Paths to Understanding Rowmotion on a Product of Two Chains

Tom Roby (UConn)

Describing joint research with Gregg Musiker[1] (UMN)

SageDays@ICERM: Combinatorics and Representation Theory
Brown University, Providence RI

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[1]Partially supported by NSF Grant DMS-1362980

Outline & Acknowledgements

1. Classical Rowmotion
2. Piecewise-linear (PL) and Birational Rowmotion
3. Formula in terms of Lattice Paths
4. Sketch of Proof
5. Applications (Periodicity and Homomesy)

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Please feel free to interrupt with questions or comments.

ご質問コメント等があれば、ご遠慮なくおっしゃって下さい。

Main Ideas

- The combinatorial rowmotion map has liftings (via a decomposition into involutions called *toggles*) to the piecewise-linear (order polytope) and then birational settings. Proving results at the birational level implies them at the other levels.
- For rectangular posets $P = [0, r] \times [0, s]$, we give a formula in terms of NILPs that allows us to compute ρ_B^k, the kth iteration of birational rowmotion.
- The key lemma is a Plücker-like relation satisfied by certain polynomials we define, proven by a colorful combinatorial bijection on pairs of NILPs (along the lines of Fulmek-Kleber).
- Using our formula, we obtain more direct proofs of the periodicity and "antipodal-reciprocity" of this system, as well as the first proof of "homomesy along files".

Classical rowmotion is the rowmotion studied by Striker-Williams (2012), who coined the term. It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).
- Propp-Roby (2015), as one of several actions that displays the homomesy phenomenon on the product of two chains.

Let P be a finite poset. **Classical rowmotion** is the map
$\mathbf{r} : J(P) \longrightarrow J(P)$

sending every order ideal S to a new order ideal $\mathbf{r}(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let S be the following order ideal

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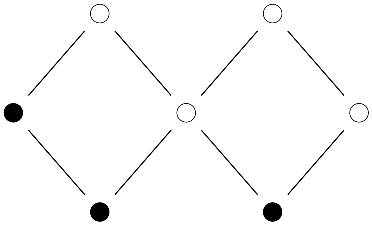

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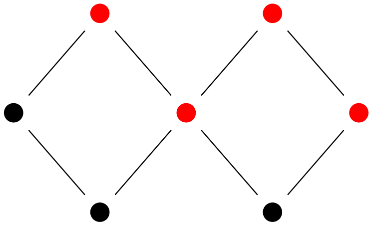
Mark the complement in red.



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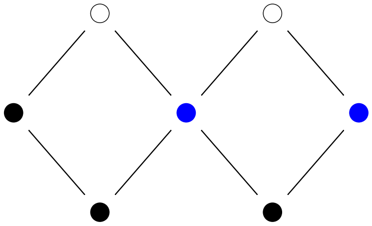
Mark M (the minimal elements of the complement) in blue.



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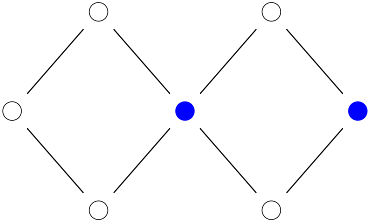
Remove the old order ideal:



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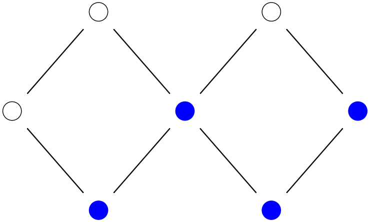
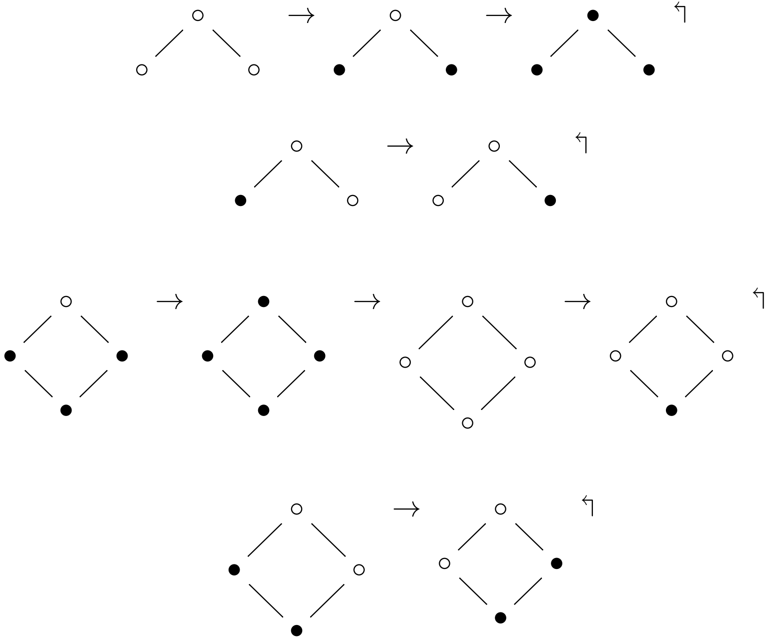
Example: Let S be the following order ideal

$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



Examples of Orbits of this Dynamic on Order Ideals.



Definition ([PR15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.

(Call f c-mesic for short.) Greek for "same-middle"

Theorem ([PR15])

For the action of rowmotion on order ideals $J(P)$ of rectangular posets $P = [p] \times [q]$, the cardinality statistic is homomesic (with average $pq/2$).

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PR15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.
- Galashin & Pylyavskyy have a very interesting extension of birational rowmotion to directed graphs and "R-systems." [GaPy17].

Classical rowmotion: Periodicity

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

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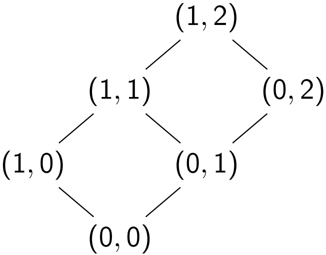
However, **for some types of** P, the order can be explicitly computed or bounded from above. See Striker-Williams [StWi11] (and the **very recent** Thomas-Williams [TW17]) for an exposition of known results.

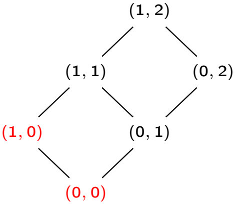
- If P is a $p \times q$-rectangle:



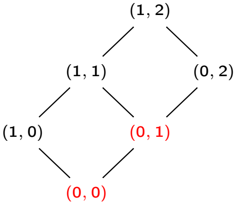
(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

Classical rowmotion: Periodicity (Example)

$S =$

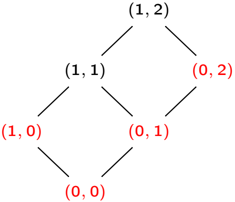


, $\mathbf{r}(S) =$

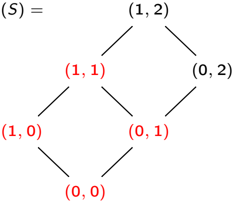


,

$\mathbf{r}^2(S) =$



, $\mathbf{r}^3(S) =$



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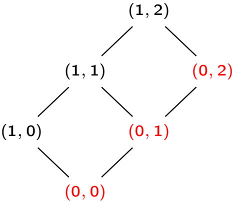
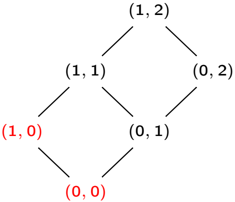


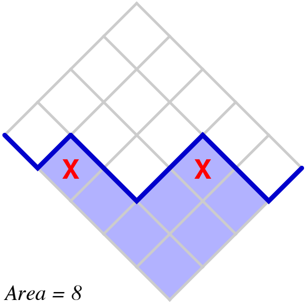
, $\mathbf{r}^5(S) =$



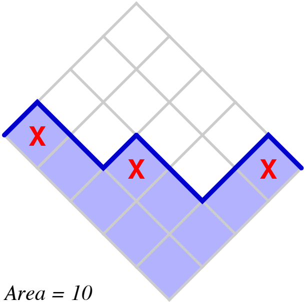
,

Example in lattice cell form

Viewing the elements of the poset as **squares** below, we would map:



$\longrightarrow$



Rowmotion on $[4] \times [2]$ A

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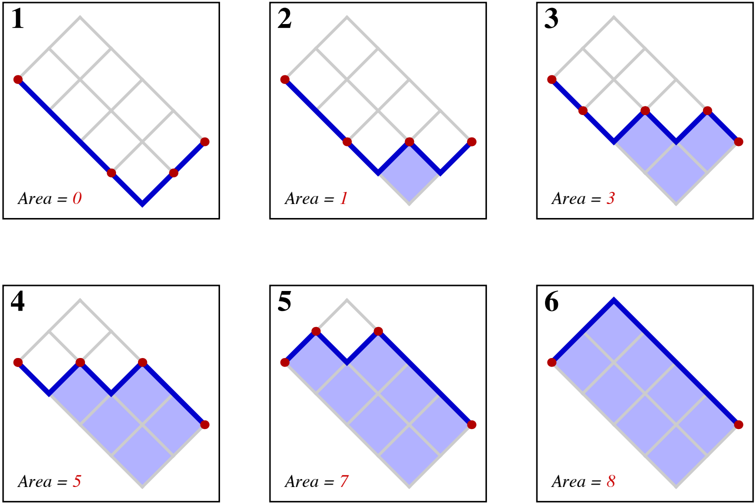

$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ B

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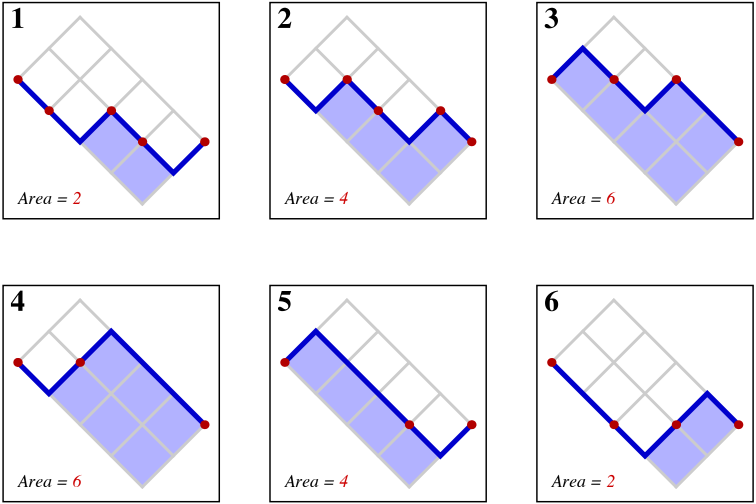

$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ C

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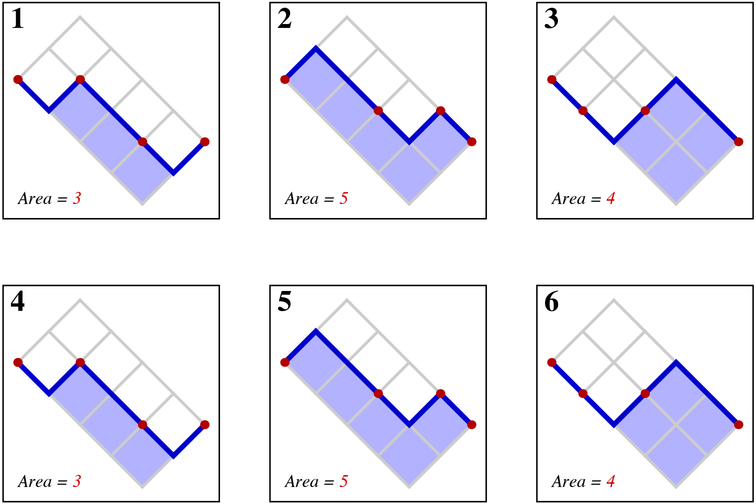
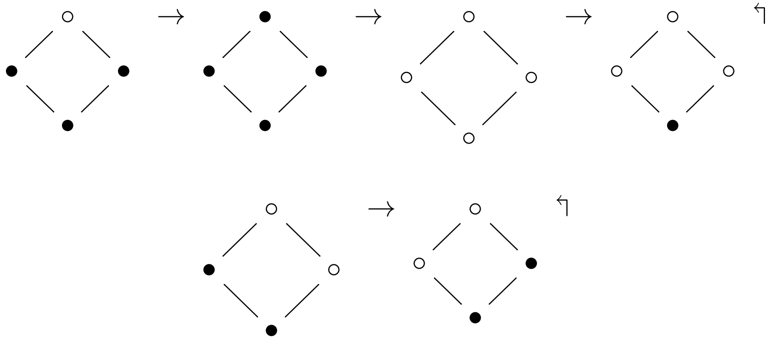


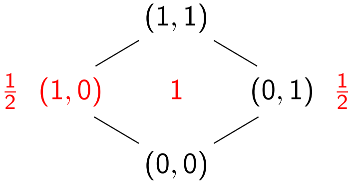
$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Classical rowmotion: Antipodal and File Homomesies



The average value along antipodal (N-S, E-W) pairs is 1 for both orbits,

and is also **constant**, as



, on files.

Rowmotion: the toggling definitions

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
    - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
    - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
    - S otherwise.
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.

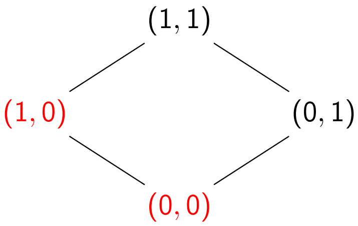
Classical rowmotion via toggling ("rowmotion in slowmotion")

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass [CaFl95] showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Example:

Start with this order ideal S:



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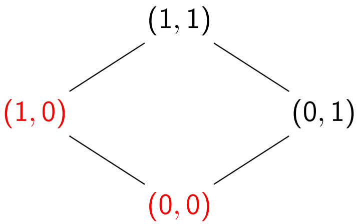
Example:

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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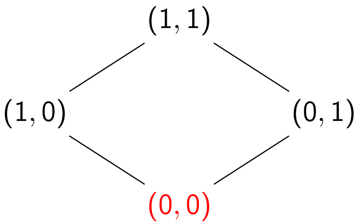
Then apply $\mathbf{t}_{(1,0)}$, which removes $(1, 0)$ from the order ideal:



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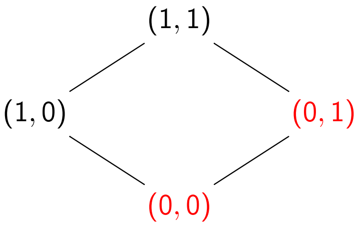

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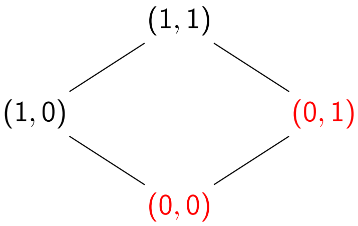

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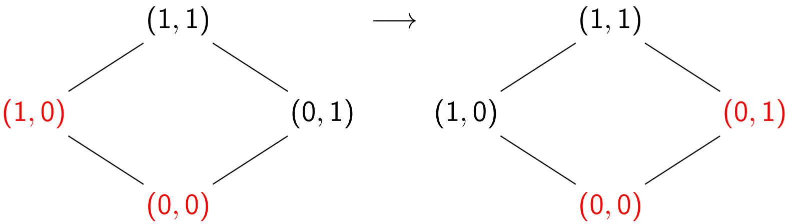
Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley [Stan86]) is the set of functions $f : P \to [0,1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$. (Compare with $J(P) = \{f : P \to \{0,1\} : f \text{ is monotone}\}$)

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

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where $z \cdot> x$ means z covers x and $w <\cdot x$ means x covers w.

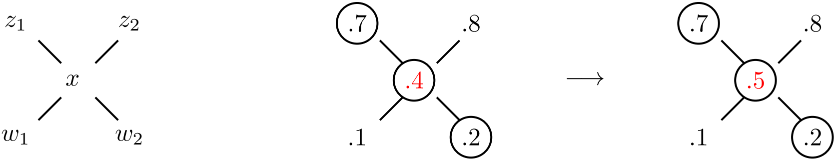
Note that the interval $[\min_{z \cdot> x} f(z), \max_{w <\cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

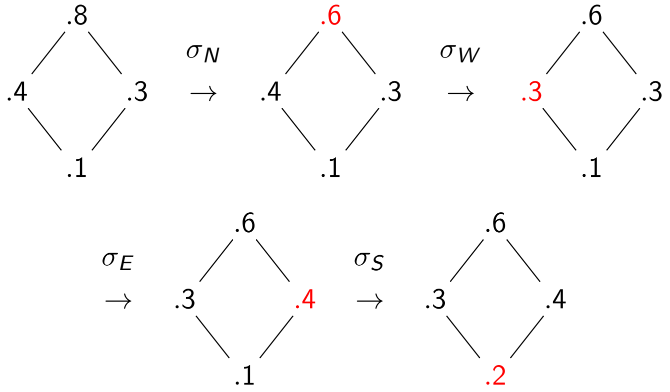
Example of flipping at a node



$$\min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order to get $\rho_{PL}(f)$.)

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0,1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can"detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements $\widehat{0}$ and $\widehat{1}$ to P and forcing
  - $\widehat{0}$ to be less than every other element, and
  - $\widehat{1}$ to be greater than every other element.
- For a *field* $\mathbb{K}$, a **$\mathbb{K}$-labelling of P** will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.
- For any $v \in P$, define the **birational v-toggle** as the rational map

  $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by $(T_v f)(w) = \frac{\sum_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.

  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.

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- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.
- We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ ... \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, ..., v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- This is originally due to Einstein and Propp [EiPr13, EiPr14], following the lead of Kirillov-Berenstein [KiBe95].

Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset

$$\begin{tikzcd}[column sep=small] & \widehat{1} \arrow[d, dash] & \\ & (1,1) \arrow[dl, dash] \arrow[dr, dash] & \\ (1,0) \arrow[dr, dash] & & (0,1) \arrow[dl, dash] \\ & (0,0) \arrow[d, dash] & \\ & \widehat{0} & \end{tikzcd}$$

labelling

$$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$$

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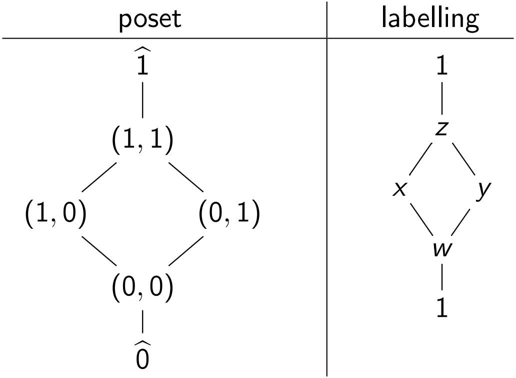

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension

$((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} f$
$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & {\color{red}\frac{(x+y)}{z}} \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$

We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ {\color{red}T_{(1,1)}}$.

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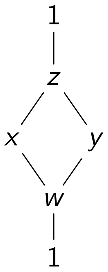

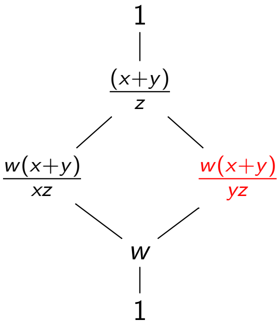

We are using $\rho_B = T_{(0,0)} \circ \textcolor{red}{T_{(0,1)}} \circ T_{(1,0)} \circ T_{(1,1)}$.

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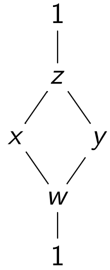
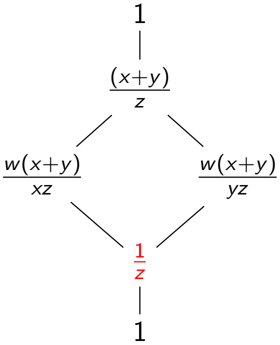
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$





We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion orbit on a product of chains

Example: Iterating this procedure we get

$$\rho_B f = \begin{tikzcd}[column sep=small] & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{(x+y)w}{xz} \arrow[dr, dash] & & \frac{(x+y)w}{yz} \arrow[dl, dash] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[column sep=small] & \frac{(x+y)w}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{1}{y} \arrow[dr, dash] & & \frac{1}{x} \arrow[dl, dash] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[column sep=small] & \frac{1}{w} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{yz}{(x+y)w} \arrow[dr, dash] & & \frac{xz}{(x+y)w} \arrow[dl, dash] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}.$$

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Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0,r] \times [0,s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Birational homomesy on files

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_B^k(f)(1,0) = \frac{(x+y)w}{xz} \quad \frac{1}{y} \quad \frac{yz}{(x+y)w} \quad (x) = 1,$$

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Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

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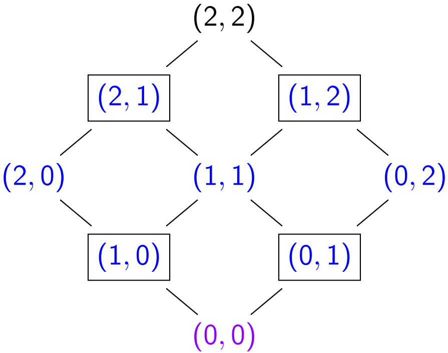
1) Let $\bigvee_{(m,n)} := \{(u,v) : (u,v) \geq (m,n)\}$ be the *principal order filter at* (m,n), $\hexagon^{k}_{(m,n)}$ be the *rank-selected subposet*, of elements in $\bigvee_{(m,n)}$ whose rank (within $\bigvee_{(m,n)}$) is at least $k-1$ and whose corank is at most $k-1$.



2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$. (For $k \leq \min\{r-m, s-n\}+1$.)

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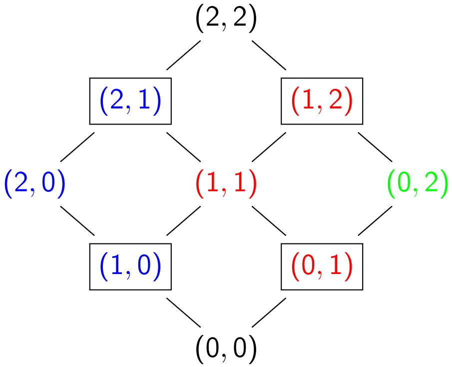
Let $A_{ij} := \frac{\sum_{z \lessdot (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1}+x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i,j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\widehat{P}$).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_{\mathbf{k}}(\mathbf{m}, \mathbf{n})$ in terms of the A_{ij}'s as follows.

Birational Rowmotion on the Rectangular Poset

We define a **lattice path of length k** within $P = [0, r] \times [0, s]$ to be a sequence $v_1, v_2, \ldots, v_k$ of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1, 0)$ or $(0, 1)$ for each $i \in [k]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex.



Birational Rowmotion on the Rectangular Poset

3) Let $S_k(m,n)$ be the set of all NILPs (non-intersecting lattice paths) in $\hexagon^k_{(m,n)}$, from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. Let $\mathcal{L} = \{L_1, L_2, \ldots L_k\} \in S_k^k(m,n)$ denote a single such k-collection of NILPs.

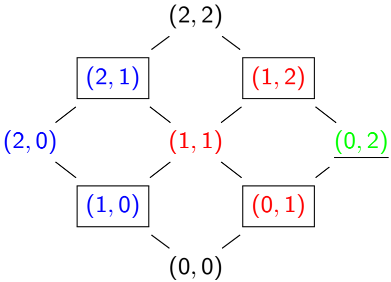
4) Define $\varphi_k(m,n) :=$
$$\sum_{\mathcal{L} \in S_k^k(m,n)} \quad \prod_{\substack{(i,j) \in \hexagon^k_{(m,n)} \\ (i,j) \notin L_1 \cup L_2 \cup \cdots \cup L_k}} A_{ij}.$$

Theorem* **(approx.):**

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

EG: $\rho_B^2(1,1) = \dfrac{\varphi_1(0,0)}{\varphi_2(0,0)}$

$$= \frac{\text{sum of 6 quartic terms in } A_{ij}}{A_{20} + A_{11} + A_{02}}.$$



(*) Caveats explained and general statement given below.

Main Theorem (M-Roby 2018)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k-i]_+ + [k-j]_+$.

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(a1) When $M = 0$, i.e., $(i-k, j-k)$ still lies in the poset $[0,r] \times [0,s]$:

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(a2) When $0 < M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+, [k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\mu^{(a,b)}$ is the operator that takes a rational function in $\{A_{(u,v)}\}$ and simply shifts each index in each factor of each term: $A_{(u,v)} \mapsto A_{(u-a, v-b)}$.

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Fix $k \in [0, r+s+1]$ and set $M = [k-i]_+ + [k-j]_+$. After $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$ we get:

(a) When $0 \leq M \leq k$:

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where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined above.

(b) When $M \geq k$: $\rho_B^{k+1}(i,j) = 1/\rho_B^{k-i-j}(r-i, s-j)$, which is well-defined by part (a).

Remark: We prove that our formulae in (a) and (b) agree when $M = k$, allowing us to give a new proof of periodicity: $\rho_B^{r+s+2+d} = \rho_B^d$; thus we get a formula for **all** iterations of the birational rowmotion map.

Corollaries of the Main Theorem

Corollary

For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$.

Corollary ([GrRo15, Thm. 30])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *is periodic, with period* $r+s+2$.

Corollary ([GrRo15, Thm. 32])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *satisfies the following reciprocity:*
$\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Corollaries of the Main Theorem

Theorem

Given a file F in $[0,r]\times[0,s]$, $\displaystyle\prod_{k=0}^{r+s+1}\prod_{(i,j)\in F}\rho_B^k(i,j)=1.$

The poset $[0,1]\times[0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

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$$\prod_{k=1}^{4} \rho_B^k(f)(0,1) = \frac{(x+y)w}{yz} \quad \frac{1}{x} \quad \frac{xz}{(x+y)w} \quad (y) = 1.$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

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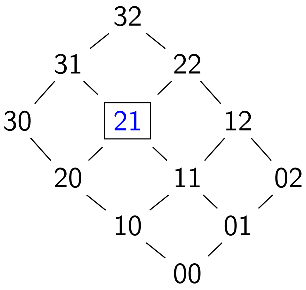
Recall that in the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get a simpler formula.

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$.



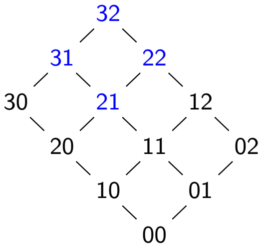
Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

Recall that in the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get a simpler formula.

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$.

When $\mathbf{k = 0}$, $M = 0$ and we get



$$\rho_B^1(2,1) = \frac{\varphi_0(2,1)}{\varphi_1(2,1)} = \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}.$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

Recall that in the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get a simpler formula.

Corollary: For $k \le \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$.

When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

For the numerator, $s_1 = (1,0)$, $t_1 = (3,2)$, and there are six lattice paths from s_1 to t_1, each of which covers 5 elements and leaves 4 uncovered.

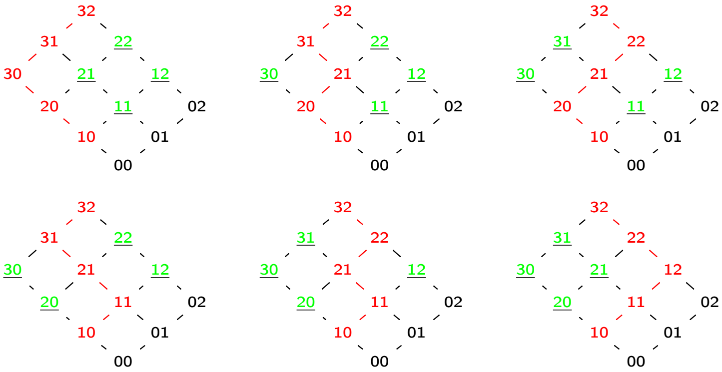
For the denominator, $s_1 = (2,0)$, $s_2 = (1,1)$, $t_1 = (3,1)$, and $t_2 = (2,2)$, and each pair of lattice paths leaves exactly one element uncovered.

Example

When $\mathbf{k} = \mathbf{1}$, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

For the numerator, $s_1 = (1,0)$, $t_1 = (3,2)$, and there are six lattice paths from s_1 to t_1, each of which covers 5 elements and leaves 4 uncovered.

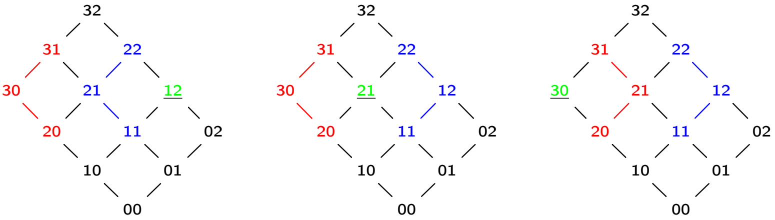


Example

When $\mathbf{k} = \mathbf{1}$, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

For the denominator, $s_1 = (2,0)$, $s_2 = (1,1)$, $t_1 = (3,1)$, and $t_2 = (2,2)$, and each pair of lattice paths leaves exactly one element uncovered.



Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2,$ and $j = 1$ throughout.

In the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) gives a point outside of P, we must also apply a μ-translation.

When k = 2, we get $M = [2-2]_+ + [2-1]_+ = 1 \leq 2 = k$. So by part (a) of the main theorem we have

$$\rho_B^3(2,1) = \mu^{(1,0)}\left[\frac{\varphi_1(1,0)}{\varphi_2(1,0)}\right] = \text{(just shifting indices in the } k = 1 \text{ formula)}$$

$$\frac{A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}}{A_{02} + A_{11} + A_{20}}.$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) gives a point outside of P, we must also apply a μ-translation.

When k = 3, we get $M = [3-2]_+ + [3-1]_+ = 3 = k$. Therefore,

$$\rho_B^4(2,1) = \mu^{(2,1)}\left[\frac{\varphi_0(2,1)}{\varphi_1(2,1)}\right] = \mu^{(2,1)}\left[\frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}\right] = \frac{A_{00}A_{01}A_{10}A_{11}}{A_{01}+A_{10}}.$$

In this situation, we can also use part (b) of the main theorem to get

$$\rho_B^4(2,1) = 1/\rho_B^{3-2-1}(3-2, 2-1) = 1/\rho_B^0(1,1) = \frac{1}{x_{11}}.$$

The equality between these two expressions is easily checked.

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P=[0,3]\times[0,2]$ for $k=0,1,2,3,4,5,6$. Here $r=3, s=2, i=2$, and $j=1$ throughout.

In the case where μ-translation would lead to negative subscripts for the φ's, i.e., $M>k$, part (a) of the Theorem does not apply.

When k = 4, we get $M=[4-2]_+ + [4-1]_+ = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^5(2,1) = 1/\rho_B^{4-2-1}(3-2,2-1) = 1/\rho_B^1(1,1) = \frac{\varphi_1(1,1)}{\varphi_0(1,1)} = \frac{A_{12}A_{22}+A_{12}A_{31}+A_{21}A_{31}}{A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}}.$$

Each term in the numerator is associated with one of the three lattice paths from $(1,1)$ to $(3,2)$ in P, while the denominator is just the product of all A-variables in the principal order filter $\bigvee(1,1)$.

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where μ-translation would lead to negative subscripts for the φ's, i.e., $M > k$, part (a) of the Theorem does not apply.

When k = 5, we get $M = [5-2]_+ + [5-1]_+ = 7 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^6(2,1) = 1/\rho_B^{5-2-1}(3-2, 2-1) = 1/\rho_B^2(1,1) = \frac{\varphi_2(0,0)}{\varphi_1(0,0)} =$$

$$(A_{02}A_{12} + A_{02}A_{21} + A_{11}A_{21} + A_{30}A_{02} + A_{30}A_{11} + A_{30}A_{20})\Big/$$

$$(A_{01}A_{11}A_{02}A_{21}A_{12}A_{22}+A_{01}A_{11}A_{02}A_{30}A_{12}A_{22}+A_{01}A_{11}A_{02}A_{30}A_{12}A_{31}+A_{01}A_{20}A_{02}A_{30}A_{12}A_{22}+A_{01}A_{20}A_{02}A_{30}A_{12}A_{31}+$$

$$A_{01}A_{20}A_{02}A_{30}A_{21}A_{31}+A_{10}A_{20}A_{02}A_{30}A_{12}A_{31}+A_{10}A_{20}A_{02}A_{30}A_{21}A_{31}+A_{10}A_{20}A_{02}A_{30}A_{12}A_{22}+A_{10}A_{20}A_{11}A_{30}A_{21}A_{31})$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where μ-translation would lead to negative subscripts for the φ's, i.e., $M > k$, part (a) of the Theorem does not apply.

When k = 6, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right]$$

$$= \mu^{(1,1)}\left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = $$

Example

When k = 6, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2,2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right]$$

$$= \mu^{(1,1)}\left[\frac{A_{12}A_{22}+A_{12}A_{31}+A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11}+A_{01}A_{20}+A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x$$

The lattice paths involved here are the same as for the $k=4$ computation.

We can deduce this by $A_{00} = 1/x_{00}, A_{10} = x_{00}/x_{10}, A_{01} = x_{00}/x_{01}$,

$A_{11} = (x_{10}+x_{01})/x_{11}, A_{20} = x_{10}/x_{20}$, and $A_{21} = (x_{20}+x_{11})/x_{21}$.

Periodicity also kicks in: $\rho_B^7(2,1) = \rho_B^0(2,1) = x_{21}$ using $(r+s+2) = 7$.

Sketch of Proof

By definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \mid\mid \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

$$A \mid\mid B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

Sketch of Proof

By definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \mid\mid \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

$$A \mid\mid B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

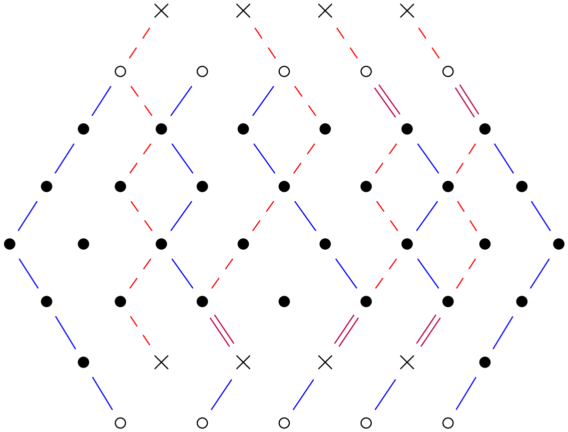
By induction on k, and the fact that we apply birational rowmotion from top to bottom, we can apply algebraic manipulations to reduce our result to proving the following **Plücker-like identity**:

$$\begin{aligned} \varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \\ \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) \\ +\varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1). \end{aligned}$$

It is sufficient to verify the following Plücker-like identity

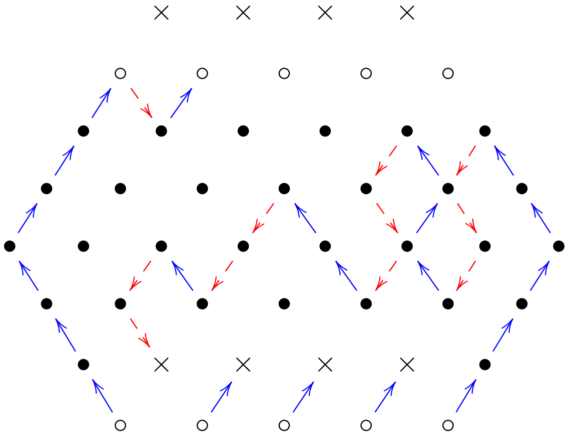
$$\varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1) = \varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k) + \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1).$$

Example (k=5):



We build **bounce paths** and **twigs** (paths of length one from $\circ$ to $\times$) starting from the bottom row of $\circ$'s.

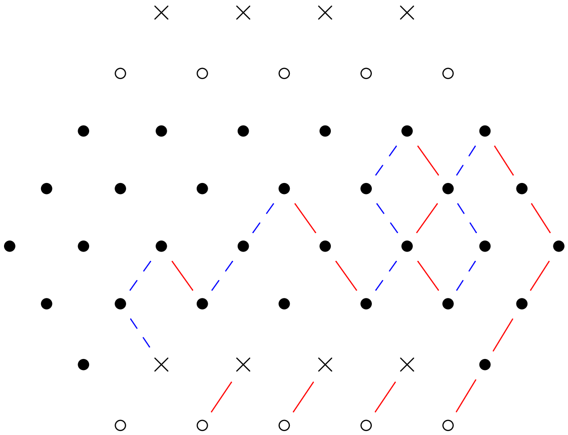
Example (k=5):



Sketch of Proof

We then reverse the colors along the $(k-2)$ **twigs** and the **one bounce path from** $\circ$ **to** $\times$ (rather than $\circ$ to $\circ$).

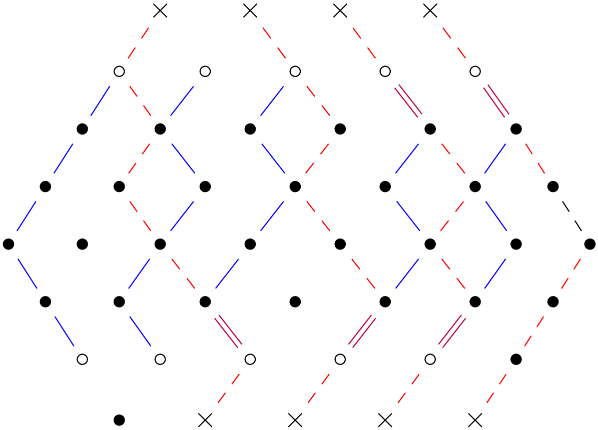
Example (k=5):



Sketch of Proof

Swap in the new colors and shift the $\circ$'s and $\times$'s in the bottom two rows.

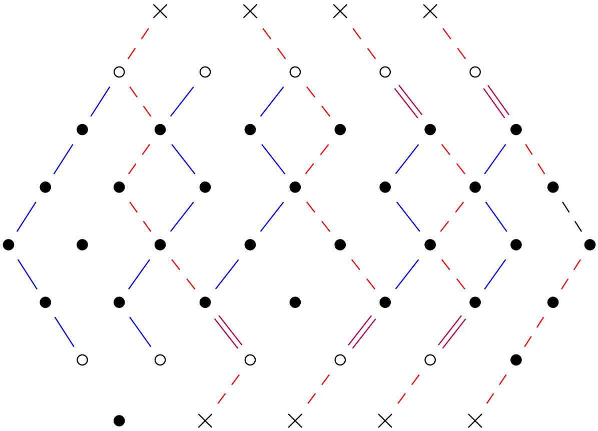
Example (k=5):



Sketch of Proof

$$\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) + \varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).$$

Example (k=5):



Theorem

Given a file F in $[0,r]\times[0,s]$, $\displaystyle\prod_{k=0}^{r+s+1}\prod_{(i,j)\in F}\rho_B^k(i,j)=1.$

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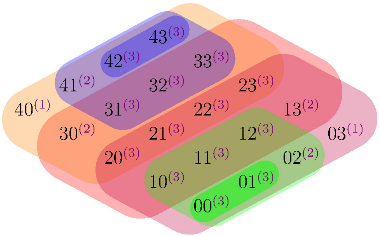
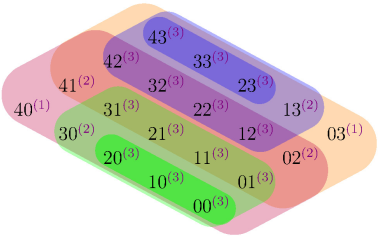
Given a file F in $[0,r]\times[0,s]$, $\displaystyle\prod_{k=0}^{r+s+1}\prod_{(i,j)\in F}\rho_B^k(i,j)=1.$

Sketch of Proof: Double-counting argument, followed by color-coded cancellations and several entries immediately equal to 1, as in ensuing table.





Further Application: Birational File Homomesy

Let $(r,s) = (4,3)$, $d = 2$, and consider the file $F = \{(4,2),(3,1),(2,0)\}$. The following table displays the values of $\rho_B^k(i,j)$ for $0 \le k \le 8$, $(i,j) \in F$.

	$(4,2)$	$(3,1)$	$(2,0)$
$k=0$	$\frac{\varphi_0(4,2)}{\varphi_1(4,2)=1}$	$\frac{\varphi_0(3,1)}{\varphi_1(3,1)}$	$\frac{\varphi_0(2,0)}{\varphi_1(2,0)}$
$k=1$	$\frac{\varphi_1(3,1)}{\varphi_2(3,1)=1}$	$\frac{\varphi_1(2,0)}{\varphi_2(2,0)}$	$\mu^{(1,0)}\left[\frac{\varphi_0(2,0)}{\varphi_1(2,0)}\right]$
$k=2$	$\frac{\varphi_2(2,0)}{\varphi_3(2,0)=1}$	$\mu^{(1,0)}\left[\frac{\varphi_1(2,0)}{\varphi_2(2,0)}\right]$	$\mu^{(2,0)}\left[\frac{\varphi_0(2,0)}{\varphi_1(2,0)}\right] = \frac{1}{x_{23}}$
$k=3$	$\mu^{(1,0)}\left[\frac{\varphi_2(2,0)}{\varphi_3(2,0)=1}\right]$	$\mu^{(2,0)}\left[\frac{\varphi_1(2,0)}{\varphi_2(2,0)}\right]$	$\frac{\varphi_1(2,3)=1}{\varphi_0(2,3)}$
$k=4$	$\mu^{(2,0)}\left[\frac{\varphi_2(2,0)}{\varphi_3(2,0)=1}\right]$	$\mu^{(3,1)}\left[\frac{\varphi_0(3,1)}{\varphi_1(3,1)}\right] = \frac{1}{x_{12}}$	$\frac{\varphi_2(1,2)=1}{\varphi_1(1,2)}$
$k=5$	$\mu^{(3,1)}\left[\frac{\varphi_1(3,1)}{\varphi_2(3,1)=1}\right]$	$\frac{\varphi_1(1,2)}{\varphi_0(1,2)}$	$\frac{\varphi_3(0,1)=1}{\varphi_2(0,1)}$
$k=6$	$\mu^{(4,2)}\left[\frac{\varphi_0(4,2)}{\varphi_1(4,2)=1}\right] = \frac{1}{x_{01}}$	$\frac{\varphi_2(0,1)}{\varphi_1(0,1)}$	$\mu^{(0,1)}\left[\frac{\varphi_3(0,1)=1}{\varphi_2(0,1)}\right]$
$k=7$	$\frac{\varphi_1(0,1)}{\varphi_0(0,1)}$	$\mu^{(0,1)}\left[\frac{\varphi_2(0,1)}{\varphi_1(0,1)}\right]$	$\mu^{(1,2)}\left[\frac{\varphi_2(1,2)=1}{\varphi_1(1,2)}\right]$
$k=8$	$\mu^{(0,1)}\left[\frac{\varphi_1(0,1)}{\varphi_0(0,1)}\right] = x_{42}$	$\mu^{(1,2)}\left[\frac{\varphi_1(1,2)}{\varphi_0(1,2)}\right] = x_{31}$	$\mu^{(2,3)}\left[\frac{\varphi_1(2,3)=1}{\varphi_0(2,3)}\right] = x_{20}$

The final slide of this talk (before the references)

- Combinatorial rowmotion is an well-studied action, that can be written as of involutions called "toggles".
- Generalizing toggling to the piecewise-linear setting, then lifting to the birational setting, gives birational rowmotion. Results (about periodicity or homomesy) at this level imply results at the PL and combinatorial of P
- We give a formula in t icity and for birational rowmotion, and use it to prove pe ational homomesy.
- Open questions includ an we g ralize this to other "nice" posets?

We're happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

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- Combinatorial rowmotion is an well-studied action, that can be written as of involutions called “toggles”.
- Generalizing toggling to the piecewise-linear setting, then lifting to the birational setting, gives birational rowmotion. Results (about periodicity or homomesy) at this level imply results at the PL and combinatorial level.
- We give a formula in terms of NILPs for birational rowmotion, and use it to prove periodicity and birational homomesy.
- Open questions include: Can we generalize this to other “nice” posets?

We’re happy to talk about this further with anyone who’s interested.

Slides for this talk are available online (or will be soon) at

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Thanks very much for coming to this talk!

どうも有り難う御座いました。

References

[BrS74] Andries E. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math. Centr. report ZW24, Amsterdam (Jun. 1974), `http://www.win.tue.nl/~aeb/preprints/zw24.pdf`.

[CaFl95] Peter J. Cameron and Dmitry G. Fon-der-Flaass, *Orbits of Antichains Revisited*, Europ. J. of Combin., **16(6)**, (1995), 545–554, `http://www.sciencedirect.com/science/article/pii/0195669895900365`.

[EiPr13] David Einstein and James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains* (2013), `arXiv:1310.5294`.

[EiPr14] David Einstein and James Propp, *Piecewise-linear and birational toggling (Extended abstract)*, DMTCS proc. FPSAC 2014, `http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAT0145/4518`.

[Flaa93] Dmitry G. Fon-der-Flaass, *Orbits of Antichains in Ranked Posets*, Europ. J. Combin., **14(1)**, (1993), 17–22, `http://www.sciencedirect.com/science/article/pii/S0195669883710036`.

[GaPy17] Pavel Galashin and Pavlo Pylyavskyy, *R-systems*, `arXiv:1709.00578v1`.

[GrRo16] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. of Combin. **23**(1), #P1.33 (2016). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i1p33

[GrRo15] Darij Grinberg, Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. of Combin. **22**(3), #P3.40 (2015). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p40.

[KiBe95] Anatol N. Kirillov and Arkadiy D. Berenstein, *Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz, **7**, #1 (1995), 92–152.

[AIM15] James Propp, Tom Roby, Jessica Striker, and Nathan Williams (organizers), Sam Hopkins (notetaker), *Notes from the AIM workshop on dynamical algebraic combinatorics*, American Institute of Math., San Jose, CA, 23–27 March 2015, http://aimath.org/pastworkshops/dynalgcomb.html, http://mit.edu/~shopkins/docs/aim_dyn_alg_comb_notes.pdf.

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22(3)** (2015), #P3.4,

Thanks for Listening http://math.umn.edu/∼musiker/Birational18.pdf

[Rob16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in Andrew Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

[Sage17] William A. Stein et. al., *Sage Mathematics Software (Version 8.1)*, The Sage Development Team (2017), http://www.sagemath.org.

[Stan86] Richard P. Stanley, *Two Poset Polytopes*, Discrete & Computational Geometry (1986), Volume 1, Issue 1, 9–23.

[StWi11] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942, http://www.sciencedirect.com/science/article/pii/S0195669812000972.

[TW17] HughThomas and Nathan Williams, *Rowmotion in Slow Motion*, arXiv:1712.10123.

[Yil17] Emine Yıldırım, *Coxeter transformation on Cominuscule Posets*, arXiv:1710.10632.

Gregg Musiker and Tom Roby, *Paths to understanding birational rowmotion on products of two chains* (2018), arXiv:1801.03877, to appear in *Algebraic Combinatorics*.