Jeu de Taquin: The Fifteen Puzzle in Research Mathematics

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Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

Who is Tom Roby?

I love to talk with people about:

- Math: Professor of Mathematics at UConn, with interests in Combinatorics, Algebra, & Math Education;
- MathEd: Worked with programs for K–12 teachers, high-ability HS students (Ross, HCSSiM, PROMYS, MathPath), and general ugrads who want help with math-intensive courses (UConn's Q Center).
- Folkdancing: Avid folk dancer in various styles (Balkan, English Country, Scandinavian, Waltz, Contra, Swing, . . .);
- Languages: Strong interests in cultures, languages & linguistics, particularly Japanese.
- More at http://www.math.uconn.edu/~troby

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Overview & Outline

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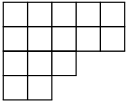
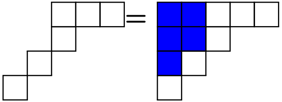
- Our slides take place in left and top-justified straight shapes  or in skew shapes  which are the difference of two straight shapes.

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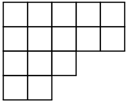
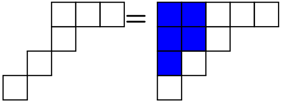
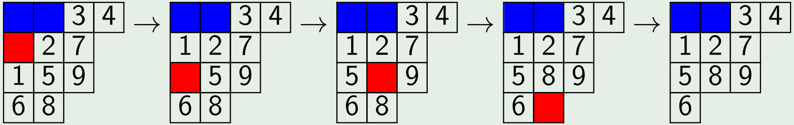
- Our slides take place in left and top-justified straight shapes  or in skew shapes  which are the difference of two straight shapes.
- Putting numbers in the squares of our shape so that they are **always** increasing from top to bottom and and from left-to-right turns it into a tableau.
- In this context, each slide is *deterministic*. Amazing things happen!

Example (Sliding to get from one skew tableau to another)



Amazing Fact 1: Confluence when sliding to straight shapes

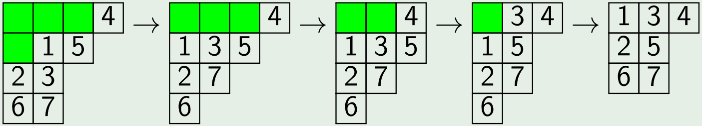
Theorem (Confluence)

Any sequence of jeu de taquin moves leading to a straight shape gives the same result (regardless of the order of the moves).

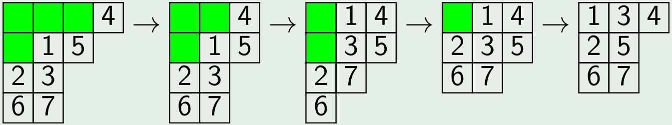
Example

There are multiple ways we can slide a given skew tableau to a straight shape tableau:

Sequence 1:



Sequence 2:

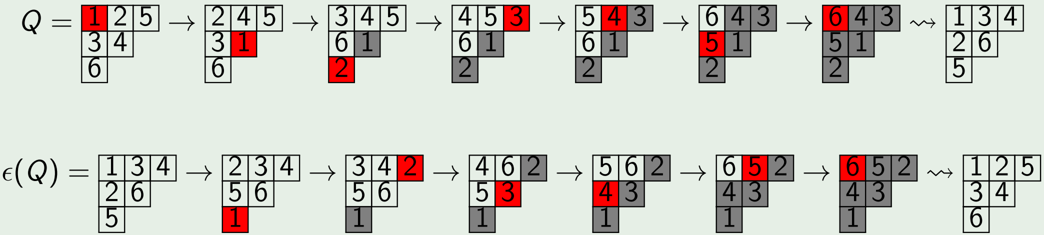


Note the different tableaux at the penultimate step.

Evacuation

Repeating this kind of sliding operation leads to interesting operations. For evacuation, we treat the upper-left cell as empty, slide it according to the rules, and keep track of where it ends up on the boundary. Reinterpret the result as a tableau (by reversing the ordering of the labels). This gives a map $\epsilon : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$.

Example

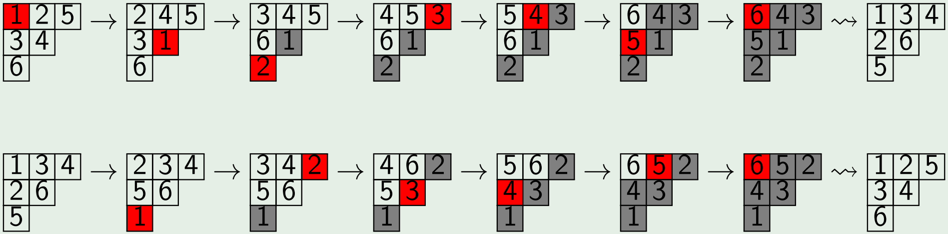


Amazing Fact 2: Evacuation is its own inverse

Theorem

For any tableau Q*,* $\epsilon(\epsilon(Q)) = Q$*, so* $\epsilon^2 = \mathrm{id}$*.*

Example



FTWK: This fact generalizes to linear extensions of any finite poset.

A useful Applet for Jeu de Taquin

Lauren K. Williams of Mercyhurst University has a number of useful applets, including one that does this type of Jeu de Taquin slide.

https://www.integral-domain.org/lwilliams/Applets/discretemath/jeudetaquin.php

I also recommend her applet for viewing 2-dimensional linear transformations, which I use with my sophomore linear algebra students:

https://www.integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php

References

[Fom] Sergey Fomin, *Knuth Equivalence, Jeu de Taquin, and the Littlewood–Richardson Rule* in Richard P. Stanley, *Enumerative Combinatorics, volume 2, Ed. 2*, Cambridge Univ Press, 2024, pp. 687–713.

[Hai] M. D. Haiman, *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math. **99** (1992), 79–113.

[Sch61] M.-P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Canad. J. Math. **13** (1961), 117–128.

[Sch72] M.-P. Schützenberger, *Promotion des morphismes d'ensembles ordonnés*, Discrete Math. **2** (1972), 73–94.

[Sl06] Jerry Slocum and Dic Sonnevald *The 15 Puzzle Book: How it Drove the World Crazy* , Slocum Puzzle Foundation (2006).

[Sta09] Richard P. Stanley, *Promotion and Evacuation*, Electron. J. Combin. **16(2)** (2009), #R9, `https://doi.org/10.37236/75`.

[SW12] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. **33** (2012), 1919–1942,

Another (related) operation on $Q \in \mathrm{SYT}(\lambda)$ via jeu de taquin is called promotion. Slide through the box with the lowest label until it gets to the boundary, where it becomes the new largest element. Then decrement the other labels by 1 to get $\partial(Q) \in \mathrm{SYT}(\lambda)$.

Example

$$Q = \begin{matrix} 1 & 3 & 6 \\ 2 & 4 & 9 \\ 5 & 7 & \\ 8 & & \end{matrix} \to \begin{matrix} 2 & 3 & 6 \\ 4 & 7 & 9 \\ 5 & 1 & \\ 8 & & \end{matrix} \rightsquigarrow \begin{matrix} 1 & 2 & 5 \\ 3 & 6 & 8 \\ 4 & 9 & \\ 7 & & \end{matrix} = \partial(Q) \to \begin{matrix} 2 & 5 & 8 \\ 3 & 6 & 1 \\ 4 & 9 & \\ 7 & & \end{matrix} \rightsquigarrow \begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 9 \\ 3 & 8 & \\ 6 & & \end{matrix} = \partial^2(Q)$$

Natural Question: How large is the period of this map? That is, what is the minimum number of times we need to apply ∂ that guarantees we end up where we started, no matter what tableau we start with?

EG [SW12]: Promotion has order 7,554,844,752 on $\mathrm{SYT}(\lambda)$ for $\lambda = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & & & & \\ \cline{1-6} \end{array}$.

Amazing Fact 3: Promotion on an $a \times b$ rectangle has order ab.

Theorem

Promotion on an $a \times b$ rectangle has order ab.

For the 5 tableaux of shape 2×3, we get an orbit of size three, and one of size two:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \Lsh \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \Lsh$$

For the 14 tableaux of shape 4×2, we get orbits of size eight, four, and two.

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&6\\\hline 4&7\\\hline 5&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline 7&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&7\\\hline 6&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline 7&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&7\\\hline 6&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&4\\\hline 2&6\\\hline 3&7\\\hline 5&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline 7&8\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&7\\\hline 6&8\\\hline\end{array} \Lsh$$

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