Dynamical Algebraic Combinatorics: Actions, Orbits, Averages

Tom Roby (UConn)
Program in Algebraic and Enumerative Combinatorics
Institut Mittag-Leffler
Djursholm, SWEDEN

25 February 2020

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html
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Abstract: Dynamical Algebraic Combinatorics explores actions on sets of discrete combinatorial objects, many of which can be built up by small local changes, e.g., Schützenberger’s promotion and evacuation, or the rowmotion map on order ideals. There are strong connections to the combinatorics of representation theory and with Coxeter groups. Birational liftings of these actions are related to the Y-systems of statistical mechanics, thereby to cluster algebras, in ways that are still relatively unexplored.

The term “homomesy” describes the following widespread phenomenon: Given a group action on a set of combinatorial objects, a statistic on these objects is called “homomesic” if its average value is the same over all orbits. Along with its intrinsic interest as a kind of “hidden invariant”, homomesy can be used to prove certain properties of the action, e.g., facts about the orbit sizes. Homomesy can often be found among the same dynamics that afford cyclic sieving. Proofs of homomesy often involve developing tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection.

This talk will be a introduction to these ideas, giving a number of examples of such actions and pointing out connections to other areas.
This seminar talk discusses work with Darij Grinberg, Mike Joseph, Jim Propp and Gregg Musiker including ideas and results from Arkady Berenstein, David Einstein, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Drew Armstrong, Anders Björner, Barry Cipra, Darij Grinberg, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to interrupt with questions or comments.

ご質問コメント等があれば、ご遠慮なくおっしゃって下さい。
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- orbit structure (how many, what sizes);
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**Themes in Dynamical Algebraic Combinatorics:**

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**Themes in Dynamical Algebraic Combinatorics:**

1) **Periodicity/order**; 1) **Orbit structure**; 1) **Homomesy** 1) **Equivariant bijections**

- Cyclic rotation of binary strings and definition of homomesy;
- Toggling independent sets of path graph; and
- Rowmotion map on antichains and order ideals of posets;
- Piecewise-linear and birational liftings;
Cyclic rotation of binary strings
Let $S_{n,k}$ be the set of length $n$ binary strings with $k$ 1s.
Let $C_R : S_{n,k} \rightarrow S_{n,k}$ be rightward cyclic rotation.

Example
Cyclic rotation for $n = 6$, $k = 2$:

$$
101000 \quad \longrightarrow \quad 010100
$$

$C_R$
Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions \((i, j)\) with \(i < j\) such that there is a 1 in position \(i\) and a 0 in position \(j\).

### Example

Orbits of cyclic rotation for \(n = 6, \ k = 2\):

<table>
<thead>
<tr>
<th>String</th>
<th>Inv</th>
<th>String</th>
<th>Inv</th>
<th>String</th>
<th>Inv</th>
</tr>
</thead>
<tbody>
<tr>
<td>101000</td>
<td>7</td>
<td>110000</td>
<td>8</td>
<td>100100</td>
<td>6</td>
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<tr>
<td>010100</td>
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<td>011000</td>
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<td>010010</td>
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<tr>
<td>010001</td>
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</tr>
</tbody>
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<td>4</td>
</tr>
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Definition of Homomesy

Given

- a set $S$, 
- an invertible map $\tau : S \to S$ such that every $\tau$-orbit is finite, 
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple $(S, \tau, f)$ exhibits homomesy if there exists a constant $c \in \mathbb{K}$ such that for every $\tau$-orbit $\emptyset \subseteq S$,

$$\frac{1}{\# \emptyset} \sum_{x \in \emptyset} f(x) = c.$$
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$$\frac{1}{\#\emptyset} \sum_{x \in \emptyset} f(x) = c.$$

In this case, we say that the function $f$ is **homomesic** with average $c$ (also called $c$-mesic) under the action of $\tau$ on $S$. 
Theorem (Propp & R. [PrRo15, §2.3])

Let $I(s)$ denote the number of inversions of $s \in S_{n,k}$.

Then the function $I : S_{n,k} \to \mathbb{Q}$ is homomesic with average $\frac{k(n-k)}{2}$ with respect to cyclic rotation on $S_{n,k}$.
Theorem (Propp & R. [PrRo15, §2.3])

Let $I(s)$ denote the number of inversions of $s \in S_{n,k}$.

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Proof.

Consider superorbits of length $n$. Show that replacing “01” with “10” in a string $s$ leaves the total number of inversions in the superorbit generated by $s$ unchanged (and thus the average since our superorbits all have the same length).
Cyclic rotation of binary strings

### Example

\( n = 6, \ k = 2 \)

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</tr>
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Average 4

Average 4

Average 4
### Example

<table>
<thead>
<tr>
<th>String</th>
<th>String</th>
<th>Inversions</th>
</tr>
</thead>
<tbody>
<tr>
<td>101000</td>
<td>011000</td>
<td>-1</td>
</tr>
<tr>
<td>010100</td>
<td>001100</td>
<td>-1</td>
</tr>
<tr>
<td>001010</td>
<td>000110</td>
<td>-1</td>
</tr>
<tr>
<td>000101</td>
<td>000011</td>
<td>-1</td>
</tr>
<tr>
<td>100010</td>
<td>100001</td>
<td>-1</td>
</tr>
<tr>
<td>010001</td>
<td>110000</td>
<td>+5</td>
</tr>
</tbody>
</table>

There are other homomesic statistics as well, e.g., Let \( \mathbb{1}_j(s) := s_j \), the \( j \)th bit of the string \( s \). Can you see why this is homomesic?
Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of SSYT;
- Rowmotion of “nice” (e.g., minuscule heap) posets [PrRo15, StWi11, Had14, RuWa15+];
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- In general, composing certain involutions called “toggles” on the set leads to operations with interesting homomesy [Str18];
- Toggling the “arcs” in noncrossing partitions [EFGJMPR16];
- Whirling functions between finite sets: injections, surjections, parking functions, etc. [JPR17+]; and
- Liftings of homomesy from combinatorial maps to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].

There are many others, including in upcoming examples.
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Coxeter Toggling
Independent Sets
of Path Graphs
Definition

An independent set of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $I_n$ denote the set of independent sets of the $n$-vertex path graph $P_n$. We usually refer to an independent set by its binary representation.

Example

\[ \begin{array}{c}
\bullet & \bigcirc & \bullet & \bigcirc & \bullet & \bigcirc & \bigcirc \\
\end{array} \]

is written 1010100.
Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $I_n$ denote the set of independent sets of the $n$-vertex path graph $P_n$. We usually refer to an independent set by its **binary representation**.

Example

```
●  ○  ●  ○  ●  ○  ○
```

is written $1010100$.

In this case, $I_n$ refers to all binary strings with length $n$ that do not contain the factor 11.
Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the toggle at vertex $i$ is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, $\tau_i$ removes $i$ from $S$,
- if $i \notin S$, $\tau_i$ adds $i$ to $S$, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$
\tau_i(S) = \begin{cases} 
S \setminus \{i\} & \text{if } i \in S \\
S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\
S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n
\end{cases}
$$
Proposition

Each toggle $\tau_i$ is an involution, i.e., $\tau_i^2$ is the identity. Also, $\tau_i$ and $\tau_j$ commute if and only if $|i - j| \neq 1$.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $I_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xrightleftharpoons{\tau_1} 00010 \xrightleftharpoons{\tau_2} 01010 \xrightleftharpoons{\tau_3} 01010 \xrightleftharpoons{\tau_4} 01000 \xrightleftharpoons{\tau_5} 01001.$$
Here is an example $\varphi$-orbit in $I_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
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<td>$S$</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>$\varphi(S)$</td>
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<td>1</td>
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<td>$\varphi^4(S)$</td>
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<td>$\varphi^6(S)$</td>
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Define $\mathbb{1}_i : \mathcal{I}_n \to \{0, 1\}$ to be the indicator function of vertex $i$.

For $1 \leq i \leq n$, $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic on $\varphi$-orbits of $\mathcal{I}_n$.

Also $2\mathbb{1}_1 + \mathbb{1}_2$ and $\mathbb{1}_{n-1} + 2\mathbb{1}_n$ are 1-mesic on $\varphi$-orbits of $\mathcal{I}_n$. 

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### Idea of the proof that $1_i - 1_{n+1-i}$ is 0-mesic:

Given a 1 in an “orbit board”, if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.
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**Idea of the proof that** \(1_i - 1_{n+1-i}\) **is 0-mesic:** This allows us to partition the 1’s in the orbit board into snakes that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called “winching” on \(k\)-element subsets of \(\{1, 2, \ldots, n\}\).
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| \( \varphi^2(S) \) | 1 0 1 0 0 0 1 0 1 0 |
| \( \varphi^3(S) \) | 0 0 0 1 0 0 0 0 0 1 |
| \( \varphi^4(S) \) | 1 0 0 0 1 0 1 0 0 0 |
| \( \varphi^5(S) \) | 0 1 0 0 0 0 0 1 0 1 |
| \( \varphi^6(S) \) | 0 0 1 1 0 1 0 0 0 0 |
| \( \varphi^7(S) \) | 1 0 0 0 0 1 0 1 0 1 |
| \( \varphi^8(S) \) | 0 1 0 1 0 0 0 0 0 0 |
| \( \varphi^9(S) \) | 0 0 0 0 1 0 1 0 1 0 |
| \( \varphi^{10}(S) \) | 1 0 1 0 0 0 0 0 0 1 |
| \( \varphi^{11}(S) \) | 0 0 0 1 0 1 0 1 0 0 |
| \( \varphi^{12}(S) \) | 1 0 0 0 0 0 0 0 1 0 |
| \( \varphi^{13}(S) \) | 0 1 0 1 0 1 0 0 0 1 |
| \( \varphi^{14}(S) \) | 0 0 0 0 0 1 0 0 0 0 |
| **Total:** | 6 3 4 4 4 4 4 4 3 6 |

**Idea of the proof that** \( \mathbb{1}_i - \mathbb{1}_{n+1-i} \) **is 0-mesic:** Each snake corresponds to a composition of \( n - 1 \) into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right
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<td>1 0 0 0 0 0 0 1 0 1</td>
<td>0 0 1 0 0 1 0 0 0 0</td>
<td>0 0 0 1 0 1 0 0 0 0</td>
<td>0 0 0 0 1 0 1 0 0 0</td>
<td>0 0 0 0 0 1 0 0 0 0</td>
<td>0 0 0 0 0 0 1 0 0 0</td>
<td>0 0 0 0 0 0 0 1 0 0</td>
<td>0 0 0 0 0 0 0 0 1 0</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>6 3 4 4 4 4 4 4 4 3 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Red snake composition: 221121
Purple snake composition: 211212
Orange snake composition: 112122
Green snake composition: 121221
Blue snake composition: 212211
Brown snake composition: 122112
Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When $n$ is even, all orbits have odd size.
- “Most” orbits in $I_n$ have size congruent to $3(n - 1) \mod 4$.
- The number of orbits of $I_n$ (OEIS A000358)
- And much more...

Using elementary Coxeter theory, it’s possible to extend our main theorem to other “Coxeter elements” of toggles. We get the same homomesy if we toggle exactly once at each vertex in any order.
Antichain Rowmotion on Posets
Rowmotion: an invertible operation on antichains

Let \( \mathcal{A}(P) \) be the set of antichains of a finite poset \( P \).

Given \( A \in \mathcal{A}(P) \), let \( \rho_A(A) \) be the set of minimal elements of the complement of the downward-saturation of \( A \).

\( \rho_A \) is invertible since it is a composition of three invertible operations:

antichains \( \longleftrightarrow \) downsets \( \longleftrightarrow \) upsets \( \longleftrightarrow \) antichains

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it rowmotion.
Rowmotion: an invertible operation on antichains

Let $A(P)$ be the set of antichains of a finite poset $P$.

Given $A \in A(P)$, let $\rho_A(A)$ be the set of minimal elements of the complement of the downward-saturation of $A$.

$\rho_A$ is invertible since it is a composition of three invertible operations:

antichains $\leftrightarrow$ downsets $\leftrightarrow$ upsets $\leftrightarrow$ antichains

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it rowmotion.
Viewing the elements of the poset as \textit{squares} below, we would map:

\begin{align*}
\text{Area} = 8 & \quad \rightarrow \quad \text{Area} = 10
\end{align*}
Let $\Delta$ be a (reduced irreducible) root system in $\mathbb{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank $n$ by decreeing that $y$ covers $x$ iff $y-x$ is a simple root.

**Theorem (Armstrong-Stump-Thomas [AST11], Conj. [Pan09])**

Let $\mathcal{O}$ be an arbitrary $\rho_A$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$  

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.
Here are the classes of posets included in Panyushev’s conjecture.

(Graphic courtesy of Striker-Williams.)
Here we have just an orbit of size 2 and an orbit of size 3:

\[
\begin{align*}
1 & & 1 \\
0 & & 2 & & 1
\end{align*}
\]

Within each orbit, the average antichain has cardinality \( n/2 = 1 \).
Example of antichain rowmotion on $A_3$ root poset

For the type $A_3$ root poset, there are 3 $\rho_A$-orbits, of sizes 8, 4, 2:

Checking the average cardinality for each orbit we find that

$$\frac{1 + 2 + 2 + 1 + 1 + 2 + 2 + 1}{8} = \frac{0 + 3 + 2 + 1}{4} = \frac{2 + 1}{2} = \frac{3}{2}.$$
Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (the type $A$ minuscule poset), where $[k] = \{1, 2, \ldots, k\}$:

**Theorem (Propp, R.)**

Let $O$ be an arbitrary $\rho_A$-orbit in $A([a] \times [b])$. Then

$$\frac{1}{\# O} \sum_{A \in O} \# A = \frac{ab}{a + b}.$$
Antichains in \([a] \times [b]\): cardinality is homomesic

**Theorem (Propp, R.)**

Let \(\mathcal{O}\) be an arbitrary \(\rho_A\)-orbit in \(A([a] \times [b])\). Then \[
\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a + b}.
\]

This proof uses an non-obvious equivariant bijection (the “Stanley–Thomas” word [Sta09, §2]) between order ideals in \([a] \times [b]\) and binary strings, which carries the \(\rho_J\) map to cyclic rotation of bitstrings.

The right figure shows the Stanley–Thomas word for a 3-element antichain in \(A([7] \times [5])\). Red and black correspond to +1 and −1 respectively.

(Graphic courtesy of Ben Young.)
Antichains in \([a] \times [b]\): the case \(a = b = 2\)

Here we have an orbit of size 2 and an orbit of size 4:

Within each orbit, the average antichain has cardinality \(ab/(a + b) = 1\).
Within each orbit, the average antichain has 

$\frac{1}{2}$ of a green element and $\frac{1}{2}$ of a blue element.
Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i, j) \in [a] \times [b]$, and $A$ an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not $A$ contains $(i, j)$.

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0, 1\}$ (the cardinality of the intersection of $A$ with the fiber $\{(i, 1), (i, 2), \ldots, (i, b)\}$ in $[a] \times [b]$), so that $|A| = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $|A| = \sum_j g_j(A)$.

**Theorem (Propp, R.)**

For all $i, j$,

$$\frac{1}{\# \mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \quad \text{and} \quad \frac{1}{\# \mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$ 

The indicator functions $f_i$ and $g_j$ are homomesic under $\rho_A$, even though the indicator functions $1_{i,j}$ aren’t.
Theorem (Propp, R.)

In any orbit, the number of $A$ that contain $(i, j)$ equals the number of $A$ that contain the opposite element $(i', j') = (a + 1 - i, b + 1 -j)$.

That is, the function $\mathbb{1}_{i,j} - \mathbb{1}_{i',j'}$ is homomesic under $\rho_A$, with average value 0 in each orbit.
Rowmotion on order ideals

We’ve already seen examples of Rowmotion on antichains $\rho_A$:

\[ \rho_A : \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \]

We can also define it as an operator $\rho_J$ on $J(P)$, the set of order ideals of a poset $P$, by shifting the waltz beat by 1:

\[ \rho_J : \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\end{array} \]

\[
\frac{(0+1+3+5+7+8)}{6} = 4
\]

1

Area = 2

2

Area = 4

3

Area = 6

4

Area = 6

5

Area = 4

6

Area = 2

$(2 + 4 + 6 + 6 + 4 + 2) / 6 = 4$

\[
\frac{(3 + 5 + 4 + 3 + 5 + 4)}{6} = 4
\]
Example of order ideal rowmotion on $A_3$ root poset

For the type $A_3$ root poset, there are 3 $\rho_J$-orbits, of sizes 8, 4, 2:

Checking the average cardinality for each orbit we find that

$$\frac{1 + 2 + 4 + 3 + 1 + 2 + 4 + 3}{8} = \frac{5}{2}; \quad \frac{0 + 3 + 5 + 6}{4} = \frac{7}{2}; \quad \frac{2 + 1}{2} = \frac{3}{2}.$$ Darn!
Example of order ideal rowmotion on $A_3$ root poset

For the type $A_3$ root poset, there are 3 $\rho_A$-orbits, of sizes 8, 4, 2:

Checking the average rank-alternating cardinality for each orbit we find:

$$\frac{1 + 2 + 2 + 1 + 1 + 2 + 2 + 1}{8} = \frac{1 + 2 + 2 + 1}{4} = \frac{2 + 1}{2} = \frac{3}{2} \quad \text{Yay!}$$
Theorem (Haddadan)

Let $P$ be the root poset of type $A_n$. If we assign an element $x \in P$ weight $wt(x) = (-1)^{rank(x)}$, and assign an order ideal $I \in J(P)$ weight $f(I) = \sum_{x \in I} wt(x)$, then $f$ is homomesic under rowmotion and promotion, with average $n/2$. 
Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:

Within each orbit, the average order ideal has cardinality $ab/2 = 2$. 
Within each orbit, the average order ideal has

$1/2$ of a violet element, $1$ red element, and $1/2$ of a brown element.
Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the $k$th file of $[a] \times [b]$ as

$$\{(i, j) : 1 \leq i \leq a, 1 \leq j \leq b, i - j = k\}.$$ 

For $1 - b \leq k \leq a - 1$, let $h_k(l)$ be the number of elements of $l$ in the $k$th file of $[a] \times [b]$, so that $\#l = \sum_k h_k(l)$.

**Theorem (Propp, R.)**

For every $\rho_J$-orbit $O$ in $J([a] \times [b])$:

1. \[
\frac{1}{\#O} \sum_{l \in O} h_k(l) = \begin{cases} 
\frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\
\frac{a(b+k)}{a+b} & \text{if } k \leq 0.
\end{cases}
\]
2. \[
\frac{1}{\#O} \sum_{l \in O} \#l = \frac{ab}{2}.
\]
Rowmotion via Toggling
Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Let $v \in P$ and $S \in J(P)$. Define $t_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - $S$ otherwise.
There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

Let $v \in P$ and $S \in J(P)$. Define $t_v(S)$ as:

- $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
- $S$ otherwise.

(“Try to add or remove $v$ from $S$, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave $S$ fixed.”)
Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Let $v \in P$ and $S \in J(P)$. Define $t_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - $S$ otherwise.

  ("Try to add or remove $v$ from $S$, as long as the result remains an order ideal, i.e. within $J(P)$; otherwise, leave $S$ fixed.")

- More formally, if $P$ is a poset and $v \in P$, then the $v$-toggle is the map $t_v : J(P) \to J(P)$ which takes every order ideal $S$ to:
  - $S \cup \{v\}$, if $v$ is not in $S$ but all elements of $P$ covered by $v$ are in $S$ already;
  - $S \setminus \{v\}$, if $v$ is in $S$ but none of the elements of $P$ covering $v$ is in $S$;
  - $S$ otherwise.

- Note that $t_v^2 = \text{id}$. 
Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
\rho_J = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

**Example:** *Re-coordinatizing* \(P = [a] \times [b] = [0, r] \times [0, s], \text{ sorry!}\)

Start with this order ideal \(S\):

```
(1, 1)
```

```
(1, 0)       (0, 1)
```

```
(0, 0)
```
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).
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\]

Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

**Example:** Re-coordinatizing \(P = [a] \times [b] = [0, r] \times [0, s]\), **sorry**!

First apply \(t_{(1,1)}\), which changes nothing:

\[
\begin{array}{c}
(1, 1) \\
(1, 0) \\
(0, 0) \\
(0, 1)
\end{array}
\]
Classical rowmotion: the toggling definition

Let \((v_1, v_2, \ldots, v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[ \rho_J = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}. \]

Hugh Thomas and Nathan Williams call this **Rowmotion in slow motion** [ThWi17].

**Example:** Re-coordinatizing \(P = [a] \times [b] = [0, r] \times [0, s]\), sorry!

Then apply \(t_{(1,0)}\), which removes \((1, 0)\) from the order ideal:

- \((1, 1)\)
- \((1, 0)\)
- \((0, 1)\)
- \((0, 0)\)
Classical rowmotion: the toggling definition

- Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).
- Cameron and Fon-der-Flaass showed that

\[ \rho_J = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}. \]

Hugh Thomas and Nathan Williams call this **Rowmotion in slow motion** [ThWi17].

**Example:** Re-coordinatizing \(P = [a] \times [b] = [0, r] \times [0, s]\), sorry!

Then apply \(t_{(0,1)}\), which adds \((0, 1)\) to the order ideal:

\[
\begin{align*}
(1,1) & \quad (0,1) \\
(1,0) & \quad (0,0)
\end{align*}
\]
Let \((v_1, v_2, ..., v_n)\) be a **linear extension** of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
\rho_J = t_{v_1} \circ t_{v_2} \circ ... \circ t_{v_n}.
\]

Hugh Thomas and Nathan Williams call this *Rowmotion in slow motion* [ThWi17].

**Example:** *Re-coordinatizing* \(P = [a] \times [b] = [0, r] \times [0, s]\), *sorry!*

Finally apply \(t_{(0,0)}\), which changes nothing:

```
(1, 1)

(1, 0)         (0, 1)

(0, 0)
```
Let \((v_1, v_2, \ldots, v_n)\) be a \textbf{linear extension} of \(P\); this means a list of all elements of \(P\) (each only once) such that \(i < j\) whenever \(v_i < v_j\).

Cameron and Fon-der-Flaass showed that

\[
\rho_J = t_{v_1} \circ t_{v_2} \circ \ldots \circ t_{v_n}.
\]

Hugh Thomas and Nathan Williams call this \textit{Rowmotion in slow motion} [ThWi17].

**Example:** Re-coordinatizing \(P = [a] \times [b] = [0, r] \times [0, s]\), sorry!

So this is \(S \rightarrow r(S)\):

\[
\begin{array}{c}
(1,0) \\
(0,0) \\
(0,1) \\
(1,1)
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
(1,0) \\
(0,0) \\
(0,1) \\
(1,1)
\end{array}
\]

Piecewise-linear and birational liftings
Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a piecewise-linear (PL) version of rowmotion acting on functions on a poset.

For a finite poset $P$, let $\hat{P}$ denote $P$ with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.
Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

For a finite poset $P$, let $\hat{P}$ denote $P$ with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The **order polytope** $O(P)$ (introduced by R. Stanley) is the set of functions $f : \hat{P} \to [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$. 
Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a piecewise-linear (PL) version of rowmotion acting on functions on a poset.

For a finite poset $P$, let $\hat{P}$ denote $P$ with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The order polytope $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : \hat{P} \to [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$
\begin{align*}
f'(y) = \begin{cases} 
f(y) & \text{if } y \neq x, \\
\min_{z\succ x} f(z) + \max_{w\preceq x} f(w) - f(x) & \text{if } y = x,
\end{cases}
\end{align*}
$$

where $z \succ x$ means $z$ covers $x$ and $w \preceq x$ means $x$ covers $w$. 
For each \( x \in P \), define the flip-map \( \sigma_x : \mathcal{O}(P) \to \mathcal{O}(P) \) sending \( f \) to the unique \( f' \) satisfying

\[
f'(y) = \begin{cases} 
  f(y) & \text{if } y \neq x, \\
  \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x) & \text{if } y = x,
\end{cases}
\]

where \( z \succ x \) means \( z \) covers \( x \) and \( w \prec x \) means \( x \) covers \( w \).

Note that the interval \([\min_{z \succ x} f(z), \max_{w \prec x} f(w)]\) is precisely the set of values that \( f'(x) \) could have so as to satisfy the order-preserving condition.

If \( f'(y) = f(y) \) for all \( y \neq x \), the map that sends

\[
f(x) \text{ to } \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x)
\]

is just the affine involution that swaps the endpoints.
Example of flipping at a node

\[
\begin{align*}
\min_{z \gg x} f(z) + \max_{w \ll x} f(w) &= .7 + .2 = .9 \\
f(x) + f'(x) &= .4 + .5 = .9
\end{align*}
\]
Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:

\[
\begin{array}{ccc}
0.8 & \rightarrow & 0.6 \\
0.4 & \rightarrow & 0.4 \\
0.1 & \rightarrow & 0.1 \\
0.6 & \rightarrow & 0.6 \\
0.3 & \rightarrow & 0.3 \\
0.1 & \rightarrow & 0.1 \\
\end{array}
\]

(We successively flip at \(N = (1, 1)\), \(W = (1, 0)\), \(E = (0, 1)\), and \(S = (0, 0)\) in order.)
Composing flips and example of PL rowmotion orbit

We can apply flip-maps from top to bottom (successively flipping at $N = (1, 1)$, $W = (1, 0)$, $E = (0, 1)$, and $S = (0, 0)$ in order.), to get piecewise-linear rowmotion:

Here’s an orbit of this map ($\tau = \sigma_S \circ \sigma_E \circ \sigma_W \circ \sigma_N$), which again has period 4.
In the so-called *tropical semiring*, one replaces the standard binary ring operations \((+, \cdot)\) with the tropical operations \((\max, +)\). In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at \(x\) replaced the value of a function \(f : P \to [0, 1]\) at a point \(x \in P\) with \(f'\), where

\[
f'(x) := \min_{z \succ x} f(z) + \max_{w \prec x} f(w) - f(x)
\]

We can “detropicalize” this flip map and apply it to an assignment \(f : P \to \mathbb{R}(x)\) of *rational functions* to the nodes of the poset, using that

\[
\min(z_i) = -\max(-z_i),
\]

to get the *birational toggle map*

\[
(T_x f)(x) = f'(x) = \frac{\sum_{w \prec x} f(w)}{f(x) \sum_{z \succ x} \frac{1}{f(z)}}
\]
For a field $\mathbb{K}$, a $\mathbb{K}$-labelling of $P$ will mean a function $f : \hat{P} \to \mathbb{K}$. We always set $f(\hat{0}) = f(\hat{1}) = 1$.

For any $v \in P$, define the birational $v$-toggle as the rational map

$$T_v : \mathbb{K}^\hat{P} \dashrightarrow \mathbb{K}^\hat{P} \text{ defined by } (T_v f)(w) = \frac{\sum_{\hat{P} \ni u < v} f(u)}{f(v) \sum_{\hat{P} \ni u > v} \frac{1}{f(u)}} \text{ for } w = v.$$

(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)
For a field $\mathbb{K}$, a $\mathbb{K}$-labelling of $P$ will mean a function $f : \hat{P} \to \mathbb{K}$. We always set $f(0) = f(1) = 1$.

For any $v \in P$, define the **birational $v$-toggle** as the rational map $T_v : \mathbb{K}^{\hat{P}} \to \mathbb{K}^{\hat{P}}$ defined by $(T_v f)(w) = \frac{\sum_{\hat{P} \ni u < v} f(u)}{f(v) \sum_{\hat{P} \ni u > v} \frac{1}{f(u)}}$ for $w = v$.

(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

This is a **local change** only to the label at $v$, and $T_v^2 = id$ (on the range of $T_v$).

We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\hat{P}} \to \mathbb{K}^{\hat{P}},$$

where $(v_1, v_2, \ldots, v_n)$ is a linear extension of $P$. 
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>poset</th>
<th>labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{1}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(1,1)$</td>
<td>$z$</td>
</tr>
<tr>
<td>$(1,0)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$(0,1)$</td>
<td>$y$</td>
</tr>
<tr>
<td>$(0,0)$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

We have $\rho_{\text{B}} = T((0,0)) \circ T((0,1)) \circ T((1,0)) \circ T((1,1))$ using the linear extension $((1,1),(1,0),(0,1),(0,0))$. That is, toggle in the order “top, left, right, bottom”.

Diagram:

- $\hat{1}$ is at the top.
- $(1,1)$ is to the right of $(0,0)$.
- $(1,0)$ is below $(1,1)$.
- $(0,1)$ is to the right of $(0,0)$.
- $(0,0)$ is at the bottom.
- The labelling is as follows:
  - $1$ is at the top.
  - $z$ is above $x$ and $y$.
  - $x$ is above $w$.
  - $y$ is above $w$.
  - $w$ is at the bottom.

- The linear extension is $((1,1),(1,0),(0,1),(0,0))$. 
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
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<th>labelling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$z$</td>
<td>$z$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$y$</td>
<td>$y$</td>
</tr>
<tr>
<td>$w$</td>
<td>$w$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension $((1,1), (1,0), (0,1), (0,0))$.

That is, toggle in the order “top, left, right, bottom”.
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
<th>original labelling $f$</th>
<th>labelling $T_{(1,1)}f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>z</td>
<td>(x+y)</td>
</tr>
<tr>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>w</td>
<td>w</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$. 
Example:

Let us “rowmote” a (generic) \( \mathbb{K} \)-labelling of the \( 2 \times 2 \)-rectangle:

<table>
<thead>
<tr>
<th>original labelling ( f )</th>
<th>labelling ( T_{(1,0)} T_{(1,1)} f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>z</td>
<td>( (x+y) )</td>
</tr>
<tr>
<td>x \quad y</td>
<td>( w(x+y) ) \quad xz</td>
</tr>
<tr>
<td>w</td>
<td>( w )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We are using \( \rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)} \).
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

<table>
<thead>
<tr>
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<th>labelling $T_{(0,1)} T_{(1,0)} T_{(1,1)} f$</th>
</tr>
</thead>
</table>
| 1 \[ \begin{array}{c}
  \downarrow \\
  \text{z} \\
  \downarrow \\
  \text{x} \\
  \downarrow \\
  \text{w} \\
  \downarrow \\
  1
\end{array} \] | 1 \[ \begin{array}{c}
  \downarrow \\
  \text{(x+y)} \\
  \downarrow \\
  \text{z} \\
  \downarrow \\
  \frac{w(x+y)}{xz} \\
  \downarrow \\
  \frac{w(x+y)}{yz} \\
  \downarrow \\
  1
\end{array} \] |

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$. 
Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

\[
\begin{array}{c|c}
\text{original labelling } f & \text{labelling } T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f \\
1 & 1 \\
\vdots & \vdots \\
z & (x+y) \\
x & w(x+y) \\
y & xz \\
w & yz \\
1 & 1 \\
\end{array}
\]

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$. 
Example: Iterating this procedure we get

\[ \rho_B f = \frac{(x+y)w}{xz} \frac{(x+y)w}{yz} \frac{1}{z} \frac{1}{w}, \]

\[ \rho_B^2 f = \frac{1}{y} \frac{1}{x} \frac{z}{x+y}, \]

\[ \rho_B^3 f = \frac{yz}{(x+y)w} \frac{xz}{(x+y)w} \frac{xy}{(x+y)w}, \]

\[ \rho_B^4 f = \frac{x}{w} \sqrt{y}. \]
**Example:** Iterating this procedure we get

\[
\rho_B f = \frac{(x+y)}{z} \quad \frac{(x+y)}{w} \quad \frac{1}{z} \quad \frac{1}{w},
\]

\[
\rho_B f = \frac{(x+y)}{w} \quad \frac{(x+y)}{w} \quad \frac{1}{y} \quad \frac{1}{x},
\]

\[
\rho_B f = \frac{y}{z} \quad \frac{x}{w} \quad \frac{1}{z} \quad \frac{1}{w}.
\]

Notice that \( \rho_B^4 f = f \), which generalizes to \( \rho_B^{r+s+2} f = f \) for \( P = [0, r] \times [0, s] \) [Grinberg-R 2015]. Notice also “antipodal reciprocity”.
Why study this generalization?

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- This generalization implies the results at the PL and combinatorial level (but not vice-versa).
- Birational rowmotion can be related to $Y$-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these maps all have natural homomesic statistics [PrRo15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.
Birational homomesy on files of $J([0, r] \times [0, s])$

The poset $[0, 1] \times [0, 1]$ has three files, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1, 0)\rho_B^2(f)(1, 0)\rho_B^3(f)(1, 0)\rho_B^4(f)(1, 0) = \frac{(x + y)w}{xz} \cdot \frac{1}{y} \cdot \frac{yz}{(x + y)w} \cdot (x) = 1,$$

Each of these products equalling one is the manifestation, for the poset of a product of two chains, of homomesy along files at the birational level.
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$$\rho_B(f)(0, 0)\rho_B(f)(1, 1)\rho_B^2(f)(0, 0)\rho_B^3(f)(1, 1)\rho_B^3(f)(0, 0)\rho_B^4(f)(0, 0)\rho_B^4(f)(1, 1) =$$

$$\frac{1}{z} \frac{x + y}{z} \frac{z}{x + y} \frac{(x + y)w}{xy} \frac{xy}{(x + y)w} \frac{1}{w} (x) (z) = 1,$$
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$$\frac{1}{z} \cdot \frac{x + y}{z} \cdot \frac{z}{x + y} \cdot \frac{(x + y)w}{xy} \cdot \frac{xy}{(x + y)w} \cdot \frac{1}{w} \quad (x) \quad (z) = 1,$$

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$$\frac{1}{z} \frac{x + y}{z} \frac{z}{x + y} \frac{(x + y)w}{xy} \frac{xy}{(x + y)w} \frac{1}{w} \quad (x) \quad (z) = 1,$$

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**Theorem ([GrRo15b, Thm. 30, 32])**

1. The birational rowmotion map $\rho_B$ on the product of two chains $P = [0, r] \times [0, s]$ is periodic, with period $r + s + 2$.

2. The birational rowmotion map $\rho_B$ on the product of two chains $P = [0, r] \times [0, s]$ satisfies the following reciprocity: $\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

**Theorem (Musiker-R [MR19])**

Given a file $F$ in $[0, r] \times [0, s]$, $\prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i, j) = 1$. 

The proof of this involves constructing a complicated formula for the $\rho_B^k$ in terms of families of non-intersecting lattice paths, from which one can also deduce periodicity and the other geometric homomesies of this map, first proved by Grinberg-R [GrRo15b, Thm. 32].
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Noncommutative Liftings

Much of this story lifts to skew fields, where the variables are not assumed to commute.

- In this setting toggles are no longer involutions, but the NC analogue of $\rho_B$ can be defined, and their inverses can be included in the study.
- Periodicity miraculously still appears to hold, though we have no proofs and computer experiments are much more challenging.
- In parallel with the lifting of $\rho_J$ to $\rho_B$, there is a lifting of $\rho_A$ via Stanley’s Chain polytope to birational (BAR-motion) and NC (NAR-motion) [JR19+].
- The Stanley–Thomas word which we used to show periodicity and homomesy for $\rho_A$ lifts all the way to the NC setting, where it still shows homomesy. However, it does not show periodicity outside the combinatorial realm, since it no longer losslessly encodes the labelings [JR20+].
Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to our THEMES:
  1) Periodicity/order; 2) Orbit structure; 3) Homomesy 4) Equivariant bijections
- Examples of cyclic sieving are also ripe for homomesy hunting.
- Situations in which maps can be built out of toggles seem particularly fruitful.
- Combinatorial objects are often discrete “shadows” of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at this level.

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

Thanks very much for coming to this talk!

Tack så mycket!

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References 2


