Math 2210Q (Roby) **Practice Midterm #2 Solutions** Fall 2017

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam **no calculators (or other electronic devices) are to be used.**, but you may use two pages (i.e., one sheet, both sides) of ordinary 8.5×11-inch (or A4) paper with any **handwritten (by you)** notes or formulae you like.

1. **REVIEW COURSE MATERIALS:**
   (a) Check all of your worksheets against the worksheet solutions;
   (b) Check all of the homework solutions;
   (c) Review all the problems on quizzes, on the first midterm, and on the first practice midterm, with particular attention to anything you got wrong the first time.
   (d) Review Ximera quizzes;
   (e) Review video lectures, especially anything you found confusing.
   (f) Ask questions in Piazza or in class!

2. Let A be the matrix $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & -2 & 1 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

   (a) Compute $\det A$. The answer is 6. This may be computed in a couple of ways: (1) by doing row reductions to transform A to triangular form, keeping track of any moves that modify the determinant or (2) expanding by cofactors (minors) along a suitable row or column.

   (b) Compute $\det(A^{-1})$ without computing A^{-1}. Since $\det(A^{-1}) = 1/(\det A)$, the answer is $1/6$.

   (c) Use Cramer's Rule to find x_4 so that $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

   To apply Cramer's rule, for x_4, we replace the fourth column of A with the output vector, take the determinant, and divide that by the determinant of the original matrix (computed above to be 6). Therefore,

$$x_4 = \frac{1}{6} \begin{vmatrix} 1 & 1 & 0 & 2 \\ 2 & -2 & 1 & 2 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{6}\left((-1) \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & -2 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 1 & 2 \\ 2 & -2 & 2 \\ 1 & 0 & 0 \end{vmatrix} \right) \quad = \frac{1}{6}(-12 - 6) = -3$$

3. Find the volume of the parallelepiped determined by the vectors $\begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Take the absolute value of the determinant of the matrix of column vectors to get:

$$\left| \begin{vmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ 7 & 1 & 4 \end{vmatrix} \right| = |-5| = 5.$$

4. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ -1 & -2 & 0 & 1 & 3 \\ 2 & 4 & 4 & 2 & 2 \end{bmatrix}$. Find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$. What should the sum of the dimensions of these two subspaces be? Does your answer check?

By row reduction we see that $A \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ has pivots in columns 1 and 3, so we use those columns of A as a basis for $\operatorname{Col} A$. For $\operatorname{Nul} A$, we parameterize the solutions in terms of the free variables to get the basis shown below.

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}, \text{ and } \operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

5. Define a transformation $T : \mathbb{P}_3 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(2) \end{bmatrix}$.

   (a) Show that T is a linear transformation.
   Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}_3$. Then

   $$T(\mathbf{p} + \mathbf{q}) = \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(2) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(2) + \mathbf{q}(2) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(2) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(2) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}).$$

   Similarly, one shows that $T(c\mathbf{p}) = cT(\mathbf{p})$ for any $c \in \mathbb{R}$.

   (b) Describe the kernel and range of this linear transformation.
   By def. $\ker T$ is the set of polys that map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $\{\mathbf{p} \in \mathbb{P}_3 : \mathbf{p}(0) = \mathbf{p}(2) = 0\}$, while range T is all of $\mathbb{R}^2$, since for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, we can always find a polynomial $\mathbf{p}$ with $\mathbf{p}(0) = a$ and $\mathbf{p} = b$ (e.g., by Lagrange interpolation, or less fancily by noting that matrix below has two pivot columns, so the dimension of range $T = \dim \operatorname{Col} A = 2$).

(c) Write the matrix A of this linear transformation in terms of the standard bases for $\mathbb{P}_3$ and $\mathbb{R}_2$.

(d) Compute a basis for Nul A.

(e) Compute a basis for Col A. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \end{bmatrix}$, which is already almost in RREF. So we get bases $\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $\text{Col}\, A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$.

6. Find the dimensions of Nul A and Col A for the matrix $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

We have $\dim \text{Col}\, A = \text{rk}\, A = \#\text{pivot cols} = 3$, and $\dim \text{Nul}\, A = \#\text{free vars} = 6-3 = 3$, by the rank-nullity theorem.

7. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? What is the smallest possible dimension? What if A is 3×4 matrix? Explain!

Since Row A is spanned by 4 vectors in $\mathbb{R}^3$, it has dimension at most 3, and $A = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}$, shows that dimension is achievable. The smallest possible dimension is 0, achieved by $A = 0$. Similar reasoning shows the same bounds for a 3×4 matrix.

8. For each statement below indicate whether it is **true** or **false**, and give **reasons** to support your answer. To show something is false, usually it is best to give a specific simple counterexample. Extra credit for "salvaging" false statements to make them correct.

(a) If A is a 2×2 matrix with a zero determinant, then one column of A is multiple of the other. T

(b) If λ is an eigenvalue of an $n \times n$ matrix M, then λ^2 is an eigenvalue of M^2. T

(c) If A and B are $n \times n$ matrices with $\det A = 2$ and $\det B = 3$, then $\det(A+B) = 5$. F

(d) $\det A^T = -\det A$. F

(e) The number of pivot columns of a matrix equals the dimension of its column space. T

(f) Any plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. F

(g) The dimension of the vector space $\mathbb{P}_4$ is 4. F

(h) If $\dim V = n$ and S is a linearly indpendent set in V, then S is a basis for V. F

(i) If there exists a linearly dependent set $\{v_1, \ldots, , v_p\}$ that spans V, then $\dim V \leq p$. T

(j) The eigenvectors of any $n \times n$ matrix are linearly independent in $\mathbb{R}^n$. F

(k) The range of a linear transformation is a vector subpace of the codomain. T

(l) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$ F

(m) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$. T

(n) The row space of A^T is the same as the column space of A. T

9. Let $\mathcal{S} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$.

(a) Is $\mathcal{S}$ linearly independent in $\mathbb{P}_2$? Explain!

(b) Is $\mathcal{S}$ a basis for $\mathbb{P}_2$? Explain!

(c) Express $\mathbf{p}(t) = 3 + t - 6t^2$ as a linear combination of elements of $\mathcal{S}$.

(d) Is the expression unique? Explain!

The standard isomorphism $\mathbb{P}_2 \to \mathbb{R}^3$ given by $1 \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $t \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $t^2 \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ takes the polynomials in S to the columns of the matrix

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

which is invertible. So the columns of C form a basis for $\mathbb{R}^3$, which means the original set $\mathcal{S}$ is a basis for $\mathbb{P}_2$ (b). In particular, this means that there is a unique way of writing any polynomial as a linear combination of the basis elements (d). The usual techniques for solving $C\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ give $\mathbf{x} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$, so

$$\mathbf{p}(t) = 3 + t - 6t^2 = 7(1 - t^2) - 3(t - t^2) - 2(2 - 2t + t^2)\,.$$

as one can easily check.

10. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain! No. This system is equivalent to $A\mathbf{x} = b$, where $\dim \operatorname{Nul} A = 2 \implies \dim \operatorname{Col} A = 8 - 2 = 6 \implies \mathbf{x} \mapsto A\mathbf{x}$ is *onto* $\mathbb{R}^6$; hence, every right hand side is obtainable.

11. Here are some specific tasks you should be able to accomplish **with demonstrated understanding**:

(a) **Everything** listed already on the first practice midterm.

(b) Know the *definitions* and *geometric interpretations* of the following basic terms:

- The **determinant** of a matrix A (recursive cofactor expansion) and its interpretation as signed volume of the parallelopiped defined by the columns (or rows) of A.
- A **vector space** (via ten axioms), a **subspace** (of a vector space).
- A **linear transformation** $T : V \to W$ between two (general) vector spaces V and W.
- **Linear (in)dependence** and **span** of sets of vectors in a general vector space V, and a **basis** for a subspace $\mathcal{S}$ of V.
- The **coordinates** of $\mathbf{x}$ with respect to a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space V, and the **coordinate mapping** $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ from V to $\mathbb{R}^n$.
- An **isomorphism** $T : V \to W$ between two vector spaces (i.e., a one-to-one and onto linear transformation).
- The **dimension** of a vector space and the **rank** of a matrix A.
- An **eigenvector** and **eigenvalue** of a square matrix A.
- **Similarity** of two square matrices A and B.

(c) Row reduce a matrix A to echelon and/or reduced echelon form. Use this process and an understanding of pivot positions to (in addition to items on PM#1):

- compute the determinant of a square matrix A;
- compute bases for $\operatorname{Col} A$, $\operatorname{Row} A$ and the dimensions of these subspaces; and
- compute eigenvectors corresponding to a given eigenvector λ of A.

(d) Understand how row operations affect $\det A$ and use them to reduce a matrix A to triangular form, in order to calculate $\det A$ (as product of diagonal entries). Use properties of determinants to compute the determinant of related matrices.

(e) Know basic properties of determinants, including:

i. $\det A^T = \det A$;
ii. $\det(AB) = (\det A)(\det B)$;
iii. A is invertible $\iff \det A \neq 0$; and
iv. $\det A^{-1} = \frac{1}{\det A}$.

(f) Use Cramer's Rule to compute the solution to a matrix system $A\mathbf{x} = \mathbf{b}$.

(g) Know and apply to specific examples: a linear transf $T : \mathbb{R}^n \to \mathbb{R}^n$ rescales the volume of a set (with finite volume) $S \subset \mathbb{R}^n$ by a factor of its determinant: $\operatorname{vol} T(S) = |\det A| \cdot \operatorname{vol} S$, where A is the matrix of T (with respect to any basis).

(h) Use definitions and theorems to determine whether a given **subset** $\mathcal{S}$ of a vector space V is in fact a **subspace** of V; in particular, explain why $\operatorname{Nul} A$ and $\operatorname{Col} A$ are subspaces.

(i) Use the **Spanning Set Theorem** to show that spanning sets always contain a basis, and linearly indepenent sets can always be extended to a basis. Know that each element can be written *uniquely* in terms of a basis.

(j) Given a matrix A, find the dimensions of and bases for $\operatorname{Col} A$, $\operatorname{Nul} A$, and $\operatorname{Row} A$. Use the relations among rank, $\dim \operatorname{Nul} A$, and size of A to understand properties of the associated linear transformation (one-to-one, onto, kernel, range).

(k) Understand that **dimension** measures the *size* of a vector space, and that a subspace H of a finite-dimensional vector space V has $\dim H \leq \dim V$. Know and apply the **Basis Theorem**, that if $\dim V = p$, then any set of p linearly independent vectors is a basis and any set of p vecotrs that spans V is a basis.

(l) Know how to prove **The Rank Theorem** (from our understanding of row reduction), that $\operatorname{rank} A + \dim \operatorname{Nul} A = \#\text{cols of } A$, and apply it to examples.

(m) Understand and apply in context additional conditions in the Invertible Matrix Theorem involving $\operatorname{Col} A$, $\operatorname{Nul} A$, and their dimensions, as well as those involving eigenvalues of A.

(n) Understand how to compute the **change of basis** matrix and how it allows one to translate between different coordinate systems for the same vector space V.

(o) Compute eigenvalues and eigenvectors in general and for special classes of matrices (e.g., triangular), using the definitions, characteristic equation, and row reduction.

(p) Prove that similar matrices have the same eigenvalues (with the same multiplicities) and disprove the converse (matrices with the same eigenvalues (counting multiplicities) need not be similar.

(q) Diagonalize square matrices when possible, and recognize when it's not possible. Understand that this is equivalent to having a **basis of eigenvectors**. Use the $A = PDP^{-1}$ factorization to calculate powers of A.

(r) Prove that eigenvectors corresponding to *distinct* eigenvalues are *linearly independent*; thus, a square matrix with distinct eigenvalues is diagonalizable.

(s) Understand the theory of the course so far well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.