Math 2210Q (Roby) **Practice Final Solutions** Fall 2017

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam **no calculators (or other electronic devices) are to be used.**, but you may use two pages (i.e., one sheet, both sides) of ordinary 8.5×11-inch (or A4) paper with any **handwritten (by you)** notes or formulae you like.

For pedagogical reasons, we sometimes give answers here that are briefer than we would expect from you on the exam, so please ask if you are unsure how to fill in the details.

1. **REVIEW COURSE MATERIALS:**

   (a) Check all of your worksheets against the worksheet solutions;

   (b) Check all of the homework solutions;

   (c) Review all the problems on the midterms and practice midterms, with particular attention to anything you got wrong the first time.

   (d) Review Ximera quizzes;

   (e) Review video lectures, especially anything you found confusing.

   (f) Ask questions in Piazza!

2. Define a linear transformation $T : \mathbb{P}_2 \to R^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

   (a) Find the image under T of $\mathbf{p}(t) = 5 + 3t$. Plug in: $T(5+3t) = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

   (b) Show that T is a linear transformation.

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-1) \\ \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix}.$$

   (c) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

   (d) Is T one-to-one? Is T onto? Explain! By row reduction or computing the determinant, one easily sees that this matrix is nonsingular; hence, by the IMT, T is both one-to-one and onto.

3. Let $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

(a) Find the characteristic polynomial and the eigenvalues of A. Best to compute $|A - \lambda I|$ by expanding along the middle row. After routine computation one gets the eigenvalues $\lambda = -4, 1,$ and 7.

(b) Is A diagonalizable? If so, give P and D such that $A = PDP^{-1}$. If not, explain why not. Yes (since A has distinct eigenvalues). D is formed from the eigenvalues of A, thus $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$. To compute the columns of P, solve $(A - \lambda I)\vec{x} = \vec{0}$ for each eigenvalue λ (rescaling columns as desired to avoid fractions); here $P = \begin{bmatrix} -1 & -3 & 3 \\ 0 & 6 & 0 \\ 3 & 8 & 2 \end{bmatrix}$.

(c) Is A orthogonally diagonalizable? If so, give an orthogonal matrix P and diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not. No. A matrix is orthogonally diagonalizable if and only if it is symmetric, and A is not symmetric.

4. Let $A_1 = \begin{bmatrix} 3 & -5 & -1 \\ 1 & 1 & -3 \\ -1 & 5 & -3 \\ 3 & -7 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & 6 \\ 3 & -8 \\ 1 & -2 \\ 1 & -4 \end{bmatrix}$.

(a) One of these matrices has linearly independent columns. Which one? Justify your answer. Call this matrix A. $A = A_2$ since it only has two columns and neither one is a multiple of the other. (Although not necessary, one can compute a vector in $\operatorname{Nul} A_1$ to show that A_1's columns are linearly dependent, e.g., $\vec{v} = (2, 1, -1)$).

(b) Use Gram-Schmidt to find an orthogonal basis of $\operatorname{Col} A$. $\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.

$\vec{v}_2 = \vec{a}_2 - \operatorname{Proj}_{\vec{v}_1} \vec{a}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-36}{12}\right) \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for $\operatorname{Col} A$.

(c) Find a QR factorization of A. Q is the matrix whose columns form an orthonormal basis for Col A, so we need to divide each vector above by its norm to get $Q = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 & 3 \\ 3 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $R = Q^T A = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 & 3 & 1 & 1 \\ 3 & 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 3 & -8 \\ 1 & -2 \\ 1 & -4 \end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix} 12 & -36 \\ 0 & 12 \end{bmatrix}$.

5. For each statement below determine whether it is **True** or **False** and give **reasons** to support your answer. To show something is false, it is usually best to give a **specific simple numeric counterexample**. Extra credit for "salvaging" false statements (when possible) to make them correct.

(a) If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$. True. If the columns of Q are orthonormal, then by definition $Q^T Q = I$. Hence, multiplying both sides of $A = QR$ by Q^T yields $Q^T A = IR = R$.

(b) If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then S is linearly independent. False, since S may contain $\mathbf{0}$, e.g., $S = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{0}\right\}$ is orthogonal. However, any orthogonal set of *nonzero* vectors is linearly independent [§6.2, Thm. 4].

(c) If A and B are invertible $n \times n$ matrices, then AB is similar to BA. True, since $A^{-1}(AB)A = BA$.

(d) Each eigenvector of a square matrix A is also an eigenvector of A^2. True. If $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero v, then $A^2\mathbf{v} = \lambda^2\mathbf{v}$ [Why?]. So $\mathbf{v}$ is an eigenvector for A^2 (corresponding to the eigenvalue λ^2).

(e) There exists a 2×2 matrix with real entries that has no eigenvectors in $\mathbb{R}^2$. True. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\det(A - \lambda I) = \lambda^2 + 1$, which has no real (only complex) roots.

(f) If A is row equivalent to the identity matrix I, then A is diagonalizable. False. The matrix A in the example above is a counterexample.

(g) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself. True. The projection of $\mathbf{y}$ onto W is the vector in W that is closest to $\mathbf{y}$. If $\mathbf{y} \in W$, then that vector will be $\mathbf{y}$ itself. One can also see this by noting that the formulae in §6.3, Thm. 8 and §6.2, Thm. 5 for expanding $\mathbf{y}$ in terms of basis for W give the same coefficients.

(h) For an $m \times n$ matrix A, vectors in Nul A are orthogonal to vectors in Row A. True. By definition, $\mathbf{v} \in \text{Nul}\, A$ means that $A\mathbf{v} = 0$. But this just says that the result of taking the inner product of each row of A with $\mathbf{v}$ is zero. Hence, $\mathbf{v}$ is orthogonal to a basis for Row A, hence to any vector in Row A.

(i) The matrices A and A^T have the same eigenvalues, counting multiplicities. True. They have the same characteristic equation: $|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I|$.

(j) A nonzero vector can correspond to two different eigenvalues of A. False. If $Av = \lambda v$ and $Av = \mu v$ with $\lambda \neq \mu$, then $(\lambda - \mu)v = 0 \implies \mathbf{v} = 0$, since $\lambda - \mu \neq 0$.

(k) The sum of two eigenvectors of a square matrix A is also an eigenvector of A. False. Take $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then e_1 is an eigenvector for $\lambda = 2$, and e_2 is an eigenvector for $\lambda = 3$. But $A(e_1 + e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, which is not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(l) The dimension of an eigenspace of a (real) symmetric matrix A equals the multiplicity of the corresponding eigenvectors. True. A (real) symmetric matrix is orthogonally diagonalizable. So we can write $A = PDP^{-1}$ where the columns of P form an orthogonal basis of eigenvectors. A basis of eigenvectors exists only when the dimension of each eigenspace equals the algebraic multiplicity of the eigenvalue.

(m) (Extra Credit topic) The singular values of an $m \times n$ matrix can be $3, 1, -1, -3$. False, all singular values are positive!

6. If a $n \times n$ matrix A satifies $A^2 = A$, what can you say about the determinant of A? Since the determinant is multiplicative, we get $D = \det A = \det A^2 = (\det A)^2$. The only solutions to $D^2 = D$ are $D = 0$ or 1, so $\det A = 0$ or 1.

7. Let $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$.

(a) One of A_1 or A_2 has orthogonal columns. Which one? Justify your answer. Call this matrix A. A_1 has orthogonal columns, as one checks by computing the three pairs of inner products of its columns. For A_2, however, the inner product of the 2nd and 3rd cols is 2.

(b) Show that $A\vec{x} = \vec{b}$ is inconsistent. The augmented matrix $[A|\vec{b}]$ has RREF $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(c) Would you expect the orthogonal projection of $\vec{b}$ onto the column space of A to equal $\vec{b}$? Why or why not? No. Since $A\vec{x} = \vec{b}$ is inconsistent, $\vec{b} \notin \operatorname{Col} A$, which is the range of the matrix transformation; however, the project of $\vec{b}$ onto $\operatorname{Col} A$ **is** in $\operatorname{Col} A$, so $\vec{b}$ cannot equal its projection.

(d) Compute $\text{Proj}_{\text{Col}\,A}\,\vec{b}$. Since the columns of A are orthogonal, we can use the simple formula in terms of inner products (Lay § 6.5) to find this projection:

$$\widehat{\mathbf{b}} = \frac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \frac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2 + \frac{\vec{b}\cdot\vec{a}_3}{\vec{a}_3\cdot\vec{a}_3}\vec{a}_3 = \frac{1}{3}\vec{a}_1 + \frac{14}{3}\vec{a}_2 + \frac{-5}{13}\vec{a}_3 = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}.$$

(e) Compute a least-squares solution to $A\vec{x} = \vec{b}$. The least squares solution $\widehat{\mathbf{x}}$ is the vector of weights computed above, so $\widehat{\mathbf{x}} = \begin{bmatrix}1/3\\14/3\\-5/3\end{bmatrix}$.

8. Assume that matrices A and B below are row equivalent:

$$A = \begin{bmatrix}1 & 1 & -2 & 0 & 1 & -2\\ 1 & 2 & -3 & 0 & -2 & -3\\ 1 & -1 & 0 & 0 & 1 & 6\\ 1 & -2 & 2 & 1 & -3 & 0\\ 1 & -2 & 1 & 0 & 2 & -1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 1 & -2 & 0 & 1 & -2\\ 0 & 1 & -1 & 0 & -3 & -1\\ 0 & 0 & 1 & 1 & -13 & -1\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Without calculations, list $\text{rank}\,A$ and $\dim\text{Nul}\,A$. Then find bases for $\text{Col}\,A$, $\text{Row}\,A$, and $\text{Nul}\,A$. We get $\text{rank}\,A = 5$, so $\dim\text{Nul}\,A = 6 - 5 = 1$. A basis for $\text{Col}\,A$ is given by columns 1, 2, 3, 5, and 6 of A, while a basis for $\text{Row}\,A$ is given by all five rows of B (not of A). To get a basis for $\text{Nul}\,A$, we further reduce B to echelon form:

$$B \sim \begin{bmatrix}1 & 1 & -2 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \Longrightarrow \text{Nul}\,A = \text{Span}\left\{\begin{bmatrix}-1\\-1\\-1\\1\\0\\0\end{bmatrix}\right\}.$$

9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (You do not need to compute a vector at which this maximum is attained.) This is equivalent to finding the maximum value of $\vec{x}^T A\vec{x}$ subject to the constraint $\vec{x}^T\vec{x} = 1$. By Theorem 6, this is the greatest eigenvalue λ_1 of the matrix of the quadratic form, namely

$$A = \begin{bmatrix}7 & -1\\ -1 & 3\end{bmatrix} \Longrightarrow \lambda_1 = 5 + \sqrt{5} \text{ and } \lambda_2 = 5 - \sqrt{5}.$$

10. Compute the singular value decomposition (SVD) of $A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$, and deduce its rank. *Explain each step* so that your procedure is clear even if your computation has errors. $A^TA = \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}$, with characteristic polynomial $\lambda^2 - 25\lambda = \lambda(\lambda - 5)$, so $\lambda_1 = 25$ and $\lambda_2 = 0$. Compute associated eigenvectors, and make them the columns of $V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{0} = 0$, so $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. The columns of U must be an orthonormal basis for $\mathbb{R}^3$, first compute $\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0. \end{bmatrix}$ Since $A\vec{v}_2 = 0$, we don't get a second vector that way. So we need to find two unit vectors that are orthogonal to $\vec{u}_1$ and each other. The first conditions translates to $\vec{u}_1 \cdot \vec{x} = 0 \iff 2x_1 + x_2 = 0$. By inspection (or the usual computations), we find $\vec{u}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}$ and $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence,

$$A = U\Sigma V^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

The rank of $A = 1$, the number of nonzero singular values.

11. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss! Let A be the matrix that represents this homogenous system in the form $A\mathbf{x} = \mathbf{0}$. In order for the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ to be **onto** $\mathbb{R}^{18}$, the matrix A must have rank 18. So by the rank-nullity theorem, $\dim \operatorname{Nul} A = 2$, which means that the solution set of the homogenous system is two-dimensional, so can be written as the span of a set of two linearly independent vectors.

12. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points. Check!

13. Here are some specific tasks you should be able to accomplish **with demonstrated understanding**:

    (a) **Everything** listed already on the two practice midterms.

    (b) Know the *definitions*, ***basic properties***, and *geometric interpretations* of the following basic terms:

- the **inner (dot) product** $\vec{u} \cdot \vec{v}$ of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, and how it is used to define **lengths**, and **unit vectors** (by **normalizing)**, and **orthogonality** of two vectors;
- the **orthogonal complement** of a subpsace of $\mathbb{R}^n$;
- **orthogonal/orthonormal sets** and **orthogonal/orthonormal bases**, including the Gram-Schmidt process for creating them;
- the **orthogonal projection** of a vector onto: (a) another vector, (b) an orthogonal basis for a subspace of $\mathbb{R}^n$; and
- an **orthogonal matrix**.

(c) Understand diagonalizing a matrix A as a process for computing an *"ideal" basis* for the matrix transformation $T_A : \vec{x} \mapsto A\vec{x}$, and writing $[T_A]$ with respect to this new basis. Orthogonally diagonalize *any* real symmetric matrix.

(d) Know the basic properties of inner products and norms (e.g., symmetry, bilinearity, positive definiteness) and that any orthogonal set of vectors (all nonzero) is linearly independent.

(e) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.

(f) Use properties of orthogonality to write a given vector in terms of orthogonal/orthonormal sets of vectors (using $\text{Proj}_W \vec{v}$), avoiding row reduction and noting the advantage of normality in avoiding fractions.

(g) Know how the fundamental subspaces of a matrix are related: $(\text{Row}\, A)^\perp = \text{Nul}\, A$ and $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$.

(h) Define an *orthogonal matrix* $U \in \mathbb{R}^{n \times n}$ as one that satisfies the property $U^{-1} = U^T$ and prove that its rows and columns are *orthonormal bases* for $\mathbb{R}^n$.

(i) Apply the *Best Approximation Theorem* that the orthogonal projection $\widehat{y} = \text{Proj}_W \vec{y}$ is the closest point in W to $\vec{y}$.

(j) Apply the *Gram-Schmidt* process to construct the *QR Factorization* of any $m \times n$ matrix A whose columns are linearly independent (so $\text{rank}\, A = n$). In $A = QR$, the columns of Q will be an ON basis for $\text{Col}\, A$, and R will be upper-triangular.

(k) Utilize the machinery of orthogonal projections to find *least-squares solutions* to inconsistent linear systems $A\vec{x} = \vec{b}$, where $\vec{b} \notin \text{Col}\, A$, which are the *best approximations* to actual solutions for overly-constrained systems.

(l) Compute the real symmetric matrix corresponding to a quadratic form (QF) and diagonalize it to simplify the QF (the *Principal Axes Theorem*). Use this to classify the QF according to the spectrum (eigenvalues) of the matrix and to find the constrained extrema of the QF on the unit sphere.

(m) Understand how to construct and use the the **singular value decomposition (SVD)** of an $m \times n$ matrix.

(n) Use various forms of the (ever expanding) Invertible Matrix Theorem in context.

(o) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.