Rowmotion in products of two chains

Tom Roby (UConn)

AMS Special Session on The Many Lives of Lattice Theory
Joint Mathematics Meetings 2022
Online

7 April 2022

Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

Abstract: The *rowmotion* operator on the order ideals (or antichains) of a finite poset is of long-standing interest, and there has recently been a flurry of research activity around understanding its dynamical and other properties. Questions of periodicity, orbit sizes, and *homomesy* (finding statistics that have the same average over all orbits) abound. There are also connections with representation theory that are still only partially understood. This talk will give an introductory overview of these ideas for the special (nontrivial) case where the poset is a product of two chains.

This talks discusses joint work, mostly with (chronologically) Jim Propp, Mike Joseph, and Matthew Plante.

I'm grateful to Matthew, Mike, and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

Thanks also to Drew Armstrong, Arkady Berenstein, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, David Einstein, Darij Grinberg, Shahrzad Haddadan, Sam Hopkins, Mike La Croix, Svante Linusson, Gregg Musiker, Nathan Williams, Vic Reiner, Bruce Sagan, Richard Stanley, Jessica Striker, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to put questions and comments in the chat, and the moderator will convey them with appropriate timing and finese. Or someone else may answer them!

Cyclic rotation of binary strings

"Immer mit den einfachsten Beispielen anfangen." — David Hilbert

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \quad \underset{C_R}{\longmapsto} \quad 010100$$

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Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6$, $k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
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Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$*.*

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$*.*

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Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Cyclic rotation of binary strings

Example

$n = 6, \ k = 2$

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Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well:
Let $\chi_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic?

Some themes in dynamical algebraic combinatorics

This example illustrates some of the main themes *dynamical algebraic combinatorics*. Our setup is a set S of combinatorial objects with an invertible map (or group action more generally).

1. Periodicity/order: C_R on $S_{n,k}$ has order $n = 6$.
2. Orbit structure: Every orbit size must divide n.
3. Homomesy: The number of inversions is $\dfrac{k(n-k)}{2}$-mesic.
4. Equivariant bijections: No need here.
5. *Lifting from combinatorial to piecewise-linear and birational settings.*

Rowmotion on Antichains of a Poset

Rowmotion: an invertible operation on antichains

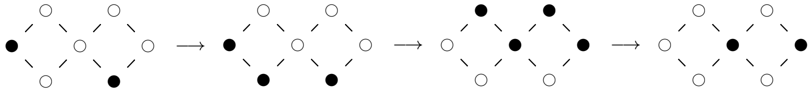
Let $\mathcal{A}(P)$ be the set of antichains (subsets of P no two elements are comparable) of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the set of minimal elements of the complement of the *downward-saturation* of A (the smallest downset containing A).

$\rho_{\mathcal{A}}$ is invertible since it is a composition of three invertible ops:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

$\rho_{\mathcal{A}}$:



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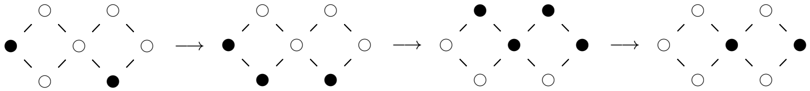


This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong–Stump–Thomas [AST11], Conj. [Pan09])

Let $\mathcal{O}$ be any $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Picture of root posets

Here are the classes of posets included in Panyushev's conjecture.

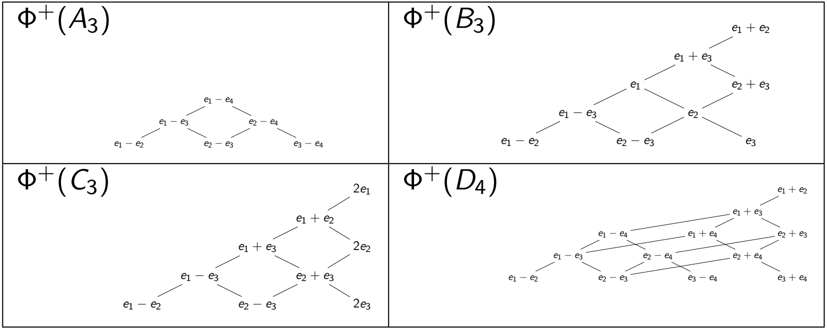
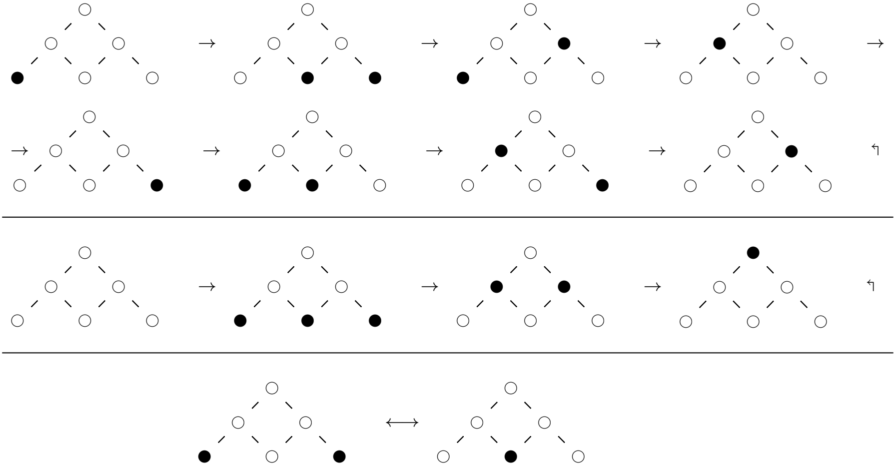


Figure: The positive root posets A_3, B_3, C_3, and D_4.

(Graphic courtesy of Striker-Williams.)

Example of antichain rowmotion on A_3 root poset

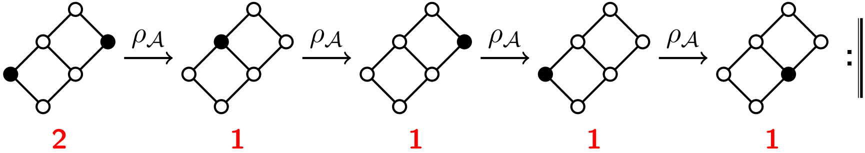
For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



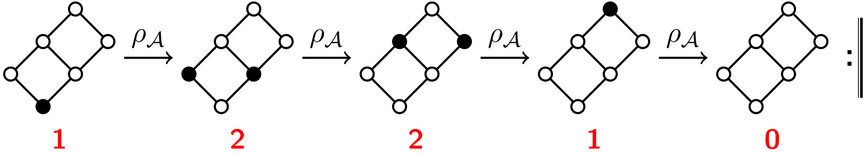
Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Orbits of rowmotion on antichains of $[2] \times [3]$



Average cardinality: 6/5



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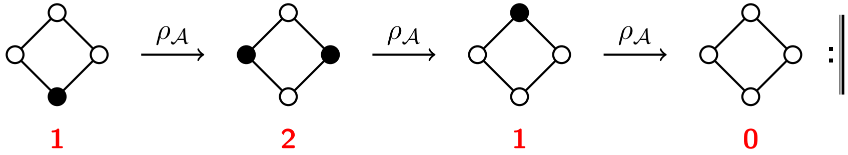

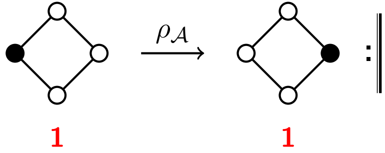

Average cardinality: 1

For antichain rowmotion on this poset, periodicity has been known for a long time:

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

Antichains in $[a] \times [b]$: cardinality is homomesic

For rectangular posets $[a] \times [b]$ (the type A *minuscule* poset, where $[k] = \{1, 2, \ldots, k\}$), the homomesy is easier to show than for root posets.

Theorem (Propp, R.)

Let $\mathcal{O}$ be any $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{A \in \mathcal{O}} \#A = \dfrac{ab}{a+b}$.

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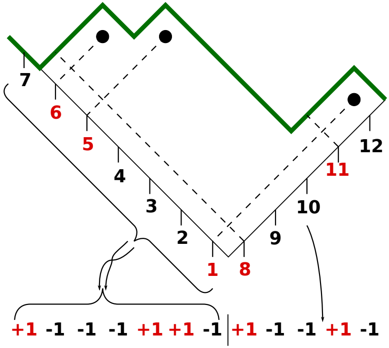

(Graphic courtesy of Ben Young.)

This proof uses an non-obvious equivariant bijection (the "Stanley–Thomas" word [Sta09, §2]) between antichains in $[a] \times [b]$ and binary strings, which carries the $\rho_{\mathcal{A}}$ map to cyclic rotation of bitstrings.

The figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

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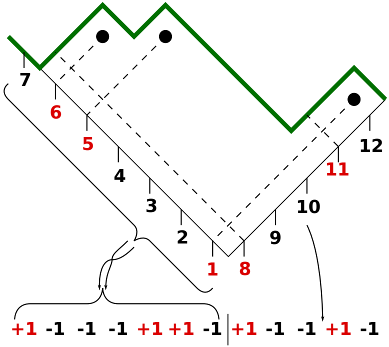

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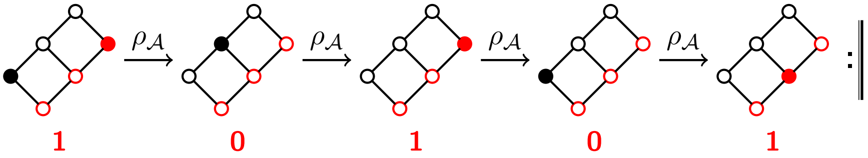
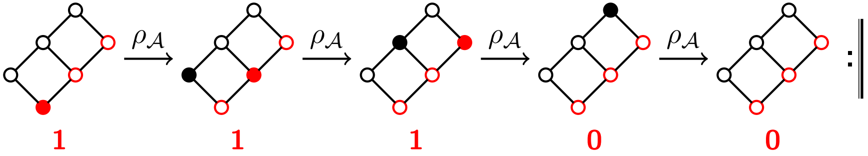
This bijection also allowed Propp–R. to derive refined homomesy results for fibers and antipodal points in $[a] \times [b]$.

Orbits of rowmotion on antichains of $[2] \times [3]$

Look at the cardinalities across a **positive fiber** such as the one highlighted in red.



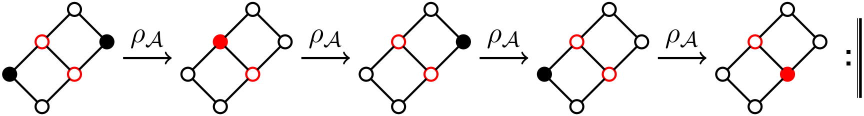
Average: 3/5



Average: 3/5

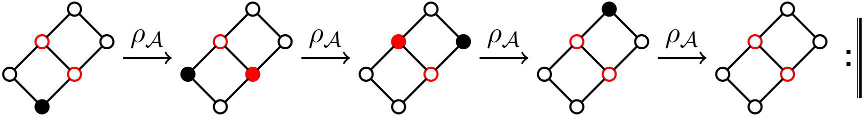
Orbits of rowmotion on antichains of $[2] \times [3]$

How about across a **negative fiber** such as the one highlighted in red.



0 **1** **0** **0** **1**

Average: 2/5



0 **1** **1** **0** **0**

Average: 2/5

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $\chi_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} \chi_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} \chi_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i, j,

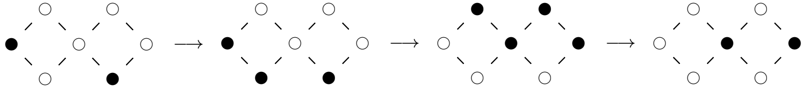
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under $\rho_{\mathcal{A}}$, even though the indicator functions $\chi_{i,j}$ aren't.

Rowmotion on order ideals

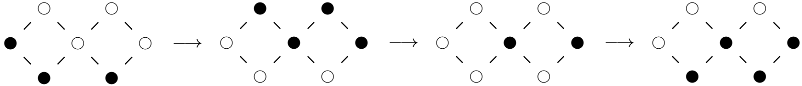
We've already seen examples of Rowmotion on antichains $\rho_{\mathcal{A}}$:

$\rho_{\mathcal{A}}$:



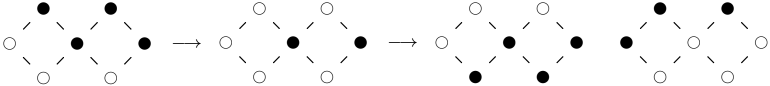
We can also define it as an operator ρ on $\mathcal{J}(P)$, the set of order ideals of a poset P, by shifting the waltz beat by 1:

ρ_J :



Or as an operator on the *up-sets (order filters)* $\mathcal{U}(P)$ of P:

$\rho_{\mathcal{U}}$:



Rowmotion on Order Ideals of a Poset

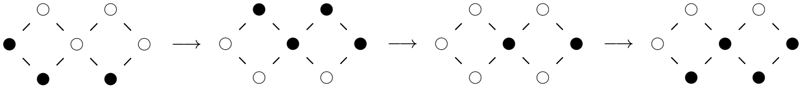
Rowmotion: an invertible operation on order ideals

We define the (cyclic) group action of **rowmotion** on the set of order ideals $\mathcal{J}(P)$ via the map $\mathrm{Row} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the following three-step process.

Start with an order ideal, and

1. Θ: Take the complement (giving an order filter)
2. ∇: Take the minimal elements (giving an antichain)
3. Δ^{-1}: Saturate downward (giving a order ideal)

ρ_J :



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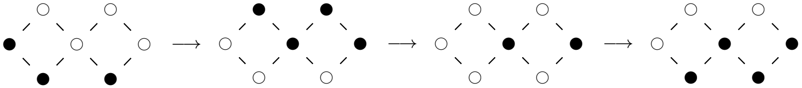

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Dynamical properties of rowmotion: cardinality is homomesic

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

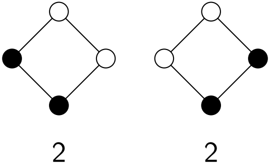
Theorem (Propp, R.)

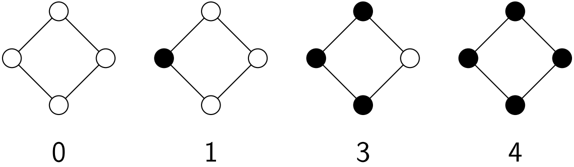
Let $\mathcal{O}$ be any rowmotion orbit in $\mathcal{J}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

We have an orbit of size 2 and an orbit of size 4:

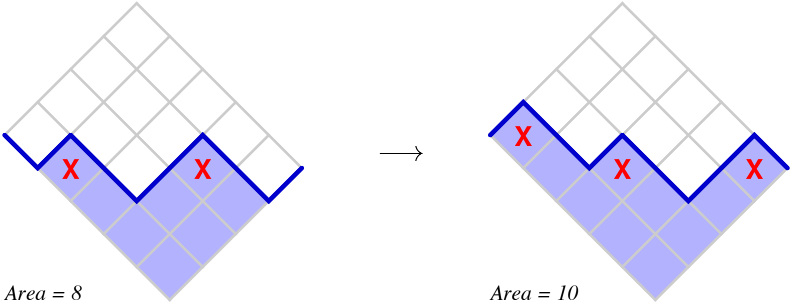




Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Example in lattice cell form

Viewing the elements of the poset as **squares** below, we would map:



Rowmotion on $[4] \times [2]$: Orbit 1

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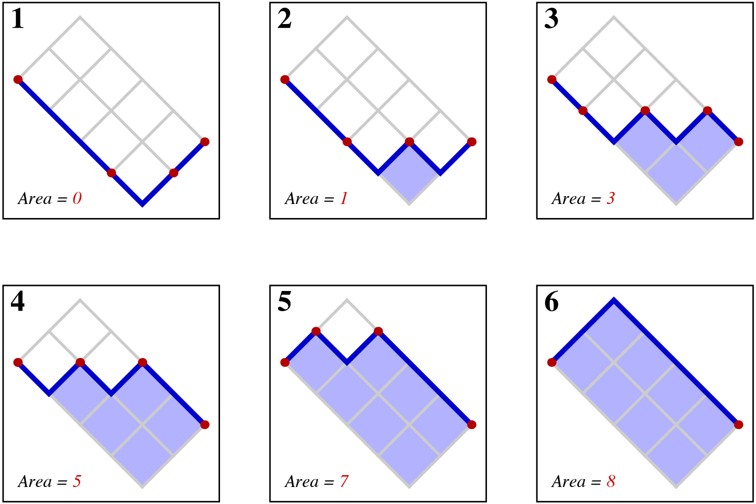

$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 2

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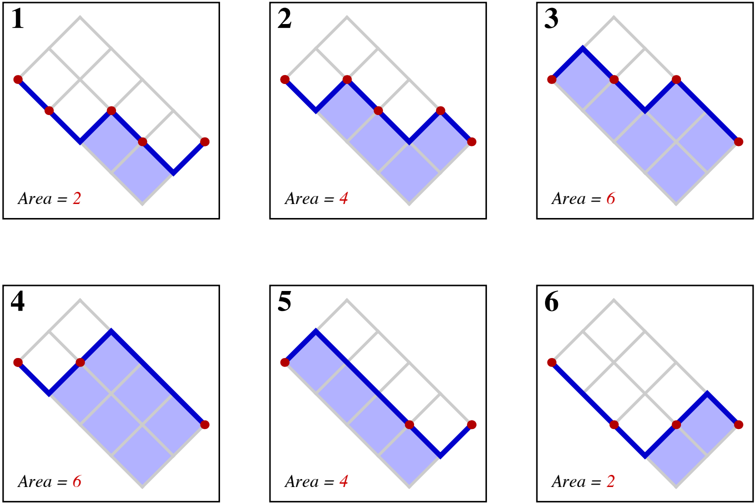

$$(2+4+6+6+4+2) / 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 3

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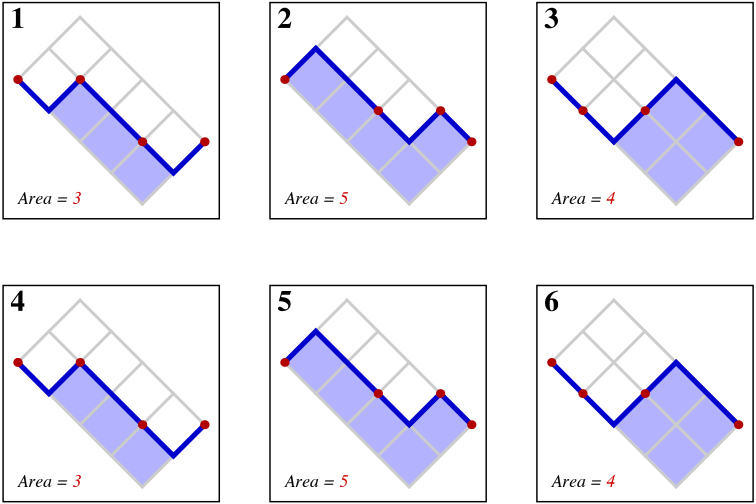


$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Toggling order ideals

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order ideals* (equivalently *order filters*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{J}(P)$ be the set of order ideals of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{J}(P) \to \mathcal{J}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{J}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{J}(P), \\ U & \text{otherwise.} \end{cases}$$

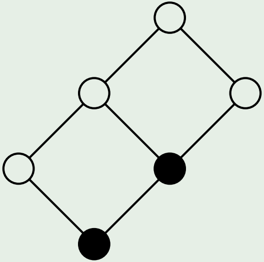
Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order ideals of P.

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

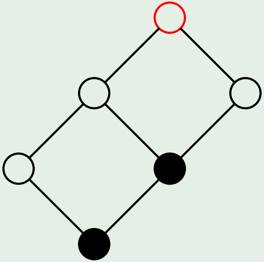
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example



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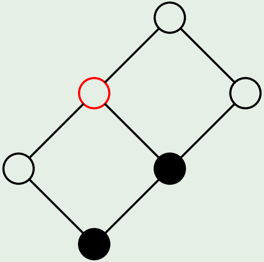

Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

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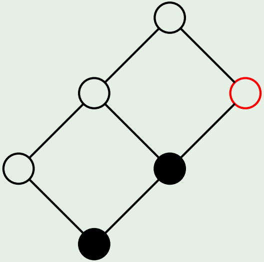
Example



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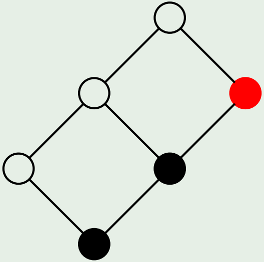

Toggling order ideals

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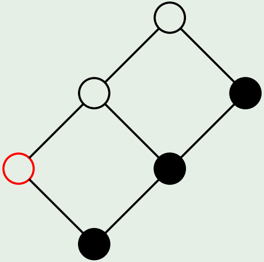

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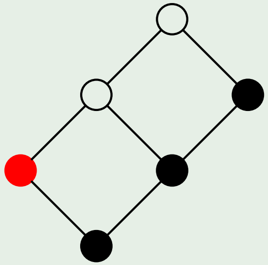

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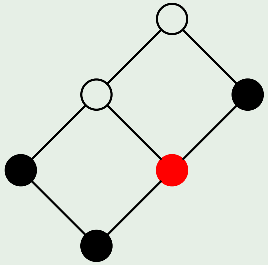

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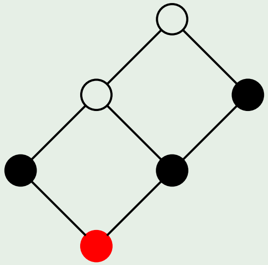

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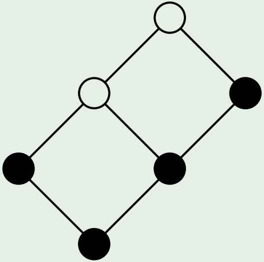
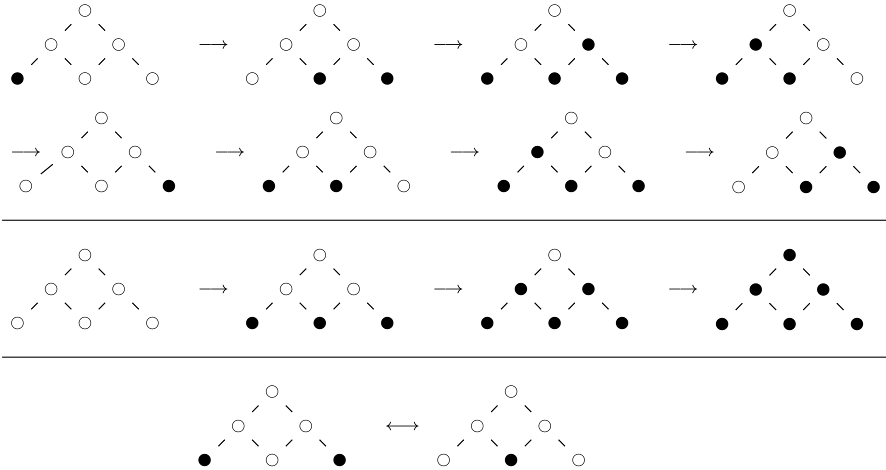


For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:

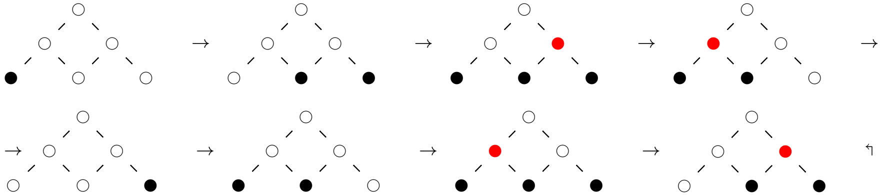


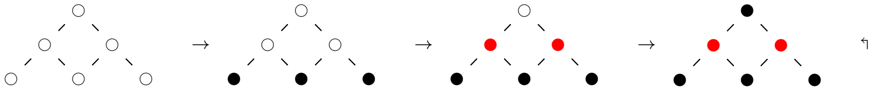
Checking the average cardinality for each orbit we find that

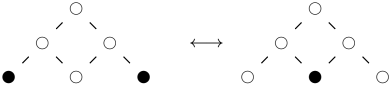
$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}; \quad \frac{0+3+5+6}{4} = \frac{7}{2}; \quad \frac{2+1}{2} = \frac{3}{2}. \text{ Darn!}$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:







Checking the average rank-alternating cardinality for each orbit we find:

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{1+2+2+1}{4} = \frac{2+1}{2} = \frac{3}{2} \text{ Yay!}$$

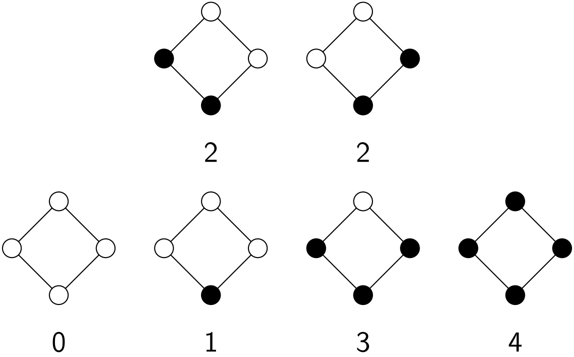
Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan)

Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign an order ideal $I \in \mathcal{J}(P)$ weight $f(I) = \sum_{x \in I} \mathrm{wt}(x)$, then f is homomesic under rowmotion and promotion, with average $n/2$.

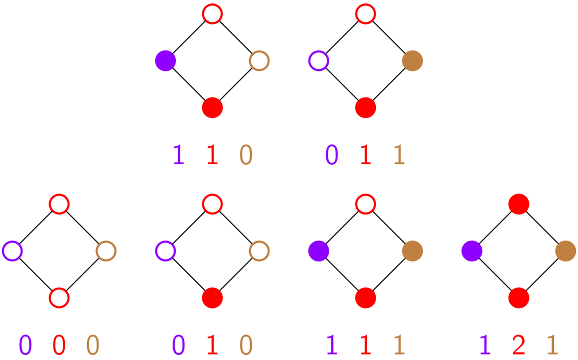
Ideals in $[a] \times [b]$: the case $a = b = 2$

We have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ of a violet element, 1 red element, and $1/2$ of a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

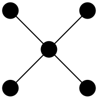
For every ρ-orbit $\mathcal{O}$ in $J([a] \times [b])$:

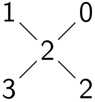
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Whirling on posets

Definition of whirling on posets

Let $\mathcal{F}_k$ be the set of order-reversing functions from P to $\{0, 1, 2, \ldots, k\}$.

$P =$ 



$\in \mathcal{F}_3(P)$

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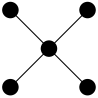
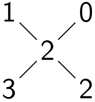
$P =$ 



$\in \mathcal{F}_3(P)$

Definition ([JPR18])

Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$ define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at x*, as follows: repeatedly add 1 $(\mathrm{mod}\ k+1)$ to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

Definition of whirling on posets

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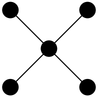
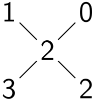
$P =$ 



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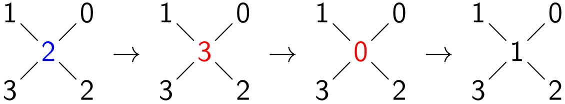


$\in \mathcal{F}_3(P)$

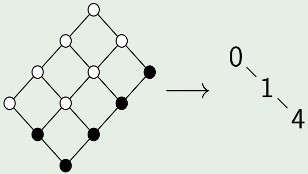
Equivariant bijection between whirling and rowmotion

Now let $\{x_1, x_2, \ldots, x_n\}$ be any linear extension of P (with $\#P = n$.) It is easy to show that w_x and w_y commute when $x, y \in P$ are *incómparable*. Thus the *whirling operator* $w := w_{x_1} w_{x_2} \cdots w_{x_n}$ is well-defined (whirling each poset element from top to bottom).

Theorem (Plante)

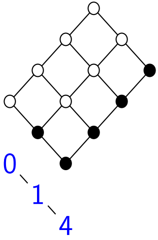
There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w to $\rho_{\mathcal{J}}$.

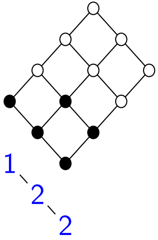
Example ($\mathcal{J}([3] \times [4])$ to $\mathcal{F}_4([3])$)

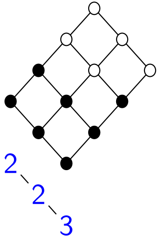


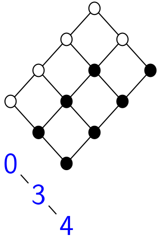
The number of order ideal elements in each fiber is recorded as an order-reversing function on [3].

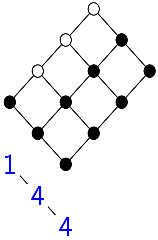
Product of two chains orbit bijection example

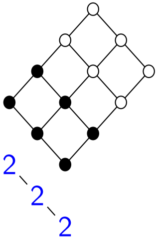


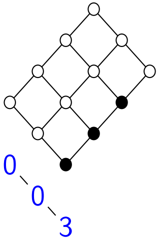












0	1	4
1	2	2
2	2	3
0	3	4
1	4	4
2	2	2
0	0	3

Theorem (Plante)

Let w denote the whirling operator on order-reversing functions $\mathcal{F}_k([m])$. Consider a superorbit board of w with length $k+m$.

1. *The board can be partitioned into m snakes of length $k+m$ under the following rules:*
   1. *Start at zero in the left column.*
   2. *Move down the column until the label does not increase then move to the right.*
   3. *End once the snake contains k in the rightmost column.*

An orbit board of $(0,1,4) \in \mathcal{F}_4([3])$:

0	1	4
1	2	2
2	2	3
0	3	4
1	4	4
2	2	2
0	0	3

Theorem (Plante)

Let w denote the whirling operator on order-reversing functions $\mathcal{F}_k([m])$. Consider a superorbit board of w with length $k+m$.

2. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the segments of a snake α, that is, α_i is the number of blocks of the snake in column i. Each snake in the board has segments which are a cyclic rotation of $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.*
3. *The average sum of values along a snake is $k(m+k)/2$.*

An orbit board of $(0,1,4) \in \mathcal{F}_4([3])$:

0	1	4
1	2	2
2	2	3
0	3	4
1	4	4
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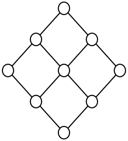
0	0	3

Orbits of a product of two chains

$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

0	0	0	0	0	3	0	3	3			
0	0	1	0	1	1	1	1	1			
0	1	2	1	1	2	0	0	2	0	2	2
1	2	3	0	2	3	0	1	3	1	1	3
2	3	3	1	3	3	1	2	2			
3	3	3	2	2	2	2	2	3			

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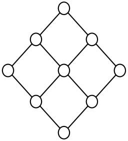
$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

0	0	0
0	0	1
0	1	2
1	2	3
2	3	3
3	3	3

0	0	3
0	1	1
1	1	2
0	2	3
1	3	3
2	2	2

0	3	3
1	1	1
0	0	2
0	1	3
1	2	2
2	2	3

0	2	2
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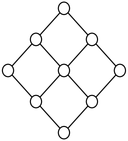
$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

0	0	0
0	0	1
0	1	2
1	2	3
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3	3	3

0	0	3
0	1	1
1	1	2
0	2	3
1	3	3
2	2	2

0	3	3
1	1	1
0	0	2
0	1	3
1	2	2
2	2	3

0	2	2
1	1	3
0	2	2
1	1	3
0	2	2
1	1	3

The $\vee \times [k]$ poset

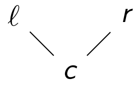
The poset $\mathsf{V} \times [k]$

- Let V be the three-element partially ordered set with Hasse diagram

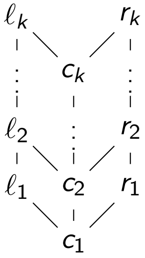
$$\begin{tikzcd}[arrows=dash]
\ell & & r \\
& c \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The poset V $\times$ $[k]$

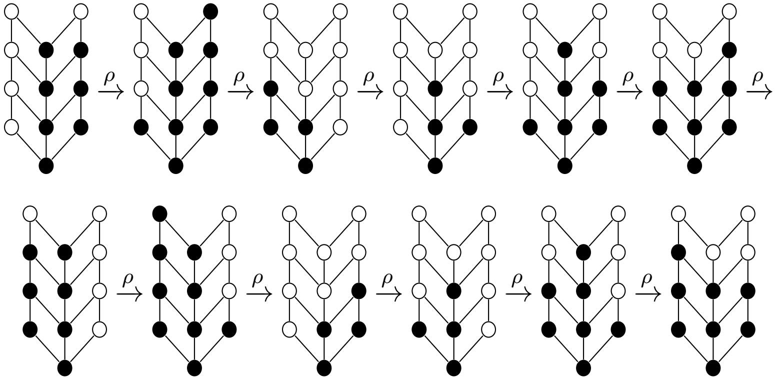
- Let V be the three-element partially ordered set with Hasse diagram



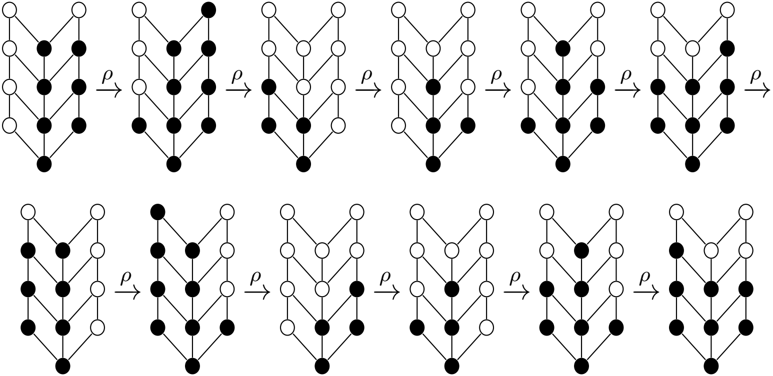
- The poset of interest is $V(k) := \mathrm{V} \times [k]$



Order-ideal rowmotion on $\mathsf{V} \times [k]$



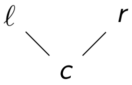
Order-ideal rowmotion on $V \times [k]$



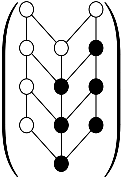
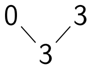
Theorem (Plante)

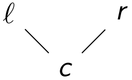
Order ideals of $V(k)$ are reflected about the center chain after $k+2$ iterations of ρ, and furthermore, the order of ρ on order ideals of $V(k)$ is $2(k+2)$.

Map to order-reversing functions on V

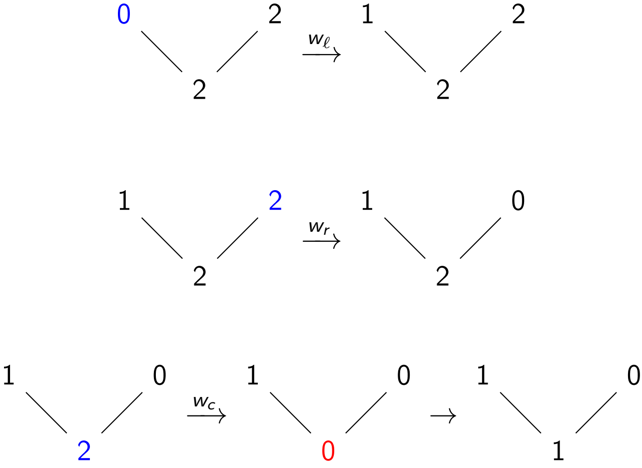


1. Define $\mathcal{F}_k(\mathsf{V}) = \{(\ell, c, r) \in \{0, \dots, k\}^3 : \ell, r \leq c\}$.
2. Define $\phi : \mathcal{J}(\mathsf{V}(k)) \to \mathcal{F}_k(\mathsf{V})$ by
$\phi(I) = \left(\sum \chi_{\ell_i}, \sum \chi_{c_i}, \sum \chi_{r_i}\right)$.

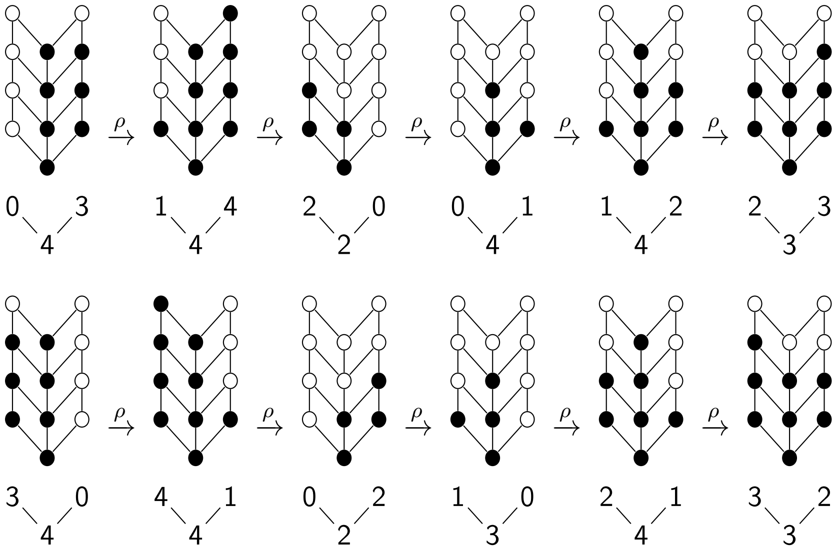
ϕ  $= (0, 3, 3) \leftrightarrow$ 

We whirl the example  first at ℓ, r, then c.

Start with $(0, 2, 2) \in \mathcal{F}_2(\mathrm{V})$.



Example of rowmotion orbit with triples



Equivariant bijection example

Alternatively we may define w on $(\ell, c, r) \in \mathcal{F}_k(\mathrm{V})$ as the process:

1. $\ell \to \ell + 1$ unless $\ell = c$, then $\ell \to 0$.

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1. $\ell \to \ell + 1$ unless $\ell = c$, then $\ell \to 0$.
2. Repeat step 1 with r instead of ℓ.
3. $c \to c + 1$ unless $c = k$, then $c \to \max(\ell, r)$.

Corollary

The map ϕ is an equivariant bijection that sends ρ to w.

$$
\begin{tikzcd}
\mathcal{J}(\mathsf{V}(k)) \arrow[r, "\rho"] \arrow[d, leftrightarrow, "\phi"'] & \mathcal{J}(\mathsf{V}(k)) \arrow[d, leftrightarrow, "\phi"'] \\
\mathcal{F}_k(\mathsf{V}) \arrow[r, "w"] & \mathcal{F}_k(\mathsf{V})
\end{tikzcd}
$$

Theorem (Plante)

Order ideals of $V(k)$ are reflected about the center chain after $k+2$ iterations of ρ, and furthermore, the order of ρ on order ideals of $V(k)$ is $2(k+2)$.

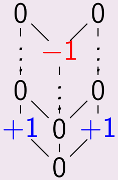
Direct inspection of order-reversing functions on V as tuples gives a straightforward proof of periodicity.

Homomesy

Theorem (Plante)

For the action of rowmotion on order ideals of $\mathsf{V}(k)$*:*

1. *The statistic* $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ *is* $\frac{2(k-1)}{k+2}$*-mesic.*



.

Theorem (Plante)

For the action of rowmotion on order ideals of $\mathsf{V}(k)$:

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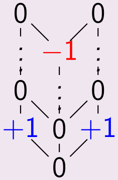
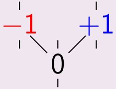


.

2. *The statistic* $\chi_{r_i} - \chi_{\ell_i}$ *is 0-mesic*



for each $i = 1, \ldots, k$, *where* χ_x *is the indicator function.*

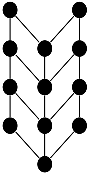
Center Seeking Snakes

We decompose the orbit board into 6 snakes of length $k + 2$. Or 2 two-tailed snakes if the order-reversing functions are symmetric. Recall that snakes start at the top of a poset and move down. Since the least element of V is in the center, we call these snakes, *center-seeking snakes*.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Sketch of Proof of Homomesy



$$\sum \chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$$

$$= (2(k+2)-3)+(2(k+2)-3)-6$$

Thus we see

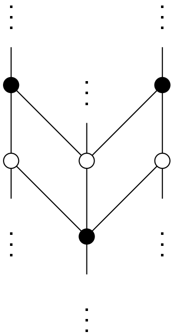
$$\frac{4(k+2)-12}{2(k+2)} = \frac{2k-2}{k+2}.$$

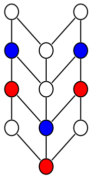
1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

$2(k+2)$

Another Potential Homomesy

There is another nice looking homomesy. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the following flux-capacitor shape in $\mathsf{V}(k)$.



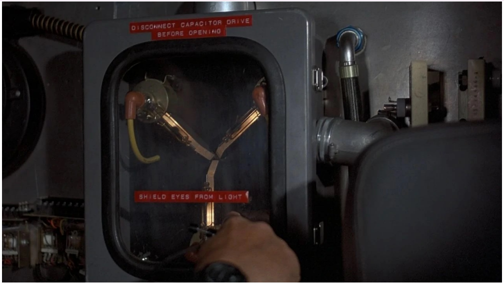


$F_3 - F_2$

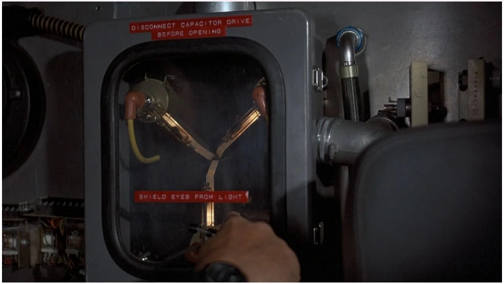
Theorem

The difference $F_i - F_j$ is $\dfrac{(i-j)}{k+2}$-mesic.

Flux Capacitor??



Flux Capacitor??



https://www.youtube.com/watch?v=VcZe8_RZO8c

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Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to questions of periodicity/order, orbit structure, homomesy, and equivariant bijections.
- Actions that can be built out of smaller, simpler actions (toggles and whirls) often have interesting and unexpected properties.
- Much more remains to be explored, perhaps for combinatorial objects or actions that you work with for other reasons.

Slides for this talk will be available online at

Google "Tom Roby".

Thanks very much for coming to this talk!