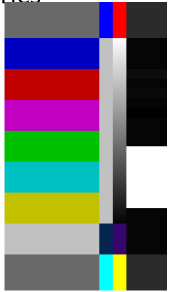
# Homomesies Lurking in the Twelvefold Way

Tom Roby (UConn)

*Describing joint research with*
*Michael Joseph & Michael LaCroix*

Special Session on Enumerative Combinatorics
AMS Central Sectional Meeting
University of St. Thomas Minneapolis, MN



28 October 2016 (Friday)

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**Abstract:** Given a group acting on a finite set of combinatorial objects, one can often find natural statistics on these objects which are *homomesic*, i.e., over each orbit of the action, the average value of the statistic is the same. Since the notion was codified a few years ago, homomesic statistics have been uncovered in a wide variety of situations within dynamical algebraic combinatorics. We discuss several examples lurking in Rota's *Twelvefold Way* related to actions on injections, surjections (joint work with Michael Joseph), and bijections/permutations (joint work with Michael LaCroix) of finite sets.

## Acknowledgments

This seminar talk discusses joint work with Michael Joseph and Michael La Croix. Thanks to James Propp for suggesting the study of whirling and of Foatic actions, as well as earlier collaborations on the homomesy phenomenon.

Please feel free to interrupt with questions or comments.

# Outline

- Actions, orbits, and homomesy;
- The Twelvefold Way;
- Foatic actions on $\mathfrak{S}_n$.
- Whirling injections and surjections;

## Subset Rotation

Among the most basic objects counted by the Twelvefold Way are subsets, counted by binomial coefficients.

Set $[n] := \{1, 2, \ldots n\}$.
$\binom{[n]}{k} = \{k\text{-element subsets of } [n]\}$.
For example, $\binom{[7]}{3}$ consists of 35 subsets (dropping braces & commas): 123, 124, 125, 126, 127, 134, 135, 136, 137, 145, 146, 147, 156, 157, 167, 234,. . .

The map that adds 1 to each element mod $n$ acts on $\binom{[n]}{k}$.

$$156 \mapsto 267 \mapsto 137 \mapsto 124 \mapsto 235 \mapsto 346 \mapsto 457 \mapsto 156$$

It's easy to see that the cardinality of each orbit of this action is a divisor of $n$.

## Rotation of bit-strings

We find another unexpected structure in the orbits if we consider $S = \binom{[n]}{k}$, as length $n$ binary strings with $k$ 1's as we cyclically shift them. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$, and count $f(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

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| $0011 \mapsto 0$ | |
| $\text{AVG} = \frac{8}{4} = 2$ | |

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We know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate $b$ (replacing a 01 somewhere in $b$ by 10 or vice versa), or one can write the number of inversions in $b$ as $\sum_{i<j} b_i(1 - b_j)$ and then perform algebraic manipulations.

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Note that in each of the three orbits average of the statistic $f$ is the same.

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## Main Definition: Homomesic statistics

### Definition ([PrRo15])

Given an (invertible) action $\tau$ on a finite set of objects $S$, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to $(S, \tau)$ iff the average of $f$ over each $\tau$-orbit $\mathcal{O}$ is the same constant $c$ for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.
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(Call $f$ $c$-mesic for short.)

Equivalently: the average of $f$ over each $\tau$-orbit $\mathcal{O}$ is the same as the average over the entire set $S$:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = \frac{1}{\#S} \sum_{s \in S} f(s).$$

So we can compute what the average should be before checking whether a statistic is homomesic.

Since its initial codification about 5 years ago, a large number of examples of the homomesy phenomenon have been identified across dynamical algebraic combinatorics. These include:

- Promotion of SSYT; Rowmotion of “nice” (e.g., minuscule heap) posets [PrRo15, StWi11, RuWa15+] ;

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- Liftings of homomesy from combinatorial actions to piecewise linear and birational maps [EiPr13, GrRo16, GrRo15b].
- There are many others.

# Rota's Twelvefold Way

The Twelvefold Way is a framework for basic combinatorial counting problems, concevied by G.-C. Rota. It counts $\#\{f : [n] \to [x]\}$, where objects may be considered distinct or indistinct, and the functions arbitrary, injective, or surjective. Quoting *EC1ed2*:[Stan11, § 1.9]

We are now ready to present the Twelvefold Way. The twelve entries are numbered and will be discussed individually. The table gives the number of inequivalent functions $f : N \to X$ of the appropriate type, where $\#N = n$ and $\#X = x$.

**The Twelvefold Way**

| **Elements of *N*** | **Elements of *X*** | **Any *f*** | **Injective *f*** | **Surjective *f*** |
|---|---|---|---|---|
| dist. | dist. | 1. $x^n$ | 2. $(x)_n$ | 3. $x!S(n,x)$ |
| indist. | dist. | 4. $\left(\binom{x}{n}\right)$ | 5. $\binom{x}{n}$ | 6. $\left(\binom{x}{n-x}\right)$ |
| dist. | indist. | 7. $S(n,0)+S(n,1)$ $+\cdots+S(n,x)$ | 8. $1$ if $n \le x$; $0$ if $n > x$ | 9. $S(n,x)$ |
| indist. | indist. | 10. $p_0(n)+p_1(n)$ $+\cdots+p_x(n)$ | 11. $1$ if $n \le x$; $0$ if $n > x$ | 12. $p_x(n)$ |

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In the rest of this talk we will consider actions on some of the basic things counted: permutations, injections, and surjections.

# The Rényi-Foata (Drop Parentheses) Map

## Definition

For $w \in \mathfrak{S}_n$, its **canonical (disjoint) cycle decomposition (CCD)** satifies:
(a) each cycle is written with its largest element first; and
(b) the cycles are written in increasing order of largest element.
The map $\mathcal{F} : \mathfrak{S}_n \to \mathfrak{S}_n$ simply removes the parentheses from the CCD of $w$ and regards the resulting word as a permutation in one-line notation.

$$w = 847296513 = (42)(6)(81)(9375) \overset{\mathcal{F}}{\mapsto} 426819375 = (2)(951487369),$$

Note that here $w$ has 4 cycles, and $\mathcal{F}(w)$ has 4 **records** (i.e., left-to-right maxima) viz., 4, 6, 8, and 9.

It is easy to see that $\mathcal{F}$ is a bijection.

# Foatic Actions of $\mathfrak{S}_n$

We consider actions $\mathfrak{S}_n$ of the following form:

$$\mathfrak{S}_n \xrightarrow{\mathcal{F}} \mathfrak{S}_n \xrightarrow{\mathcal{A}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{B}} \mathfrak{S}_n$$

where $\mathcal{A}$ and $\mathcal{B}$ are **dihedral involutions**, defined below.

- (a) $\mathcal{C} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \dots w_n$ to its **complement** whose value in position $i$ is $n + 1 - w_i$;
- (b) $\mathcal{R} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \dots w_n$ to its **reversal** whose value in position $i$ is $w_{n+1-i}$;
- (c) $\mathcal{Q}^2 : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \dots w_n$ to its **rotation by 180-degrees**, whose value in position $i$ is $n + 1 - w_{n+1-i}$.
- (d) $\mathcal{I} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w$ to its **inverse** $w^{-1}$;
- (e) $\mathcal{D} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w$ to its its **rotated-inverse** $\mathcal{Q}^2(\mathcal{I}(w))$.

We call such fourfold compositions, where $\mathcal{A}$ and $\mathcal{B}$ are from the above list, **Foatic**.

# Foatic example

$\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{I}$ gives the Foatic map $\gamma := \mathcal{I} \circ \mathcal{F}^{-1} \circ \mathcal{C} \circ \mathcal{F}$.
If $n = 5$, then $\gamma[(4213)(5)] = (2)(4)(513)$ as follows

$$(4213)(5) \overset{\mathcal{F}}{\mapsto} 42135 \overset{\mathcal{C}}{\mapsto} 24531 \overset{\mathcal{F}^{-1}}{\mapsto} (2)(4)(531) \overset{\mathcal{I}}{\mapsto} (2)(4)(513)$$

The orbit (of size six) generated by (4213)(5) is

$$(4213)(5) \overset{\gamma}{\mapsto} (2)(4)(513) \overset{\gamma}{\mapsto} (412)(53) \overset{\gamma}{\mapsto} (2)(5314) \overset{\gamma}{\mapsto} (431)(52) \overset{\gamma}{\mapsto} (2)(3)(541)$$

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**Conjecture**

*The statistic* $\mathrm{Fix}$ *is homomesic with respect to the Foatic complement-inversion action on* $\mathfrak{S}_n$.

# Reversal-Inversion

The Foatic action with the nicest orbit structures and properties appears to be *Reversal-Inversion*:

$$\overline{\varphi} : \mathfrak{S}_n \xrightarrow{\mathcal{F}} \mathfrak{S}_n \xrightarrow{\mathcal{R}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{I}} \mathfrak{S}_n.$$

$$\varphi : \mathfrak{S}_n \xrightarrow{\mathcal{R}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{I}} \mathfrak{S}_n. \xrightarrow{\mathcal{F}} \mathfrak{S}_n$$

$$\begin{aligned}
w = &(2)(43)(51) \mapsto 24351 \mapsto 15342 \mapsto (1)(5342) \mapsto (1)(5243) &= \overline{\varphi}(w)\\
&(1)(5243) \mapsto 15243 \mapsto 34251 \mapsto (3)(42)(51) \mapsto (3)(42)(51) &= \overline{\varphi}^2(w)\\
&(3)(42)(51) \mapsto 34251 \mapsto 15243 \mapsto (1)(5243) \mapsto (1)(5342) &= \overline{\varphi}^3(w)\\
&(1)(5342) \mapsto 15342 \mapsto 24351 \mapsto (2)(43)(51) \mapsto (2)(43)(51) &= \overline{\varphi}^4(w)
\end{aligned}$$

This example also displays (down the second column) the conjugate orbit of $\varphi$, also of size 4.

$$24351 \xrightarrow{\varphi} 15243 \xrightarrow{\varphi} 34251 \xrightarrow{\varphi} 15342 \Lsh$$

# Data on Orbit Sizes for Foatic Reversal-Inversion

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| # of orbits: | 1 | 1 | 2 | 5 | 19 | 84 | 448 | 2884 | 21196 | 17416 |
| LCM of orbit sizes: | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| GCD of orbit sizes: | 1 | 2 | 2 | 4 | 4 | 4 | 4 | 8 | 8 | 8 |
| Longest orbit size: | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| Shortest orbit size: | 1 | 2 | 2 | 4 | 4 | 4 | 4 | 8 | 8 | 8 |
| Size of id's orbit: | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |

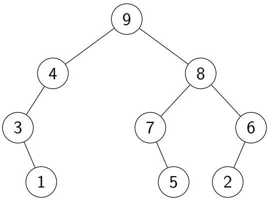
# Heap representation of a permutation

## Definition

Recursively define the **heap** of $w \in \mathfrak{S}_n$, $H(w)$ as follows: Set $H(\emptyset) = \emptyset$ (the empty word). If $w \neq \emptyset$, let $m$ be the largest element of $w$, so $w$ can be written uniquely as $umv$, where $u$ and $v$ are partial permutations (possibly empty). Set $m$ to be the root of $H(w)$, with $H(u)$ its left subtree and $H(v)$ its right subtree.

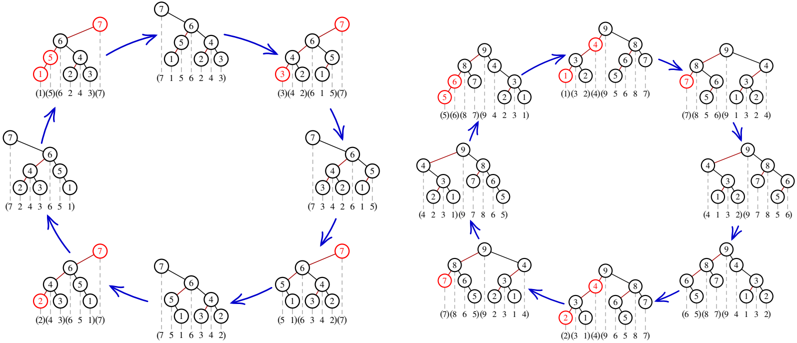
The heap of a permutation will turn out to be a *decreasing binary tree*, (labels decrease along any path from root).

The heap associated with $w = 314\mathbf{9}75826$ is shown below.



# Orbits represented as heaps

Two orbits (one for $\mathfrak{S}_7$, one for $\mathfrak{S}_9$) of the Foatic reversal-inversion map $\overline{\varphi}$ with associated heaps, with fixed points marked in red. Each orbit has an average of one fixed point per permutation.



# Reversal-Inversion Results

## Theorem (La Croix-R.)

*The orbits of the action of $\overline{\varphi}$ (or $\varphi$) on $\mathfrak{S}_n$, satisfy the following:*

1. *The size of a $\varphi$-orbit $\mathcal{O}$ (equivalently $\overline{\varphi}$-orbit) is $2^h$, where $h$ is the number of edges in a maximal path from the root (to a leaf) for any $w \in \mathcal{O}$.*
2. *Let $\mathrm{Fix}\, w$ denote the number of fixed points, i.e., 1-cycles, of $w$. Then the statistic $\mathrm{Fix}$ is 1-mesic with respect to the action of $\overline{\varphi}$; (Equivalently, $\mathrm{Rasc} =$#record-ascents is 1-mesic with respect action of $\varphi$.)*
3. *For fixed $i \neq j$ in $[n]$, let $\mathbb{1}_{i<j}(u)$ denote the indicator statistic of whether $i$ occurs to the left of $j$ in the one-line notation of $u$. Then $\mathbb{1}_{i<j}$ is $\frac{1}{2}$-mesic with respect to the action of $\varphi$.*
4. *Similarly for fixed $i \in [n]$, let $\mathbb{1}_{(i,n)}$ denote the indicator statistic of whether $i$ and $n$ lie in the same cycle of $w$. Then $\mathbb{1}_{(i,n)}$ is $\frac{1}{2}$-mesic with respect to the action of $\overline{\varphi}$.*

All the results listed above follow without difficulty from the following key lemma.

**Lemma**

*Let $w \in \mathfrak{S}_n$ have the form $AnB$ (in one-line notation), where $A$ and $B$ are (possibly empty) partial permutations of $n$. Then the action of $\varphi$ satisfies $\varphi(AnB) = \varphi(B)nA$. Thus, $H(\varphi(AnB))$ is the heap interchanging the left and right subtrees at $v$, leaving the former unchanged and applying $\varphi$ recursively to the latter. In particular, the action of $\varphi$ preserves the underlying unlabeled graph of the corresponding heaps.*

## Whirling action on injections/surjections

We write functions $f \in [k]^{[n]}$ in one-line notation $f(1)f(2)\cdots f(n)$.

**Definition**

Let $\mathcal{J}$ denote either $\mathrm{Inj}(n,k)$ or $\mathrm{Sur}(n,k)$ for a given $n, k \in \mathbb{P}$. Define a map $\mathrm{wh}_i : \mathcal{J} \to \mathcal{J}$, called **whirling at index** $i$ in the following way. Given $f \in \mathcal{J}$, repeatedly add 1 (mod $k$) to the value of $f(i)$ until we get a function in $\mathcal{J}$. The new function is $\mathrm{wh}_i(f)$.

EG: $f = 124 \in \mathrm{Inj}(3,6) \implies \mathrm{wh}_1(124) = 324$, $\mathrm{wh}_2(124) = 134$, and $\mathrm{wh}_3(124) = 125$.

These generalize *toggle operations*, which are involutions. The composition $\mathrm{wh} := \mathrm{wh}_n \, \mathrm{wh}_{n-1} \cdots \mathrm{wh}_2 \, \mathrm{wh}_1$ is called **whirling**.

EG: $\mathrm{wh}(124) = (324 \cdots 354 \cdots 356) = 356$. 124 generates the whirling orbit

$$124 \mapsto 356 \mapsto 412 \mapsto 534 \mapsto 651 \mapsto 263 \mapsto 415 \mapsto 621 \mapsto 342 \mapsto 563$$

# Whirling surjections

EG: Let $v = 21444323 \in \mathrm{Sur}(8,4)$. Then $\mathrm{wh}_1(v) = 31444323$, while $\mathrm{wh}_2(v) = v$. The orbit generated by $v$ is:

$21444323 \mapsto 31114424 \mapsto 32211134 \mapsto 43222141 \mapsto 13332242 \mapsto 14433312$ ↰

**Theorem (Joseph-R.)**

*Fix $\mathcal{J}$ to be either $\mathrm{Inj}(n,k)$ or $\mathrm{Sur}(n,k)$ for a given $n, k \in \mathbb{P}$. For $i \in [k]$, define $\eta_i(f) = \#f^{-1}(\{i\})$ to be the number of times $i$ appears as an output of the function $f$. Then $\eta_i$ is $\frac{n}{k}$-mesic for any $i \in [k]$.*

Equivalently, $\eta_i - \eta_j$ is 0-mesic for any $i, j \in [k]$, i.e., $i$ and $j$ appear as outputs of functions the same number of times across any orbit.

# Proving homomesy for injections

Key Idea: Partition the orbit into $[k]$-chunks. If a value $j$ appears in $i$th spot, then $j+1 \bmod k$ must occur directly below, unless it was already in the row when $w_i$ was applied. Thus, the next $j+1$ occurs no later than the $n$th letter after $j$. Color these the same.

**124**
**356**
**412**
**534**
**651**
**263**
**415**
**621**
**342**
**563**

It's easy to see this relation goes backwards as well as forwards, so partitions the orbit into chunks each containing all of $[k]$. (The chunks can wrap around from bottom to top.)

# Proving homomesy for surjections

This uses a somewhat different partitioning argument. Since $v_i$ and $\mathrm{wh}(v)_i$ either agree or differ by one, we could just partition into vertical chunks, except when $v_i = \mathrm{wh}(v)_i$ (i.e., same values on top of one another). So it suffices to show that the number of pairs of a $j$ directly below another $j$ is the same for all $j \in [k]$. The tops of such pairs are circled below in red.

Finally, one shows that every circled $j$ is followed within the next $n$ slots by a circled $j+1$, allowing these to be partitioned as well.

$$\begin{array}{cccccccc}
2 & 1 & 4 & 4 & 4 & 3 & 2 & 3 \\
3 & 1 & 1 & 1 & 4 & 4 & 2 & 4 \\
3 & 2 & 2 & 1 & 1 & 1 & 3 & 4 \\
4 & 3 & 2 & 2 & 2 & 1 & 4 & 1 \\
1 & 3 & 3 & 3 & 2 & 2 & 4 & 2 \\
1 & 4 & 4 & 3 & 3 & 3 & 1 & 2
\end{array}$$

# A consequence of homomesy for orbits

From this homomesy we can deduce information about orbit sizes (that we currently don't know by any other means).

Let $\ell(\mathcal{O})$ be the length of the orbit $\mathcal{O}$.

If we consider surjective functions from [7] to [4], then across every orbit , the numbers 1, 2, 3, 4 all appear as outputs the same number of times; hence, $4 \mid 7\ell(\mathcal{O}) \implies 4 \mid \ell(\mathcal{O})$.

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On the other hand, if we consider surjective functions from [8] to [4], then across every orbit, $4 \mid 8\ell(\mathcal{O})$, which gives no new information.

A function $f : S \to T$ between two sets $S$ and $T$ is $m$-**injective** if $\#f^{-1}(t) \leq m$ for every $t \in T$ and $m$-**surjective** if $\#f^{-1}(t) \geq m$ for every $t \in T$. Let $\mathrm{Inj}_m(n,k)$ and $\mathrm{Sur}_m(n,k)$ denote the set of $m$-injective (resp. $m$-surjective) functions from $[n]$ to $[k]$.

**Conjecture (Joseph)**

*Fix $\mathcal{J}$ to be either $\mathrm{Inj}_m(n,k)$ or $\mathrm{Sur}_m(n,k)$ for fixed $n,k,m \in \mathbb{P}$. For $i \in [k]$, define $\eta_i(f) = \#f^{-1}(\{i\})$ to be the number of times $i$ appears as an output of the function $f$. Then $\eta_i$ is $\frac{n}{k}$-mesic for any $i \in [k]$.*

## The final slide of this talk (before the references)

We're happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

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Thanks very much for coming to this talk!

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