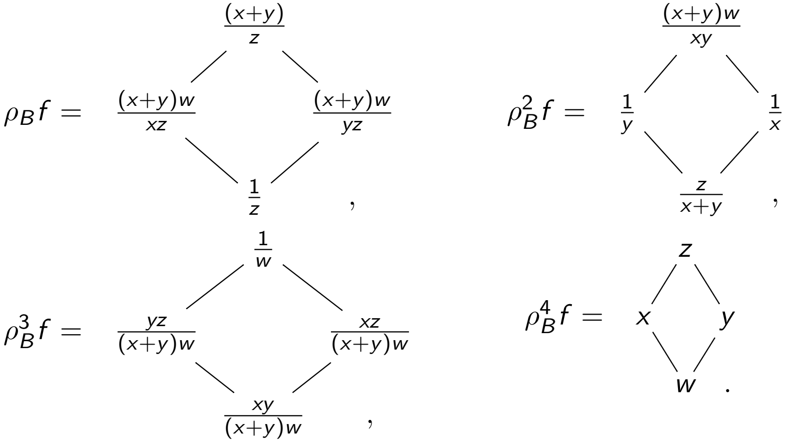

Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Why study this generalization?

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- This generalization implies the results at the PL and combinatorial level (but not vice-versa).
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these maps all have natural homomesies [PrRo15, EiPr13/21].
- Periodicity of these systems is generally nontrivial to prove.

Proof of periodicity via Grassmannian embedding

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  - Show that "almost all" labellings of P are in the image of a certain map Grasp_0 from the matrix space $\mathbb{K}^{p \times (p+q)}$ to $\mathbb{K}^{\widehat{P}}$. Explicitly, if $A \in \mathbb{K}^{p \times (p+q)}$ is any matrix, then $(\operatorname{Grasp}_0 A)(0) = (\operatorname{Grasp}_0 A)(1) = 1$ and

$$(\operatorname{Grasp}_0 A)(i, j) = \frac{\det(A[1 : i \mid i + j - 1 : p + j])}{\det(A[0 : i \mid i + j : p + j])}$$

    for all $(i, j) \in P$, where the $A[a : b \mid c : d]$s are certain submatrices of A. (Note that this map Grasp_0 actually factors through the Grassmannian.)

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  - Construct a commutative diagram

$$\begin{array}{ccc} \mathbb{K}^{p\times(p+q)} & \xrightarrow{\mathrm{Grasp}_0} & \mathbb{K}^{\widehat{P}} \\ \downarrow \rho & & \downarrow R \\ \mathbb{K}^{p\times(p+q)} & \xrightarrow{\mathrm{Grasp}_0} & \mathbb{K}^{\widehat{P}} \end{array},$$

    where ρ is (more or less) rotating the matrix horizontally (last column to front).

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  - Conclude that $R^{p+q} = \mathrm{id}$ because $\rho^{p+q} = \mathrm{id}$.

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    where ρ is (more or less) rotating the matrix horizontally (last column to front).
  - Conclude that $R^{p+q} = \mathrm{id}$ because $\rho^{p+q} = \mathrm{id}$.
  - Reciprocity also easy using Grasp_0.

Birational homomesy on files of $J([0,r] \times [0,s])$

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_B^k(f)(1,0) = \frac{(x+y)w}{xz} \quad \frac{1}{y} \quad \frac{yz}{(x+y)w} \quad (x) = 1,$$

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$$\prod_{k=1}^{4} \rho_B^k(f)(0,0)\rho_B^k(f)(1,1) = \frac{1}{z} \ \frac{x+y}{z} \ \frac{z}{x+y} \ \frac{(x+y)w}{xy} \ \frac{xy}{(x+y)w} \ \frac{1}{w} \ (w) \ (z) = 1,$$

$$\prod_{k=1}^{4} \rho_B^k(f)(0,1) = \frac{(x+y)w}{yz} \ \frac{1}{x} \ \frac{xz}{(x+y)w} \ (y) = 1.$$

Each of these products equaling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

Birational homomesy on files of $J([0,r] \times [0,s])$

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *is periodic, with period* $r+s+2$*.*

(2) The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *satisfies the following reciprocity:*
$\rho_B^{i+j+1} f(i,j) = 1/\rho_B^0 f(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem (Musiker–R [MR19])

Given a file F *in* $[0,r] \times [0,s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k f(i,j) = 1.$

Birational homomesy on files of $J([0,r] \times [0,s])$

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ is periodic, with period $r+s+2$.

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Theorem (Musiker–R [MR19])

Given a file F in $[0,r] \times [0,s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k f(i,j) = 1.$

The proof of this involves constructing a complicated formula for the ρ_B^k in terms of families of non-intersecting lattice paths, from which one can also deduce periodicity and the other geometric homomesies of this map, first proved by Grinberg-R [GrRo15b, Thm. 32].

Noncommutative Liftings

Much of this story lifts to skew fields, where the variables are not assumed to commute.

- In this setting toggles are no longer involutions, but the NC analogue of ρ_B can be defined, and their inverses can be included in the study.
- A version of periodicity miraculously still holds, though even checking this computationally with rational expressions in noncommutative variables was a challenge.
- In parallel with the lifting of ρ to ρ_B, there is a lifting of $\rho_{\mathcal{A}}$ via Stanley's Chain polytope to birational (*BAR-motion*) and NC (*NAR-motion*) [JR20].
- The Stanley–Thomas word which we used to show periodicity and homomesy for $\rho_{\mathcal{A}}$ lifts all the way to the NC setting, where it still shows homomesy. However, it does not show periodicity outside the combinatorial realm, since it no longer losslessly encodes the labelings [JR21].
- Quite recently Darij Grinberg and I have found a proof of periodicity for rectangles, which requires its own talk.

Noncommutative Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the partial map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

for all $w \in \widehat{P}$.
Here (and in the following), $\overline{m}$ means m^{-1} whenever $m \in \mathbb{K}$.

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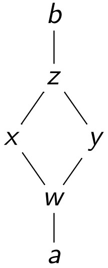
- We define **(noncommutative) birational rowmotion** as the partial map

$$R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

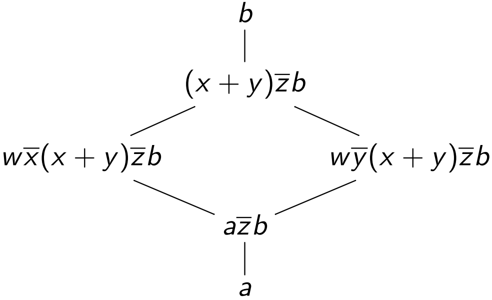
  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.

Noncommutative rowmotion on a 2×2 rectangle

Here is $R^0 f$:



Here is $R^1 f$:


b
$(x+y)\overline{z}b$
$w\overline{x}(x+y)\overline{z}b$
$w\overline{y}(x+y)\overline{z}b$
$a\overline{z}b$
a

Here is $R^2 f$:

$$b$$

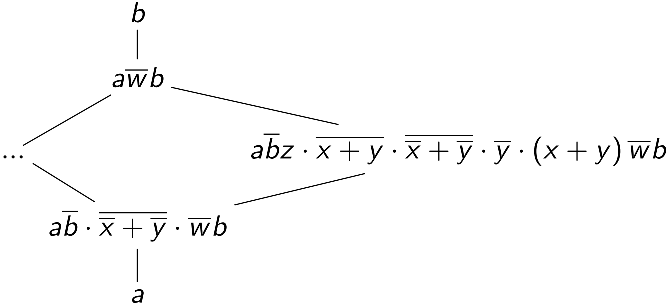
$$w\left(\overline{x}+\overline{y}\right)b$$

$$a \cdot \overline{x+y} \cdot x\left(\overline{x}+\overline{y}\right)b \qquad\qquad a \cdot \overline{x+y} \cdot y\left(\overline{x}+\overline{y}\right)b$$

$$a\overline{b}z \cdot \overline{x+y} \cdot b$$

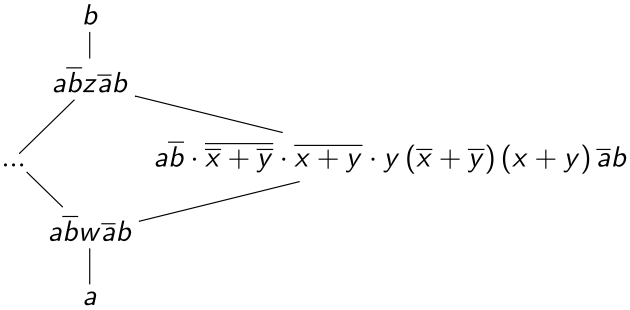
$$a$$

Here is $R^3 f$:

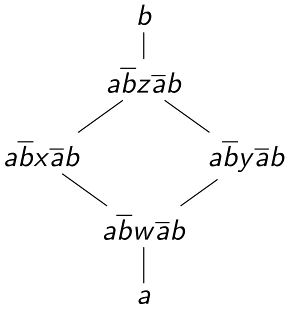

b
$a\overline{w}b$
...
$a\overline{b}z \cdot \overline{x+y} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{y} \cdot (x+y)\,\overline{w}b$
$a\overline{b} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{w}b$
a

Noncommutative rowmotion on a 2×2 rectangle

Here is $R^4 f$:



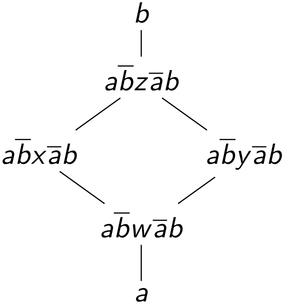
Here is $R^4 f$:



(after nontrivial simplifications).

Noncommutative rowmotion on a 2×2 rectangle

Here is $R^4 f$:



This confirms the "periodicity" theorem for NCBR when $p = q = 2$.

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Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to questions of periodicity, orbit structure, homomesy, and equivariant bijections
- Actions that can be built out of smaller, simpler actions (e.g., toggles) often have interesting and unexpected properties.
- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at higher levels.
- Much more remains to be explored, perhaps for combinatorial objects or actions that you work with for other reasons.

Slides for this talk will be available online at: Google "Tom Roby".

Thanks very much for coming to this talk!

Danke schön für eure Aufmerksamkeit!

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- We now believe this approach is a dead end.