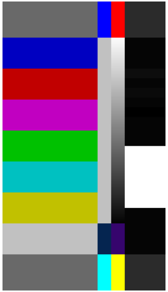
Dynamical Algebraic Combinatorics: Actions, Orbits, Averages

Tom Roby (UConn)

Program in Algebraic and Enumerative Combinatorics
Institut Mittag-Leffler
Djursholm, SWEDEN



25 February 2020

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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Abstract

Abstract: Dynamical Algebraic Combinatorics explores actions on sets of discrete combinatorial objects, many of which can be built up by small local changes, e.g., Schützenberger's promotion and evacuation, or the rowmotion map on order ideals. There are strong connections to the combinatorics of representation theory and with Coxeter groups. Birational liftings of these actions are related to the Y-systems of statistical mechanics, thereby to cluster algebras, in ways that are still relatively unexplored.

The term "homomesy" describes the following widespread phenomenon: Given a group action on a set of combinatorial objects, a statistic on these objects is called "homomesic" if its average value is the same over all orbits. Along with its intrinsic interest as a kind of "hidden invariant", homomesy can be used to prove certain properties of the action, e.g., facts about the orbit sizes. Homomesy can often be found among the same dynamics that afford cyclic sieving. Proofs of homomesy often involve developing tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection.

This talk will be a introduction to these ideas, giving a number of examples of such actions and pointing out connections to other areas.

Acknowledgments

This seminar talk discusses work with Darij Grinberg, Mike Joseph, Jim Propp and Gregg Musiker including ideas and results from Arkady Berenstein, David Einstein, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Drew Armstrong, Anders Björner, Barry Cipra, Darij Grinberg, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to interrupt with questions or comments.

ご質問コメント等があれば、ご遠慮なくおっしゃって下さい。

Analogy & Outline

Given a dynamical system (moving among discrete possibilities or swinging pedulum) we can identify:

- orbit structure (how many, what sizes);

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Themes in Dynamical Algebraic Combinatorics:

1) *Periodicity/order*; ***1)****Orbit structure*; ***1)****Homomesy* ***1)****Equivariant bijections*

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Themes in Dynamical Algebraic Combinatorics:

1) Periodicity/order; 1)Orbit structure; 1)Homomesy 1)Equivariant bijections

- Cyclic rotation of binary strings and definition of *homomesy*;
- Toggling independent sets of path graph; and
- Rowmotion map on antichains and order ideals of posets;
- Piecewise-linear and birational liftings;

Cyclic rotation of binary strings

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \underset{C_R}{\longmapsto} 010100$$

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6$, $k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathcal{I}(s)$ denote the number of inversions of $s \in S_{n,k}$.

Then the function $\mathcal{I} : S_{n,k} \to \mathbb{Q}$ is homomesic with average $\frac{k(n-k)}{2}$ with respect to cyclic rotation on $S_{n,k}$.

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Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Cyclic rotation of binary strings

Example

$n = 6, \ k = 2$

String	Inv	String	Inv	String	Inv
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Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well, e.g., Let $\mathbb{1}_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic?

Homomesy

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of SSYT;
- Rowmotion of "nice" (e.g., minuscule heap) posets [PrRo15, StWi11, Had14, RuWa15+];

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- Liftings of homomesy from combinatorial maps to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].
- There are many others, including in upcoming examples.

Coxeter Toggling

Independent Sets

of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

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Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the factor 11.

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex i** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases}.$$

Toggles

Proposition

Each toggle τ_i is an involution, i.e., τ_i^2 is the identity. Also, τ_i and τ_j commute if and only if $|i-j| \neq 1$.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Homomesy

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

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$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph-R.[JR18])

Define $\mathbb{1}_i : \mathcal{I}_n \to \{0,1\}$ to be the indicator function of vertex i.

For $1 \leq i \leq n$, $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic on φ-orbits of $\mathcal{I}_n$.

Also $2\mathbb{1}_1 + \mathbb{1}_2$ and $\mathbb{1}_{n-1} + 2\mathbb{1}_n$ are 1-mesic on φ-orbits of $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
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$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
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Total:	6	3	4	4	4	4	4	4	3	6

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Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
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$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
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$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
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Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
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$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1) \bmod 4$.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using elementary Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Antichain Rowmotion on Posets

Rowmotion: an invertible operation on antichains

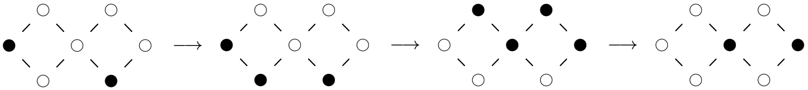
Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the set of minimal elements of the complement of the downward-saturation of A.

$\rho_{\mathcal{A}}$ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

$\rho_{\mathcal{A}}$:



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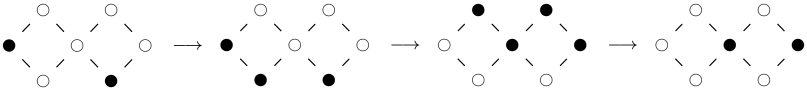
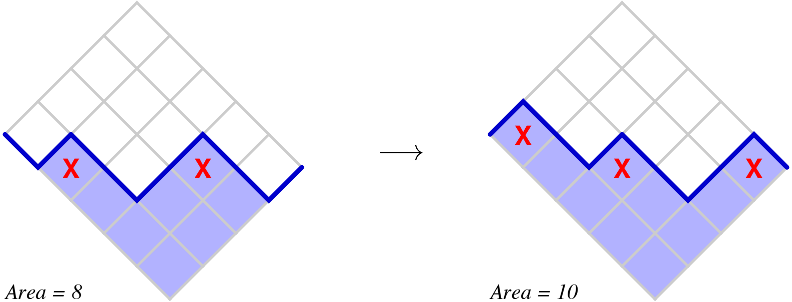


This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Example in lattice cell form

Viewing the elements of the poset as **squares** below, we would map:



Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

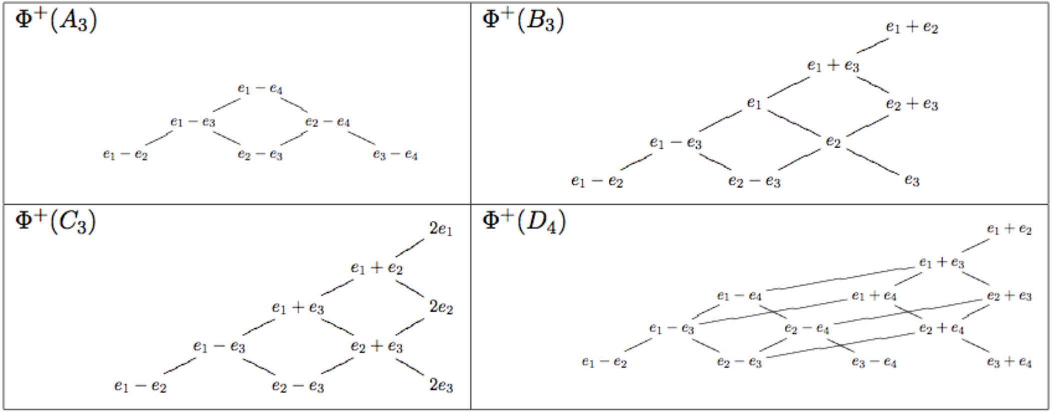
Theorem (Armstrong-Stump-Thomas [AST11], Conj. [Pan09])

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

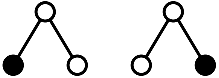
Here are the classes of posets included in Panyushev's conjecture.



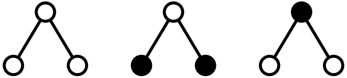
(Graphic courtesy of Striker-Williams.)

Panyushev's conjecture: The A_n case, $n = 2$

Here we have just an orbit of size 2 and an orbit of size 3:



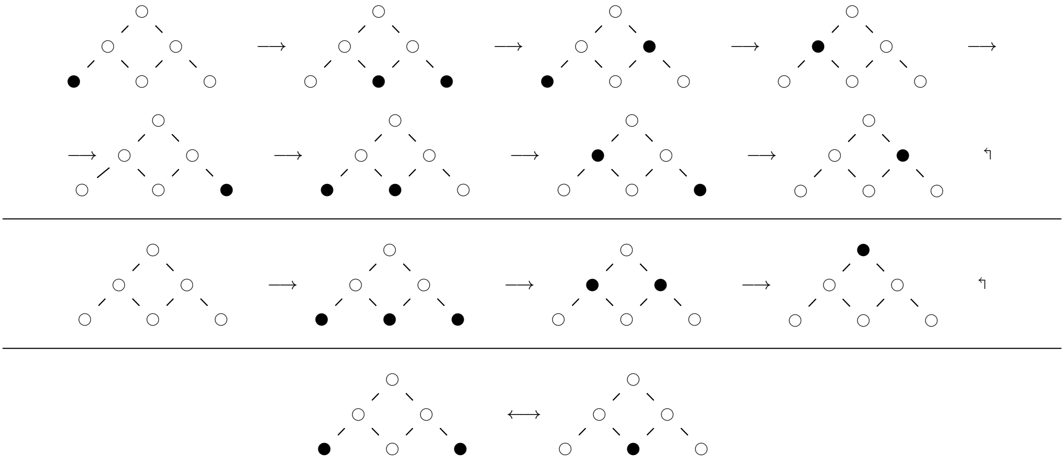
$1 \quad 1$



$0 \quad 2 \quad 1$

Within each orbit, the average antichain has cardinality $n/2 = 1$.

For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (the type A *minuscule* poset), where $[k] = \{1, 2, \ldots, k\}$:

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{A \in \mathcal{O}} \#A = \dfrac{ab}{a+b}$.

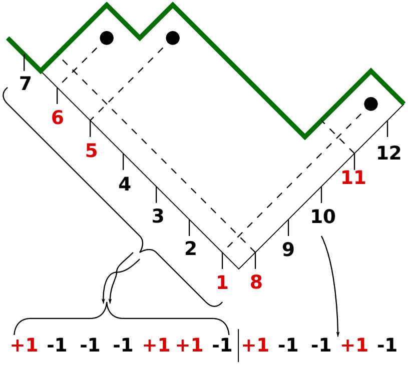
Antichains in $[a] \times [b]$: cardinality is homomesic

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\displaystyle\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}$.

This proof uses an non-obvious equivariant bijection (the "Stanley–Thomas" word [Sta09, §2]) between order ideals in $[a] \times [b]$ and binary strings, which carries the ρ_J map to cyclic rotation of bitstrings.

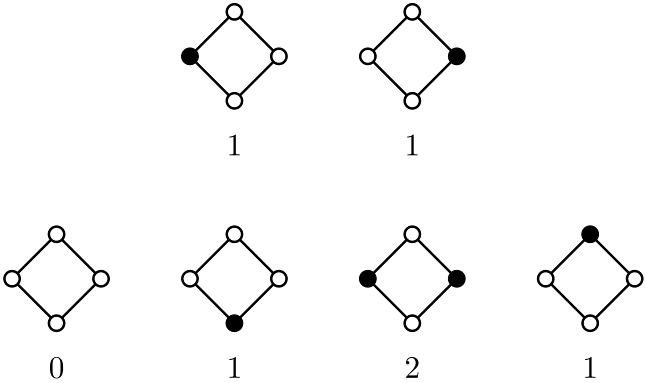
The right figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red and black correspond to $+1$ and -1 respectively.



(Graphic courtesy of Ben Young.)

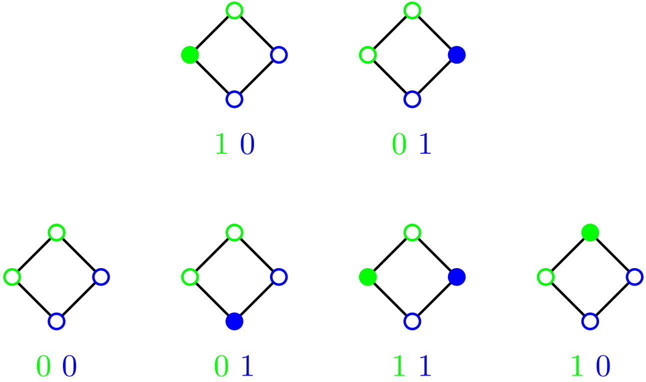
Antichains in $[a] \times [b]$: the case $a = b = 2$

Here we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average antichain has cardinality $ab/(a + b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has

$1/2$ of a green element and $1/2$ of a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $\mathbb{1}_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} \mathbb{1}_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} \mathbb{1}_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i,j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under $\rho_{\mathcal{A}}$, even though the indicator functions $\mathbb{1}_{i,j}$ aren't.

Theorem (Propp, R.)

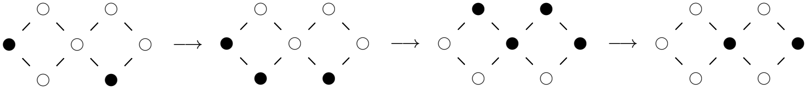
In any orbit, the number of A that contain (i,j) equals the number of A that contain the opposite element $(i',j') = (a+1-i, b+1-j)$.

That is, the function $\mathbb{1}_{i,j} - \mathbb{1}_{i',j'}$ is homomesic under $\rho_{\mathcal{A}}$, with average value 0 in each orbit.

Rowmotion on order ideals

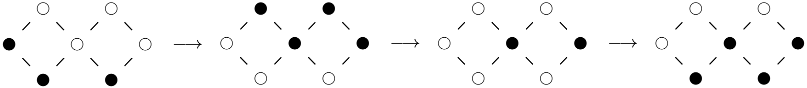
We've already seen examples of Rowmotion on antichains $\rho_{\mathcal{A}}$:

$\rho_{\mathcal{A}}$:



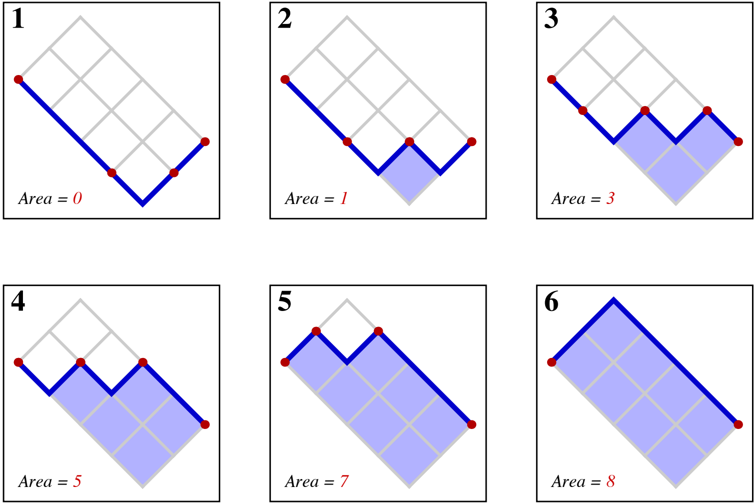
We can also define it as an operator ρ_J on $J(P)$, the set of order ideals of a poset P, by shifting the waltz beat by 1:

ρ_J :



Rowmotion on $[4] \times [2]$: Orbit 1

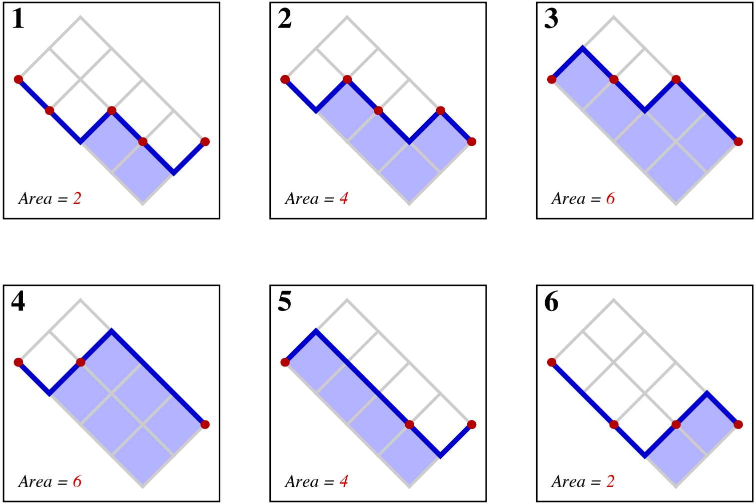
Rowmotion on $[4] \times [2]$: Orbit 1



$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 2

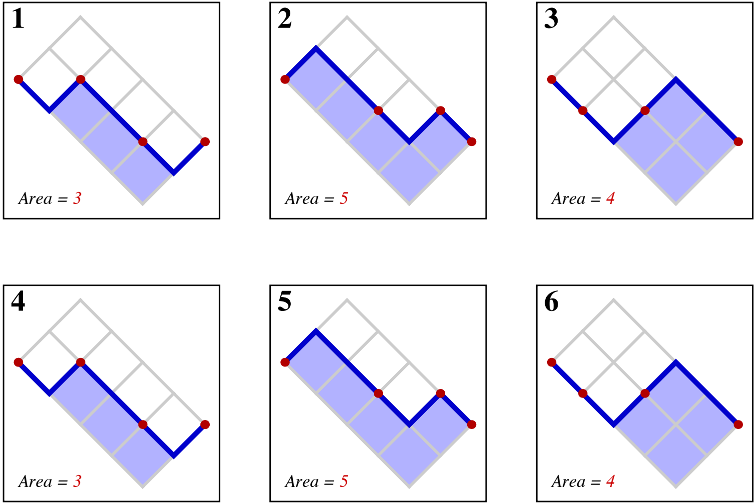
Rowmotion on $[4] \times [2]$: Orbit 2



$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 3

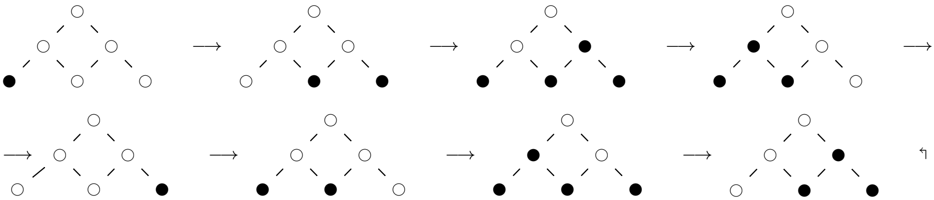
Rowmotion on $[4] \times [2]$: Orbit 3

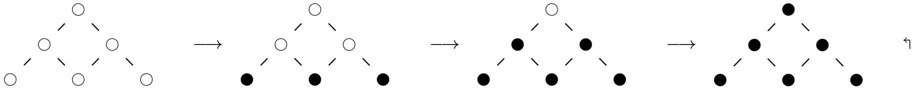


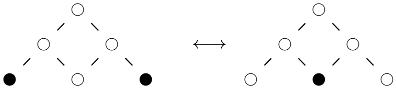
$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ_J-orbits, of sizes 8, 4, 2:





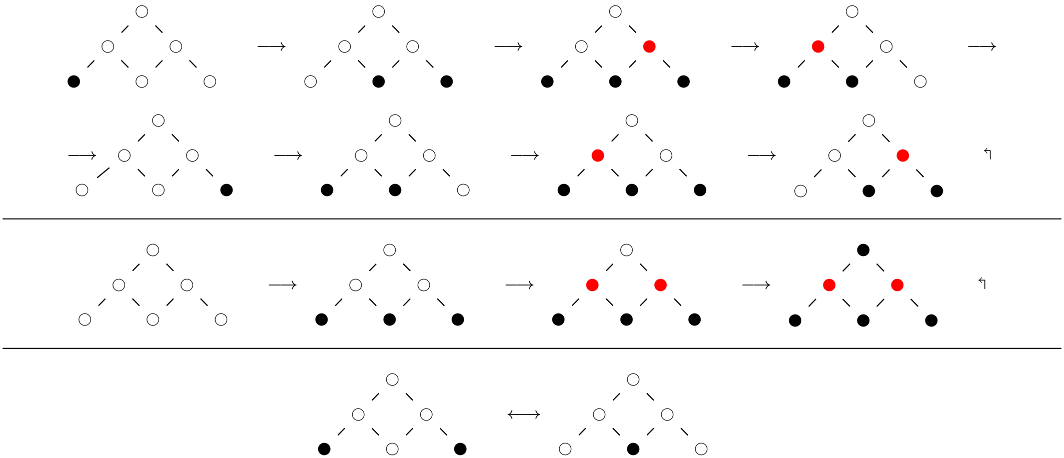


Checking the average cardinality for each orbit we find that

$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}; \quad \frac{0+3+5+6}{4} = \frac{7}{2}; \quad \frac{2+1}{2} = \frac{3}{2}. \text{ Darn!}$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



Checking the average rank-alternating cardinality for each orbit we find:

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{1+2+2+1}{4} = \frac{2+1}{2} = \frac{3}{2} \text{ Yay!}$$

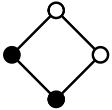
Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan)

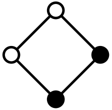
Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $wt(x) = (-1)^{rank(x)}$, and assign an order ideal $I \in J(P)$ weight $f(I) = \sum_{x \in I} wt(x)$, then f is homomesic under rowmotion and promotion, with average $n/2$.

Ideals in $[a] \times [b]$: the case $a = b = 2$

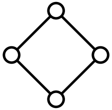
Again we have an orbit of size 2 and an orbit of size 4:



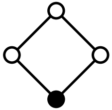
2



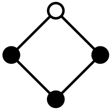
2



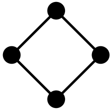
0



1



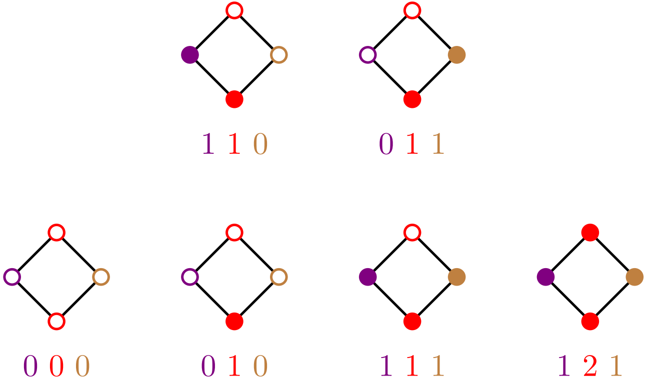
3



4

Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ of a violet element, 1 red element, and $1/2$ of a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every ρ_J-orbit $\mathcal{O}$ in $J([a] \times [b])$:

- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Rowmotion via Toggling

Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Let $v \in P$ and $S \in J(P)$. Define $\mathbf{t}_v(S)$ as:
  - $S \bigtriangleup \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

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  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.

Classical rowmotion: the toggling definition

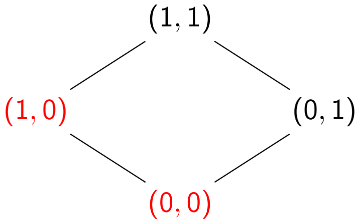
- Let $(v_1, v_2, \ldots, v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\rho_J = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \ldots \circ \mathbf{t}_{v_n}.$$

Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

Example: Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!

Start with this order ideal S:



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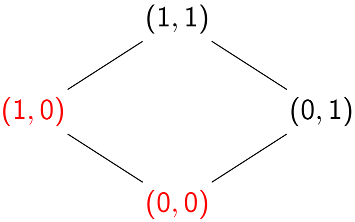
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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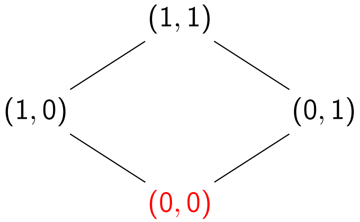
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1, 0)$ from the order ideal:



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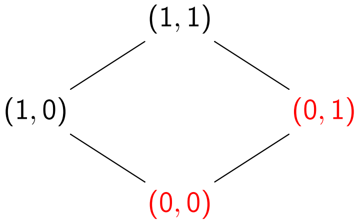
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Then apply $\mathbf{t}_{(0,1)}$, which adds $(0, 1)$ to the order ideal:



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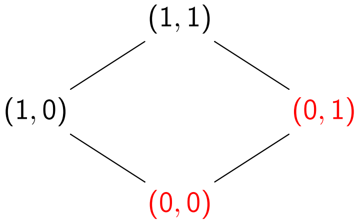
Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

Finally apply $\mathbf{t}_{(0,0)}$, which changes nothing:



Classical rowmotion: the toggling definition

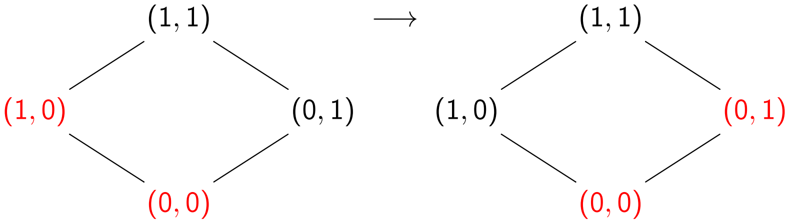
- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\rho_J = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

Example: **Re-coordinatizing $P = [a] \times [b] = [0, r] \times [0, s]$, sorry!**

So this is $S \longrightarrow \mathbf{r}(S)$:



Piecewise-linear and birational liftings

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

For a finite poset P, let $\widehat{P}$ denote P with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : \widehat{P} \to [0, 1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

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where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

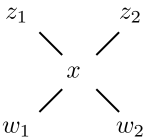
Note that the interval $[\min_{z \gtrdot x} f(z), \max_{w \lessdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

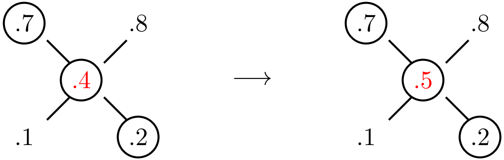
if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

Example of flipping at a node



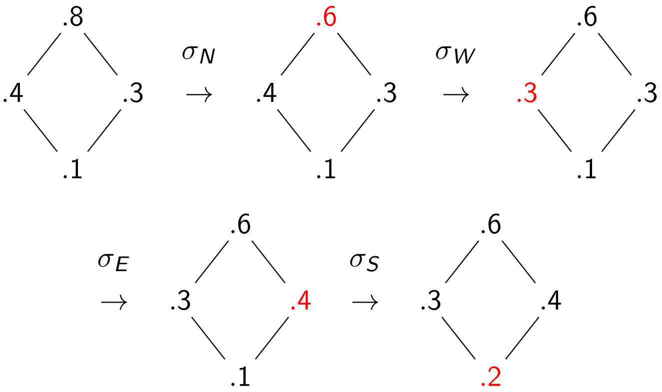


$$\min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

Composing flips and example of PL rowmotion orbit

We can apply flip-maps from top to bottom (successively flipping at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.), to get *piecewise-linear rowmotion*:

$$\begin{tikzcd}
& .8 \arrow[dl, dash] \arrow[dr, dash] & & & & .6 \arrow[dl, dash] \arrow[dr, dash] & & & & .6 \arrow[dl, dash] \arrow[dr, dash] & & & & .6 \arrow[dl, dash] \arrow[dr, dash] & & & & .6 \arrow[dl, dash] \arrow[dr, dash] & \\
.4 \arrow[dr, dash] & & .3 \arrow[dl, dash] \arrow[rr, "\sigma_N"] & & .4 \arrow[dr, dash] & & .3 \arrow[dl, dash] \arrow[rr, "\sigma_W"] & & .3 \arrow[dr, dash] & & .3 \arrow[dl, dash] \arrow[rr, "\sigma_E"] & & .3 \arrow[dr, dash] & & .4 \arrow[dl, dash] \arrow[rr, "\sigma_S"] & & .3 \arrow[dr, dash] & & .4 \arrow[dl, dash] \\
& .1 & & & & .1 & & & & .1 & & & & .1 & & & & .2 &
\end{tikzcd}$$

Here's an orbit of this map ($\tau = \sigma_S \circ \sigma_E \circ \sigma_W \circ \sigma_N$), which again has period 4.

$$\begin{tikzcd}
& .8 \arrow[dl, dash] \arrow[dr, dash] & & & & .6 \arrow[dl, dash] \arrow[dr, dash] & & & & .8 \arrow[dl, dash] \arrow[dr, dash] & & & & .9 \arrow[dl, dash] \arrow[dr, dash] \arrow[llllllllllll, bend right, "\tau"'] & \\
.4 \arrow[dr, dash] & & .3 \arrow[dl, dash] \arrow[rr, "\tau"] & & .3 \arrow[dr, dash] & & .4 \arrow[dl, dash] \arrow[rr, "\tau"] & & .7 \arrow[dr, dash] & & .6 \arrow[dl, dash] \arrow[rr, "\tau"] & & .6 \arrow[dr, dash] & & .7 \arrow[dl, dash] \\
& .1 & & & & .2 & & & & .4 & & & & .2 &
\end{tikzcd}$$

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w \lessdot x} f(w)}{f(x) \sum_{z \gtrdot x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- For a field $\mathbb{K}$, a **$\mathbb{K}$-labelling of P** will mean a function $f : \widehat{P} \to \mathbb{K}$. We always set $f(\widehat{0}) = f(\widehat{1}) = 1$.
- For any $v \in P$, define the **birational v-toggle** as the rational map

$$T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}} \text{ defined by } (T_v f)(w) = \frac{\sum\limits_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum\limits_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}} \text{ for } w = v.$$

(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

- This is a **local change** only to the label at v, and $T_v^2 = id$ (on the range of T_v).
- We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset	labelling
$\begin{tikzcd}[column sep=small] & \widehat{1} \arrow[d, dash] & \\ & (1,1) \arrow[dl, dash] \arrow[dr, dash] & \\ (1,0) \arrow[dr, dash] & & (0,1) \arrow[dl, dash] \\ & (0,0) \arrow[d, dash] & \\ & \widehat{0} & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

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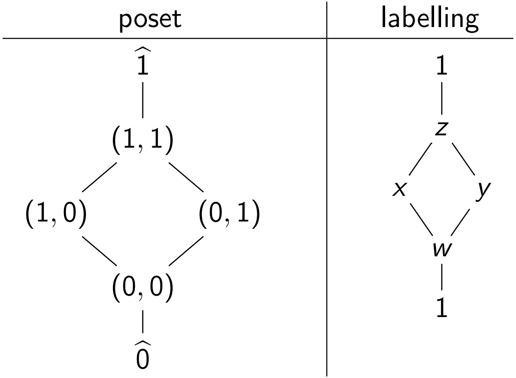

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension $((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} f$
$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion: example

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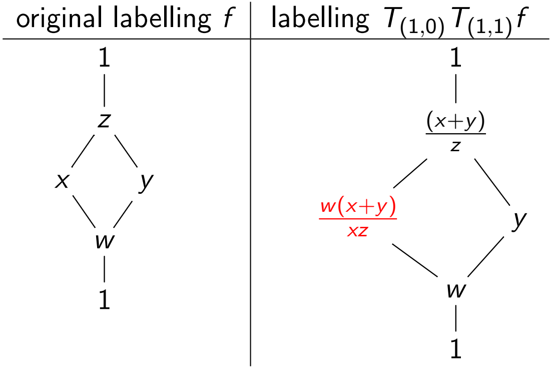

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original labelling f	labelling $T_{(0,1)} T_{(1,0)} T_{(1,1)} f$
$\begin{tikzcd}[column sep=small] & 1 & \\ & z \arrow[u, dash] & \\ x \arrow[ur, dash] & & y \arrow[ul, dash] \\ & w \arrow[ul, dash] \arrow[ur, dash] & \\ & 1 \arrow[u, dash] & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 & \\ & \frac{(x+y)}{z} \arrow[u, dash] & \\ \frac{w(x+y)}{xz} \arrow[ur, dash] & & \frac{w(x+y)}{yz} \arrow[ul, dash] \\ & w \arrow[ul, dash] \arrow[ur, dash] & \\ & 1 \arrow[u, dash] & \end{tikzcd}$

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)}\, T_{(0,1)}\, T_{(1,0)}\, T_{(1,1)} f = \rho_B f$

$$\begin{tikzcd}
& 1 \arrow[d, dash] & \\
& z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[d, dash] & \\
& \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{w(x+y)}{xz} \arrow[dr, dash] & & \frac{w(x+y)}{yz} \arrow[dl, dash] \\
& {\color{red}\frac{1}{z}} \arrow[d, dash] & \\
& 1 &
\end{tikzcd}$$

We are using $\rho_B = {\color{red}T_{(0,0)}} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion orbit on a product of chains

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd}[arrows=dash] & \frac{(x+y)}{z} \arrow[dl] \arrow[dr] & \\ \frac{(x+y)w}{xz} \arrow[dr] & & \frac{(x+y)w}{yz} \arrow[dl] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[arrows=dash] & \frac{(x+y)w}{xy} \arrow[dl] \arrow[dr] & \\ \frac{1}{y} \arrow[dr] & & \frac{1}{x} \arrow[dl] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[arrows=dash] & \frac{1}{w} \arrow[dl] \arrow[dr] & \\ \frac{yz}{(x+y)w} \arrow[dr] & & \frac{xz}{(x+y)w} \arrow[dl] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[arrows=dash] & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & w & \end{tikzcd}.$$

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Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0,r] \times [0,s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- This generalization implies the results at the PL and combinatorial level (but not vice-versa).
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these maps all have natural *homomesic* statistics [PrRo15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz} \quad \frac{1}{y} \quad \frac{yz}{(x+y)w} \quad (x) = 1,$$

Birational homomesy on files of $J([0,r] \times [0,s])$

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$$\frac{1}{z} \quad \frac{x+y}{z} \quad \frac{z}{x+y} \quad \frac{(x+y)w}{xy} \quad \frac{xy}{(x+y)w} \quad \frac{1}{w} \quad (x) \quad (z) = 1,$$

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Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

Birational homomesy on files of $J([0,r]\times[0,s])$

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B *on the product of two chains* $P=[0,r]\times[0,s]$ *is periodic, with period* $r+s+2$*.*

(2) The birational rowmotion map ρ_B *on the product of two chains* $P=[0,r]\times[0,s]$ *satisfies the following reciprocity:* $\rho_B^{i+j+1}=1/\rho_B^0(r-i,s-j)=\frac{1}{x_{r-i,s-j}}$.

Theorem (Musiker-R [MR19])

Given a file F *in* $[0,r]\times[0,s]$, $\displaystyle\prod_{k=0}^{r+s+1}\prod_{(i,j)\in F}\rho_B^k(i,j)=1.$

Birational homomesy on files of $J([0,r] \times [0,s])$

Theorem ([GrRo15b, Thm. 30, 32])

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Theorem (Musiker-R [MR19])

Given a file F *in* $[0,r] \times [0,s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1.$

The proof of this involves constructing a complicated formula for the ρ_B^k in terms of families of non-intersecting lattice paths, from which one can also deduce periodicity and the other geometric homomesies of this map, first proved by Grinberg-R [GrRo15b, Thm. 32].

Much of this story lifts to skew fields, where the variables are not assumed to commute.

- In this setting toggles are no longer involutions, but the NC analogue of ρ_B can be defined, and their inverses can be included in the study.
- Periodicity miraculously still appears to hold, though we have no proofs and computer experiments are much more challenging.
- In parallel with the lifting of ρ_J to ρ_B, there is a lifting of $\rho_{\mathcal{A}}$ via Stanley's Chain polytope to birational (*BAR-motion*) and NC (*NAR-motion*) [JR19+].
- The Stanley–Thomas word which we used to show periodicity and homomesy for $\rho_{\mathcal{A}}$ lifts all the way to the NC setting, where it still shows homomesy. However, it does not show periodicity outside the combinatorial realm, since it no longer losslessly encodes the labelings [JR20+].

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to our THEMES:
1) *Periodicity/order*; 2) *Orbit structure*; 3) *Homomesy* 4) *Equivariant bijections*

- Examples of cyclic sieving are also ripe for homomesy hunting.

- Situations in which maps can be built out of toggles seem particularly fruitful.

- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at this level.

Slides for this talk are available online (or will be soon) at

`http://www.math.uconn.edu/~troby/research.html`

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to our THEMES:
1) *Periodicity/order*; 2) *Orbit structure*; 3) *Homomesy* 4) *Equivariant bijections*
- Examples of cyclic sieving are also ripe for homomesy hunting.
- Situations in which maps can be built out of toggles seem particularly fruitful.
- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at this level.

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

Thanks very much for coming to this talk!

Tack så mycket! どうも有り難う御座いました。

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