Math 2210Q (Roby) **Practice Midterm #1** Solutions Fall 2017

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam **no calculators are to be used.**, but you may use one page of ordinary 8.5 × 11-inch (or A4) paper with any **handwritten** notes or formulae you like.

1. **REVIEW COURSE MATERIALS:**

   (a) Check all of your worksheets against the worksheet solutions;

   (b) Check all of the homework solutions;

   (c) Review Ximera quizzes;

   (d) Review video lectures, especially anything you found confusing.

   (e) Ask questions in Piazza!

2. Compute all solutions to the following linear system by reducing its associated augmented matrix to Reduced Row-Echelon Form.

$$\begin{array}{rcrcrcrl} x_1 & + & & & x_3 & & & = 2 \\ x_1 & + & 2x_2 & + & 5x_3 & + & 2x_4 & = 0 \\ 2x_1 & + & x_2 & + & 4x_3 & & & = 2 \\ x_1 & + & x_2 & + & 3x_3 & + & x_4 & = 1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 1 & 2 & 5 & 2 & 0 \\ 2 & 1 & 4 & 0 & 2 \\ 1 & 1 & 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|r} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ -2 - 2x_3 \\ x_3 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. Write each of the following vectors as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, or show that this cannot be done.

(a) $\begin{bmatrix} -2 \\ 14 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$

(a) Row reduce as follows:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 2 & -2 & 14 \\ 3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} -2 \\ 14 \\ 9 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 3 \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

(b)

$$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 2 & -2 & 4 \\ 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 1 & 8/3 \\ 0 & 1 & 29/5 \end{array}\right]$$

which is inconsistent, so no solution.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x - 3y \\ x + xy \end{bmatrix}$. Is T a **linear** transformation or not? Give a complete and careful explanation.

We compute $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$ while $T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 6 \end{bmatrix} \neq 2 \cdot T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so T is not linear.

5. Perform the indicated matrix operations, where $A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 & -1 \end{bmatrix}$, and $D = \begin{bmatrix} -2 & -3 \end{bmatrix}$. If any operation is not possible, explain why.

a) A^2 $\begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix}$ b) $B^T A$ $\begin{bmatrix} 3 & 4 \\ 1 & 3 \\ -4 & 8 \end{bmatrix}$ c) DAB $\begin{bmatrix} -25 & -13 & -4 \end{bmatrix}$

d) BC Incompatible dimensions: 2×3 and 1×3. e) D^3 Ditto: 1×3 and 1×3.

6. Let R be the linear transformation on two-dimensional real vectors that multiplies the vector components by 2. Let S be the linear tranformation which projects vectors onto the first coordinate (i.e., the x-axis). Let T be the linear transformation which rotates a vector by $\pi/2$ radians.

a) Write a matrix for the composite linear tranformation which performs R, then S, then T (in that order)

For each transformation, compute its action on the standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and write the images as the columns of the matrix. This leads to

$$R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and } T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies TSR = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

b) Use your answer to the previous part to compute the image of the vector $\mathbf{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ under this composite tranformation.

One easily computes that $TSR\mathbf{u} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$. Or one can apply each transformation in turn (just based on its geometrical description, if you like) to $\mathbf{u}$, getting $R\mathbf{u} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$, $SR\mathbf{u} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$, etc.

c) Is the composite transformation injective (one-to-one)? You may give your answer in clear English if you feel that no calculation is necessary.

No. One convincing argument is to note that if $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, then $TSR\mathbf{v} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} = TSR\mathbf{u}$. Or you could state that since the transformation S always zeroes out the second coordinate of any vector, that the image of any vector only depends on its first coordinate.

7. For each statement below indicate whether it is **true** or **false**, and give **reasons** to support your answer. To show something is false, usually it is best to give a specific simple counterexample. Extra credit for "salvaging" false statements to make them correct.

   (a) A homogeneous system of equations can be inconsistent. False. The zero vector is always a solution.

   (b) The columns of the standard matrix for a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix under T. True, See Theorem 10 in Section 1.9.

   (c) If A and B are matrices such that both products AB and BA are defined, then A and B must be square matrices of the same size.

   False. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then AB and BA are both defined, since either way the number of columns in the first factor equals the number of rows in the second. (Note that the entries are irrelevant here; only the dimensions matter.)

(d) If A, B, and C are matrices such that $AB = AC$, then $B = C$.

False. Let $A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$. Then $AB = AC = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, but $B \neq C$.

(e) If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

True. See proof in text.

(f) If two vectors in $\mathbb{R}^n$ are linearly dependent, then they both lie on the same line through the origin.

True. Let $c\mathbf{v} + d\mathbf{w} = 0$, where $\mathbf{v}, \mathbf{w}$ are vectors and $c, d \in \mathbb{R}$ are not both zero; WLOG (without loss of generality) say $d \neq 0$. Then we can write $\mathbf{w} = \frac{-c}{d}\mathbf{v}$, so $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$, and must lie on the line through $\mathbf{v}$ and the origin. (In the special case that $\mathbf{v} = 0$, then $\mathbf{w} = 0$ as well, and the statement clearly holds.)

8. Compute the inverse of each of the following matrices (if they exist). Show your work. (If you use a special formula, state the formula before applying it.) If an inverse does not exist, explain clearly why.

a) $\begin{bmatrix} 1 & 6 \\ -1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

(a) By the formula for inverting a 2×2 matrix, we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2 - (-6)} \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & -3/4 \\ 1/8 & 1/8 \end{bmatrix}.$$

b) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -1 & 0 & 1 \end{bmatrix}$ c) DNE.

9. Show that if a product of two $n \times n$ matrices AB is invertible, then so is A.

If AB is invertible, then $\exists$ a matrix W s.t. $(AB)W = I \implies A(BW) = I$ by associativity. Hence (BW) is a right inverse for A, so by IMT, A is invertible.

Note that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \implies AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so the statement is *false* if we remove the condition that A and B must be square.

10. Here are some specific tasks you should be able to accomplish **with demonstrated understanding**:

   (a) Know the *definitions* and *geometric interpretations* of the following basic terms:
   - **linear (in)dependence** and **span** of sets of vectors;
   - **trivial** versus **nontrivial** solutions;
   - **homogeneous** versus **inhomogeneous** linear systems;
   - **domain**, **range**, and **image** of **linear transformations** and what it means for a transformation to be **one-to-one** or **onto**;
   - the **transpose** and **inverse** of a matrix; **elementary matrices**;

   (b) Row reduce matrices to echelon and/or reduced echelon form. Use this process and an understanding of pivot positions to:
   - solve linear equations;
   - determine whether a given (column) vector is in the span of a given set of vectors;
   - determine whether a given set of vectors is linearly independent or dependent;
   - (given a vector $\mathbf{b}$ in the range of a linear transformation T) find $\mathbf{u}$ in the domain such that $T(\mathbf{u}) = \mathbf{b}$;
   - determine whether a matrix transformation is one-to-one and/or onto; and
   - compute the inverse of a matrix.

   (c) Fluently convert between all of the following: systems of linear equations, vector equations, matrix-vector equations, and augmented matrices.

   (d) Give solutions to linear systems in vector form, including parametric vector form if applicable.

   (e) Perform basic vector and matrix addition and scalar multiplication and know the underlying algebraic properties.

   (f) Interpret the product of a matrix A by a vector $\mathbf{x}$ as the linear combination of the columns: $x_1\mathbf{a}_1 \cdots x_n\mathbf{a}_n$.

   (g) Multiply two matrices A and B in two different ways: (1) column-by-column matrix-vector multiplication, and (2) row-column rule.

   (h) Describe how 2×2 matrices transform vectors in $\mathbb{R}^2$, including stretches, shrinks, shears, and rotations. Compute inverses of such matrices using the simple formula.

   (i) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.

   (j) Use various forms of the Invertible Matrix Theorem in context.