Math 2210Q (Roby) **Practice Final** Fall 2017

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam **no calculators (or other electronic devices) are to be used.**, but you may use two pages (i.e., one sheet, both sides) of ordinary 8.5×11-inch (or A4) paper with any **handwritten (by you)** notes or formulae you like.

1. **REVIEW COURSE MATERIALS:**
   (a) Check all of your worksheets against the worksheet solutions;
   (b) Check all of the homework solutions;
   (c) Review all the problems on the midterms and practice midterms, with particular attention to anything you got wrong the first time.
   (d) Review Ximera quizzes;
   (e) Review video lectures, especially anything you found confusing.
   (f) Ask questions in Piazza!

2. Define a linear transformation $T : \mathbb{P}_2 \to R^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.
   (a) Find the image under T of $\mathbf{p}(t) = 5 + 3t$.
   (b) Show that T is a linear transformation.
   (c) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.
   (d) Is T one-to-one? Is T onto? Explain!

3. Let $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.
   (a) Find the characteristic polynomial and the eigenvalues of A.
   (b) Is A diagonalizable? If so, give P and D such that $A = PDP^{-1}$. If not, explain why not.
   (c) Is A orthogonally diagonalizable? If so, give an orthogonal matrix P and diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

4. Let $A_1 = \begin{bmatrix} 3 & -5 & -1 \\ 1 & 1 & -3 \\ -1 & 5 & -3 \\ 3 & -7 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & 6 \\ 3 & -8 \\ 1 & -2 \\ 1 & -4 \end{bmatrix}$.

(a) One of these matrices has linearly independent columns. Which one? Justify your answer. Call this matrix A.

(b) Use Gram-Schmidt to find an orthogonal basis of $\operatorname{Col} A$.

(c) Find a QR factorization of A.

5. For each statement below determine whether it is **True** or **False** and give **reasons** to support your answer. To show something is false, it is usually best to give a **specific simple numeric counterexample**. Extra credit for "salvaging" false statements (when possible) to make them correct.

   (a) If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$.

   (b) If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then S is linearly independent.

   (c) If A and B are invertible $n \times n$ matrices, then AB is similar to BA.

   (d) Each eigenvector of a square matrix A is also an eigenvector of A^2.

   (e) There exists a 2×2 matrix with real entries that has no eigenvectors in $\mathbb{R}^2$.

   (f) If A is row equivalent to the identity matrix I, then A is diagonalizable.

   (g) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

   (h) For an $m \times n$ matrix A, vectors in $\operatorname{Nul} A$ are orthogonal to vectors in $\operatorname{Row} A$.

   (i) The matrices A and A^T have the same eigenvalues, counting multiplicities.

   (j) A nonzero vector can correspond to two different eigenvalues of A.

   (k) The sum of two eigenvectors of a square matrix A is also an eigenvector of A.

   (l) The dimension of an eigenspace of a (real) symmetric matrix A equals the multiplicity of the corresponding eigenvectors.

   (m) (Extra Credit topic) The singular values of an $m \times n$ matrix can be $3, 1, -1, -3$.

6. If a $n \times n$ matrix A satifies $A^2 = A$, what can you say about the determinant of A?

7. Let $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$.

   (a) One of A_1 or A_2 has orthogonal columns. Which one? Justify your answer. Call this matrix A.

   (b) Show that $A\vec{x} = \vec{b}$ is inconsistent.

   (c) Would you expect the orthogonal projection of $\vec{b}$ onto the column space of A to equal $\vec{b}$? Why or why not?

   (d) Compute $\operatorname{Proj}_{\operatorname{Col} A} \vec{b}$.

   (e) Compute a least-squares solution to $A\vec{x} = \vec{b}$.

8. Assume that matrices A and B below are row equivalent:

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Without calculations, list $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$. Then find bases for $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Nul} A$.

9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (You do not need to compute a vector at which this maximum is attained.)

10. Compute the singular value decomposition (SVD) of $A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$, and deduce its rank. *Explain each step* so that your procedure is clear even if your computation has errors.

11. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss!

12. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.

13. Here are some specific tasks you should be able to accomplish **with demonstrated understanding**:

    (a) **Everything** listed already on the two practice midterms.

    (b) Know the *definitions*, ***basic properties***, and *geometric interpretations* of the following basic terms:

    - the **inner (dot) product** $\vec{u}\cdot\vec{v}$ of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, and how it is used to define **lengths**, and **unit vectors** (by **normalizing)**, and **orthogonality** of two vectors;
    - the **orthogonal complement** of a subpsace of $\mathbb{R}^n$;
    - **orthogonal/orthonormal sets** and **orthogonal/orthonormal bases**, including the Gram-Schmidt process for creating them;
    - the **orthogonal projection** of a vector onto: (a) another vector, (b) an orthogonal basis for a subspace of $\mathbb{R}^n$; and
    - an **orthogonal matrix**.

    (c) Understand diagonalizing a matrix A as a process for computing an *"ideal" basis* for the matrix transformation $T_A : \vec{x} \mapsto A\vec{x}$, and writing $[T_A]$ with respect to this new basis. Orthogonally diagonalize *any* real symmetric matrix.

(d) Know the basic properties of inner products and norms (e.g., symmetry, bilinearity, positive definiteness) and that any orthogonal set of vectors (all nonzero) is linearly independent.

(e) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.

(f) Use properties of orthogonality to write a given vector in terms of orthogonal/orthonormal sets of vectors (using $\mathrm{Proj}_W \vec{v}$), avoiding row reduction and noting the advantage of normality in avoiding fractions.

(g) Know how the fundamental subspaces of a matrix are related: $(\mathrm{Row}\,A)^{\perp} = \mathrm{Nul}\,A$ and $(\mathrm{Col}\,A)^{\perp} = \mathrm{Nul}\,A^T$.

(h) Define an *orthogonal matrix* $U \in \mathbb{R}^{n\times n}$ as one that satisfies the property $U^{-1} = U^T$ and prove that its rows and columns are *orthonormal bases* for $\mathbb{R}^n$.

(i) Apply the *Best Approximation Theorem* that the orthogonal projection $\widehat{y} = \mathrm{Proj}_W \vec{y}$ is the closest point in W to $\vec{y}$.

(j) Apply the *Gram-Schmidt* process to construct the *QR Factorization* of any $m \times n$ matrix A whose columns are linearly independent (so $\mathrm{rank}\,A = n$). In $A = QR$, the columns of Q will be an ON basis for $\mathrm{Col}\,A$, and R will be upper-triangular.

(k) Utilize the machinery of orthogonal projections to find *least-squares solutions* to inconsistent linear systems $A\vec{x} = \vec{b}$, where $\vec{b} \notin \mathrm{Col}\,A$, which are the *best approximations* to actual solutions for overly-constrained systems.

(l) Compute the real symmetric matrix corresponding to a quadratic form (QF) and diagonalize it to simplify the QF (the *Principal Axes Theorem*). Use this to classify the QF according to the spectrum (eigenvalues) of the matrix and to find the constrained extrema of the QF on the unit sphere.

(m) Understand how to construct and use the the **singular value decomposition (SVD)** of an $m \times n$ matrix.

(n) Use various forms of the (ever expanding) Invertible Matrix Theorem in context.

(o) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.