Dynamical Algebraic Combinatorics and the Homomesy Phenomenon

Tom Roby

Abstract We survey recent work within the area of algebraic combinatorics that has the flavor of discrete dynamical systems, with a particular focus on the *homomesy phenomenon* codified in 2013 by James Propp and the author. In these situations, a group action on a set of combinatorial objects partitions them into orbits, and we search for statistics that are *homomesic*, i.e., have the same average value over each orbit. We give a number of examples, many very explicit, to illustrate the wide range of the phenomenon and its connections to other parts of combinatorics. In particular, we look at several actions that can be defined as a product of *toggles*, involutions on the set that make only local changes. This allows us to lift the well-known poset maps of rowmotion and promotion to the piecewise-linear and birational settings, where periodicity becomes much harder to prove, and homomesy continues to hold. Some of the examples have strong connections with the representation theory of semisimple Lie algebras, and others to cluster algebras via Y-systems.

Key words: antichains, cluster algebras, homomesy, minuscule heaps, non-crossing partitions, orbits, order ideals, Panyushev complementation, permutations, poset, product of chains, promotion, root systems, rowmotion, Suter's symmetry, toggle groups, tropicalization, Y-systems, Young's Lattice, Young tableaux, Zamolodchikov periodicity.

Mathematics Subject Classifications: 05E18, 06A11.

1 Introduction

The term *Dynamical Algebraic Combinatorics* is meant to convey a range of phenomena involving actions on sets of discrete combinatorial objects, many of which can be built up by small local changes. Schützenberger's operations of promotion and evacuation on Young tableaux are well-known classic examples [Sch72, Gans80, KiBe95], but there are many others. Questions concerning periodicity and orbit structure naturally arise in this setting, and frequently there is a surprising interplay between algebraic and bijective methods of discovery and proof. Connections with representation theory arise even when the original motivations for studying a certain action were purely combinatorial.

The term **homomesy** (Greek for "same middle") was coined by James Propp and the author to describe the following situation. Given a group action on a set of combinatorial objects, a statistic on these objects is called *homomesic* if its average value is the same over all orbits. There are many examples of this phenomenon, spread over large swaths of combinatorics, and of varying degrees of difficulty.

This survey discusses the work of a number of authors, attempting to convey the range of situations in which homomesy occurs. The field is rapidly evolving at present and has already grown too large for this paper to contain all the generalizations and extensions that might appropriately have been included. Many concrete examples and diagrams are provided to aid the reader in quickly grasping the results and key ideas.

Tom Roby
Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, e-mail: tom.roby@uconn.edu

1.1 Acknowledgments

This exposition discusses recent work of a number of different authors. The author is grateful to James Propp for the initial ideas that led to the formalization of homomesy and for continuing conversations as it has developed. Collaborations with Propp, Darij Grinberg, Mike LaCroix, Roger Behrend, and Gregg Musiker have been great extended-learning opportunities.

The American Institute of Mathematics hosted an NSF-sponsored workshop on this subject in March 2015 (http://aimath.org/pastworkshops/dynalgcomb.html), which greatly facilitated the author's understanding of the scope of this area. Thanks go to the extremely efficient AIM staff, the organizers (James Propp, Jessica Striker, Nathan Williams and the author), and the enthusiastic group of participants. The notes taken by Sam Hopkins [AIM15] were particularly helpful. A number of active projects were presented or jump-started at this workshop, and some of the results are mentioned here. Dynamical algebraic combinatorics and homomesy are currently relatively new and active research areas, so much of what is discussed here is (at the time of this writing, June 2015) in arXiv preprints or still "in preparation".

Mike LaCroix wrote fantastic postscript code to generate animations and pictures, which are available on my website http://www.math.uconn.edu/~troby/research.html and in earlier publications. Code contributed to Sage [Sage14, Sage08] by various authors has been crucial for understanding many examples.

Several people carefully read draft versions and made numerous helpful comments, particularly Karen Edwards, Darij Grinberg, Michael Joseph, Gregg Musiker, James Propp, Victor Reiner, and Elizabeth Sheridan-Rossi. Undoubtedly errors remain, for which the author takes full responsibility.

For productive conversations and valuable insights, thanks go to Drew Armstrong, Anders Björner, Barry Cipra, Keith Conrad, Karen Edwards, Robert Edwards, Miriam Farber, Sam Hopkins, Michael Joseph, Svante Linusson, Gregg Musiker, Alexander Postnikov, Pavlo Pylyavskyy, Vic Reiner, Ralf Schiffler, Richard Stanley, Dennis Stanton, Jessica Striker, Hugh Thomas, Nathan Williams, Pete Winkler, and Ben Young.

2 Homomesy

The notion of homomesy was originally motivated by the work of James Propp and others on chip-firing and roter-routing [H+08, HP10]. Another early example in the context of an action (rowmotion) on antichains in posets was the conjecture (Theorem 2) of Dmitri Panyushev [Pan09], later proved by Drew Armstrong, Christian Stump, and Hugh Thomas [AST11]. James Propp and the author [PR15] first codified the notion and coined the term, giving a wide-ranging collection of examples and studying in detail a situation similar to that of Panyushev. The initial discussion here and choice of examples unavoidably has some overlap with [PR15], although different numerical examples are given where possible.

Definition 1. Given a set $\mathscr{S}$, an invertible map τ from $\mathscr{S}$ to itself such that each τ-orbit is finite, and a function (or "statistic") $f : \mathscr{S} \to \mathbb{K}$ taking values in some field $\mathbb{K}$ of characteristic zero, we say the triple $(\mathscr{S}, \tau, f)$ exhibits **homomesy** iff there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathscr{O} \subset \mathscr{S}$

$$\frac{1}{\#\mathscr{O}} \sum_{x \in \mathscr{O}} f(x) = c. \tag{1}$$

In this situation we say that the function $f : \mathscr{S} \to \mathbb{K}$ is **homomesic** under the (cyclic) action of τ on $\mathscr{S}$, or more specifically ***c*-mesic**.

When $\mathscr{S}$ is a finite set, homomesy can be restated equivalently as all orbit-averages being equal to the global average:

$$\frac{1}{\#\mathscr{O}} \sum_{x \in \mathscr{O}} f(x) = \frac{1}{\#\mathscr{S}} \sum_{x \in \mathscr{S}} f(x). \tag{2}$$

In upcoming sections we will also see how to relax the definition of homomesy to include actions of general (not merely cyclic) groups (Section 2.1), or even non-invertible actions (Section 2.3).

Example 1. **Number of inversions under cyclic rotation of binary strings** Let $\mathscr{S} = \binom{[n]}{k}$ thought of as length n binary strings with exactly k 1's. Define $\tau := C_R : \mathscr{S} \to \mathscr{S}$ by $s = s_1 s_2 \cdots s_n \mapsto s_n s_1 s_2 \cdots s_{n-1}$ (rightward cyclic shift). Let $f(s) := \mathrm{inv}(s) := \#\mathrm{inversions}(s) := \#\{i < j : s_i > s_j\}$ (the usual inversion statistic on multiset permutations).Then it is straightforward to show (though not immediately obvious) that with respect to this action, the statistic inv is $\frac{k(n-k)}{2}$-mesic [PR15, § 2.3]. For example, if $n = 4$ and $k = 2$, we get two orbits: $(0011, 1001, 1100, 0110)$ with inversion statistic $(0, 2, 4, 2)$ and $(0101, 1010)$, with inversion statistic $(1, 3)$. So over each orbit, the average is $2 = \frac{2(4-2)}{2}$. Similarly, the reader can easily check that among the $\binom{6}{2} = 15$ bitstrings of length 6 with two 1's, this cyclic action gives three orbits with sizes 6, 6, and 3, each with average number of inversions equal to 4.

One important feature of homomesy is that we get constant averages over orbits even though the orbits in general may be of different lengths. It is sometimes useful to think of aggregating a shorter orbit multiple times into a *superorbit* whose length is the order of the action on the entire set (i.e., the size of the cyclic group it generates). This does not affect homomesy, since the average size of a statistic over such a superorbit will be the same as the average over the actual (shorter) orbit. For example, for purposes of homomesy we could replace the orbit $(0101, 1010)$ above with $(0101, 1010, 0101, 1010)$.

Example 2. **Indicator functions of coordinates under cyclic rotation of binary strings** There are other statistics that are more obviously homomesic with respect to the above action. For example, let $\mathbb{1}_i(s) := s_i$ denote the indicator function of the bitstring at position i. Then for any i, $\mathbb{1}_i$ is $\frac{k}{n}$-mesic. This is simply because each value in the string cycles around to each position exactly once over n iterations of the action; thus, in each superorbit, location i is occupied by a 1 k times.

It is instructive to compare the above examples with the "protoexample" given in [RSW14]. See Section 3.1 for more on this example and the relation between homomesy and cyclic sieving.

Example 3. **Sums of centrally symmetric entries under promotion of SSYT** For a fixed Young diagram λ, let $\mathrm{SSYT}_k(\lambda)$ denote the set of **semistandard Young tableaux** of shape λ and ceiling k, i.e., fillings of the cells of λ with elements of $[k] := \{1, 2, \ldots, k\}$ that are *weakly* increasing in each row and *strictly* increasing in each column. These objects and their variants are used to describe and count various things of interest in representation theory and algebraic geometry, such as irreducible representations of GL_n and cohomology classes of flag varieties. (See, e.g., [Sta99, § 7.10] and [Ful97] for more information about these objects and their relationship to symmetric functions.) Schützenberger defined an interesting operation on $\mathrm{SSYT}_k(\lambda)$, which can be generalized to an action on the set of all linear extensions of any finite poset [Sta09, BPS13, and the references therein]. Readers unfamiliar with promotion will find a self-contained definition in terms of simpler operations at the beginning of Section 3.2.

In the particular case where $\lambda = (n^m) := (n, n, \ldots, n)$ is a rectangular shape with m parts, all equal to n, the Schützenberger promotion operator π satisfies $\pi^k = \mathrm{id}$ [Rho10, Cor. 5.6]. (Simpler proofs are available for *standard Young tableaux*; see e.g., [Sta09, Thm. 4.1(a)].) Now fix any subset R of the cells of (n^m), and for $T \in \mathrm{SSYT}_k(n^m)$ set $\sigma_R(T)$ to be the sum of the entries of T whose cells lie in R.

Theorem 1 (Bloom-Pechenik-Saracino). *Let k be a positive integer and suppose that $R \subseteq (n^m)$ is symmetric with respect to 180-degree rotation about the center of (n^m). Then the statistic σ_R is c-mesic with respect to the action of promotion on $\mathrm{SSYT}_k(n^m)$, with $c = |R|\left(\frac{k+1}{2}\right)$.*

For example, consider the following promotion orbit within $\mathrm{SSYT}_5(3^2)$ (where our tableaux are here drawn "English" style, using matrix coordinates, and the symbol ↰ points back to the start of the current orbit):

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \hline \end{array} \overset{\pi}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 5 & 5 \\ \hline \end{array} \overset{\pi}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 4 & 5 & 5 \\ \hline \end{array} \overset{\pi}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array} \overset{\pi}{\mapsto} \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 3 & 4 & 5 \\ \hline \end{array} \overset{\pi}{\Lsh}$$

Then the sum of the values in the upper left and lower right cells (shown in red) across the orbit is $(5,6,6,6,7)$, which averages to $6 = 2\left(\frac{5+1}{2}\right)$. Similarly, the sum of the blue entries in the lower left and upper right corners across the orbit is $(4,5,8,7,6)$, with average 6, and the sum of the black entries in the middles of the two rows across the orbit is $(4,6,7,7,6)$, with average 6.

This result was stated as a conjecture in several talks given by the authors, and was proved by Jonathan Bloom, Oliver Pechenik, and Dan Saracino [BPS13]. The latter also proved a version of the result for *cominuscule posets*. For the action of *K-promotion* on *increasing tableaux* of rectangular shapes, they prove an analogous result for two-rowed shapes, and show that it fails in general when λ is a rectangle with more than two rows.

2.1 General group actions

Although the original definition called for the action to be given by an invertible map τ, which is equivalent to the action of a *cyclic* group, the definition of homomesy makes perfect sense if one considers the action of *any* finite group, cyclic or not. There are examples of homomesies under the action of non-cyclic groups G, and homomesy can always be "lifted" from a cyclic subgroup of G to all of G.

Example 4. **Number of inversions under 90-degree rotation of permutation matrices** Let S_n denote the set of permutation matrices, and $\mathcal{Q}$ the ("quarter turn") map which rotates a matrix W by ninety degrees clockwise. Set $\operatorname{inv} W :=$ the number of inversions of the corresponding permutation, which can also be written as

$$\operatorname{inv} W = \sum_{\substack{1 \le i < i' \le n \\ 1 \le j' \le j \le n}} W_{ij} W_{i'j'}.$$

Then inv is homomesic with respect to this action, with average $\frac{n(n-1)}{4}$. For example, when $n = 3$ we get two orbits;

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) \text{ and } \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

with respective inversion numbers $(1,2,1,2)$ and $(0,3)$.

The proof of homomesy is easy: $\mathcal{Q}$ takes inversions to non-inversions, and vice-versa. A more interesting homomesy with respect to the action of $\mathcal{Q}$ on the full set of alternating sign matrices (generalizing the above result) is given in [BR15+].

The question then naturally arises: What happens when one considers the full dihedral group acting on the set of matrices? Do we still get homomesy? The answer is yes, and it is straightforward to show this directly. But we also have the following general lemma that shows the relationship between homomesy for the action of a group G and for the action of H a subgroup of G.

Lemma 1. *Let G be a group acting on the set S, and let H be a subgroup of G. If the triple (S,H,f) exhibits homomesy, then so does the triple (S,G,f).*

The proof is simple: aggregating together orbits where a statistic has the same average always gives a larger orbit with that property.

2.2 Suter's action on Young diagrams

Recall the poset *Young's Lattice*, whose elements are all integer partitions (thought of as Young diagrams), ordered by inclusion of diagrams [Sta99, § 7.2]. In [Su02], Ruedi Suter described an action of the dihedral group D_n ($n \ge 1$) on a

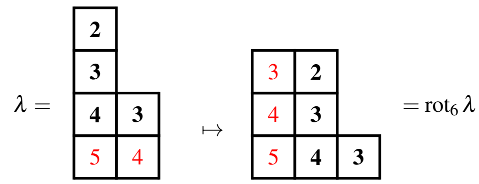


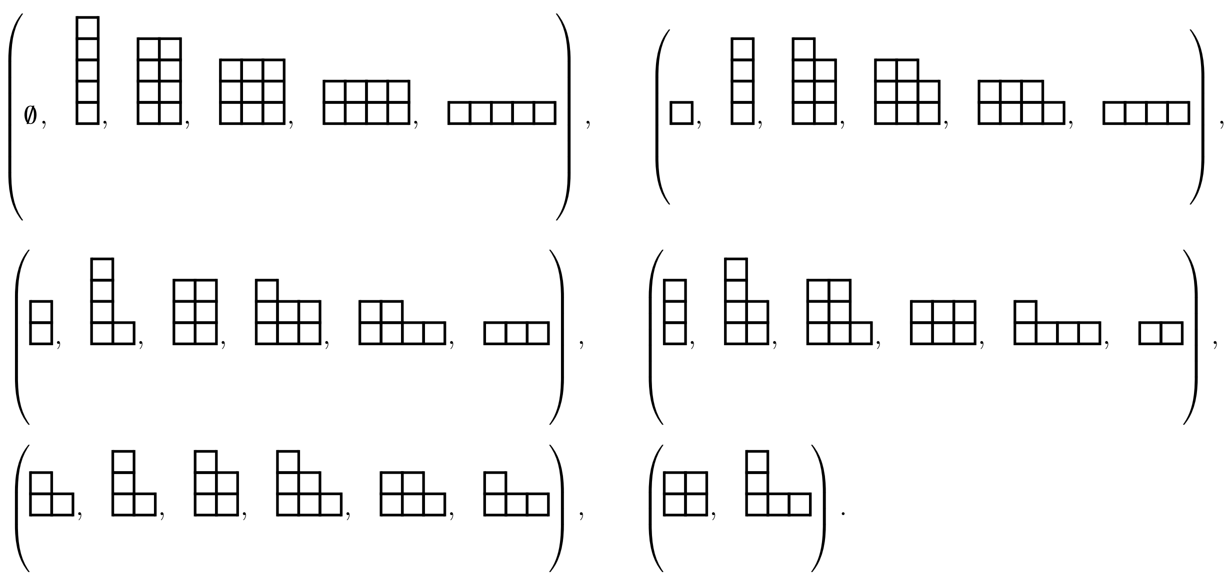
Fig. 1: An example of Suter's map rot_n for $n = 6$ (with $k = 2$); each cell (i, j) is labeled with its weight, $n+1-i-j$.

particular subgraph Y_n of the Hasse diagram of Young's lattice, defined as follows. Let the **hull** of a Young diagram be the smallest rectangular diagram that contains it, and let Y_n denote the set of all Young diagrams whose hulls are contained in the staircase diagram $(n-1, n-2, \ldots, 1)$. (Equivalently, $Y_n = \{\text{partitions } \lambda : \lambda_1 + \ell(\lambda) \leq n\}$, where $\ell(\lambda)$ denotes the *length*, i.e., number of (nonzero) parts, of λ.) It is not hard to see that there are exactly 2^{n-1} such diagrams.

Given a Young diagram $\lambda \in Y_n$ (which, following Suter, we draw "French" style as rows of boxes in the first quadrant) we discard the boxes in the bottom (first) row (let us say there are k of them), move all the remaining boxes one step downward and to the right, then insert a column of $n-1-k$ boxes at the left. Suter shows that the resulting diagram μ is again in Y_n and that the map $\mathrm{rot}_n : \lambda \mapsto \mu$ is invertible of order n. Thus, rot_n generates a cyclic action on Y_n, and the full Suter action is generated by rot_n along with the *conjugation* map that takes a Young diagram to its transpose in the usual way.

Figure 1 shows an example with $n = 6$ and $k = 2$, where boldface black numbers correspond to boxes that get shifted when one passes from $\lambda = (2,2,1,1)$ to $\mathrm{rot}_6(\lambda) = (3,2,2)$.

Example 5. **Suter rotation on Young diagrams** The action of rot_6 on Y_6 produces the following six orbits:



Let f be the statistic on Y_n that sends each Young diagram to the sum of the weights of its constituent boxes, where the box at the lower left has weight $n-1$, its two neighbors have weight $n-2$, and generally the cell (i, j) is assigned weight $f(i, j) = n+1-i-j$. The boxes in Figure 1 have been marked with their weights, so we can see that $f(\lambda) = 5+4+4+3+3+2 = 21$ while $f(\mathrm{rot}_6(\lambda)) = 5+4+4+3+3+3+2 = 24$. David Einstein and James Propp have shown (unpublished) that this statistic is homomesic with respect to Suter's action.

Proposition 1 (Einstein-Propp). *Let Y_n denote the subset of Young's lattice consisting of all shapes whose hulls fit inside the staircase shape $(n-1, n-2, \ldots, 1)$. Under the action of the cyclic group generated by rot_n on Y_n, the function f is c-mesic with $c = (n^3 - n)/12$.*

Example 6. **A homomesic weight function for Suter rotation** The values of f corresponding to each orbit in Example 5 are

$$\begin{array}{ll} (0,15,24,27,24,15), & (5,14,23,26,23,14), \\ (9,18,21,23,21,12), & (12,21,24,21,18,9), \\ (13,16,19,22,19,16), & (16,19) \end{array}$$

each of which has average $17.5 = (6^3-6)/12$.

As is often the case, there are more refined homomesies lurking underneath the one displayed above. Given positive integers i, j with $i+j=n$, we define statistics $f_{i,j}$ whose sum is f as follows:

$$f_{i,j}(\lambda) := \text{sum of all weights in } \lambda \text{ equal to } i \text{ or } j.$$

Claim. The statistic $f_{i,j}$ is homomesic with average ij with respect to the action of rot_n on Y_n.

Then since $f = \frac{1}{2}(f_{1,n-1} + f_{2,n-2} + \cdots + f_{n-1,1})$, it follows that f is homomesic with average

$$\frac{1}{2}\big((1)(n-1)+(2)(n-2)+\cdots+(n-1)(1)\big) = \frac{1}{12}(n-1)(n)(n+1) = \frac{1}{12}(n^3-n).$$

See [PR15, § 2.8] for the proof of the claim.

For the example in Figure 1, we get $f_{1,5}(\lambda)=5$, $f_{2,4}(\lambda)=10$, and $f_{3,3}(\lambda)=6$, while $f_{1,5}(\mathrm{rot}_6\,\lambda)=5$, $f_{2,4}(\mathrm{rot}_6\,\lambda)=10$, and $f_{3,3}(\mathrm{rot}_6\,\lambda)=9$. Also, the values of $f_{2,4}$ corresponding to each orbit of Example 5 are

$$\begin{array}{ll} (0,6,12,12,12,6), & (0,6,12,12,12,6), \\ (4,8,10,10,10,4), & (4,8,10,10,10,4) \\ (8,8,8,8,8,8), & (8,8) \end{array}$$

each of which has average $8 = 2\cdot 4$. The idea that homomesies of interest can be built up as linear combinations of simpler homomesies is very useful.

Finally, we can apply Lemma 1 to immediately obtain

Proposition 2. *Consider the full Suter action of the dihedral group D_n on Y_n. Define the weight of box (i,j) in λ to be $n+1-i-j$, and the statistic $f(\lambda)$ to be the sum of the weights of the boxes in λ. Then the triple (Y_n, D_n, f) is homomesic with average $\frac{n(n^2-1)}{12}$. Furthermore, for positive integers i, j with $i+j=n$, the statistic $f_{i,j}(\lambda) :=$ sum of the weights equal to i or to j in λ is ij-mesic under this action.*

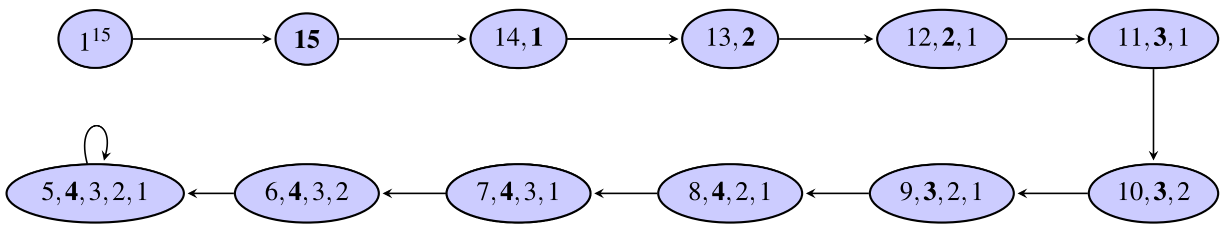
Example 7. **Full dihedral action of Suter on Young diagrams** Note that in Example 5 above, all orbits are closed under the action of conjugation, except for the third and fourth, which combine into a single orbit of size ten under the full D_6 action.

2.3 Homomesy for self-maps & Bulgarian solitaire

In certain situations it is reasonable to generalize the notion of homomesy to non-invertible actions on finite sets, although this comes at a cost of algebraic simplicity. An interesting example is the discrete dynamical system known as *Bulgarian solitaire* [Too81, Gard83, Hop12].

Consider a partition $\lambda \vdash n$ as representing a collection of n identical chips (or uninteresting playing cards) separated into $\ell = \ell(\lambda)$ (unordered) piles containing $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ chips. The game proceeds by taking one chip from each pile and placing them together to form a new pile. We set $\mathfrak{b}(\lambda)$ to be the partition obtained in this way, whose parts are the nonzero elements among $\ell, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_\ell - 1$. Note that the newly created part of size ℓ can range in size from 1 to n, making it hard to write a concise formula for $\mathfrak{b}(\lambda)$ in terms of the parts of λ.

Example 8. **Bulgarian solitaire** For $n = 15$, one trajectory of Bulgarian solitaire is:



This process first surfaced as a puzzle in Russia around 1980, and a solution by Andrei Toom was published in *Kvant* [Too81]. A few years later it was popularized in one of Martin Gardner's *Mathematical Games* columns [Gard83]. The puzzle was to show that no matter which of the 176 partitions of 15 one selects for the initial sizes of the piles, one always eventually ends up at the "staircase" partition $(5, 4, 3, 2, 1)$, which is a fixed point of the action (as in the above example). It turns out that if n is a triangular number (so such a staircase partition exists), then any sequence of moves eventually leads to this fixed point of the action; however, in general the action can exhibit more complex dynamical behavior. (See Figure 2.) Some pointers to more recent literature and more information about the history of this problem, including the fanciful, inaccurate (but easily googlable) name, are available in Brian Hopkins's expository survey [Hop12].

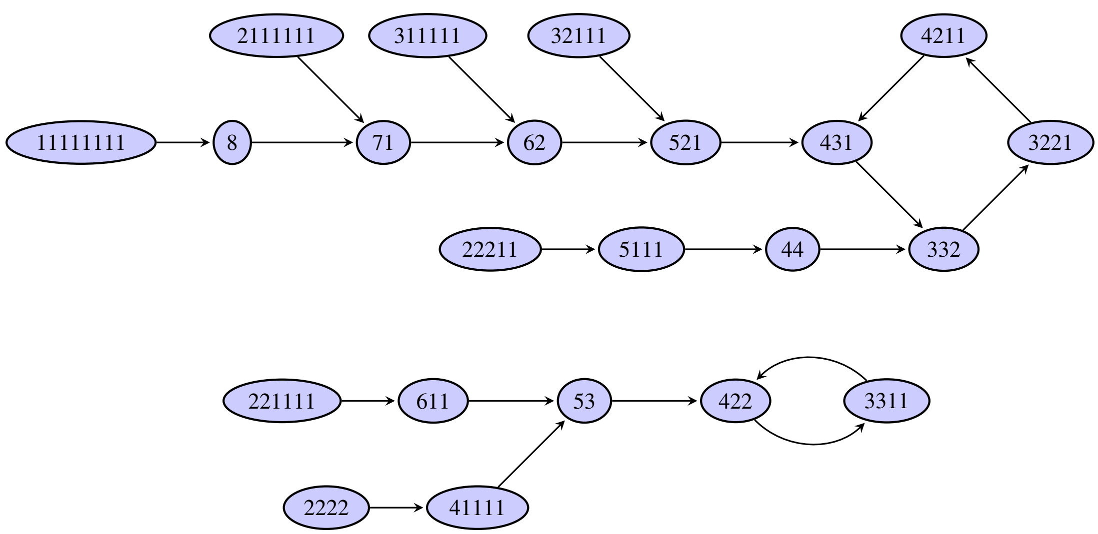


Fig. 2: The action of Bulgarian solitaire on partitions of $n = 8$

Definition 2. Let $\mathscr{S}$ be a *finite* set with a (not necessarily invertible) map $\tau : \mathscr{S} \to \mathscr{S}$ (called a **self-map**). Applying the map iteratively to any $x \in \mathscr{S}$ eventually yields a *recurrent cycle*, and the *recurrent set* is the union of these cycles. (See Figure 2.) We call a statistic $f : \mathscr{S} \to \mathbb{K}$ **homomesic** if the average of f is the same over every recurrent cycle. It is clear that if τ is an invertible action on a finite set S, then this definition specializes to the original one.

Example 9. **Number of parts under Bulgarian solitaire on partitions of n** Consider the example of Bulgarian solitaire for $n = 8$ as displayed in Figure 2. Let the statistic $f(\lambda) := \ell(\lambda)$, the number of parts. We claim that this is homomesic in the sense of Definition 2. No matter where one starts, one eventually ends up in one of two recurrent cycles, namely $(431, 332, 3221, 4211)$ or $(422, 3311)$, with average statistics

$$\frac{3+3+4+4}{4} = \frac{7}{2} \text{ and } \frac{3+4}{2} = \frac{7}{2}.$$

So for Bulgarian solitaire acting on partitions of 8, ℓ is $\frac{7}{2}$-mesic.

It is not hard to show that the general situation is as follows:

Proposition 3. *Let $n = k(k-1)/2 + j$ with $0 \leq j < k$, and consider the action of Bulgarian solitaire on the set of partitions of n. Then the length statistic ℓ which computes the number of parts of λ is homomesic with average $(k-1) + j/k$.*

Note that in Example 9, $n = 8$ corresponds to $k = 4$, $j = 2$, while in the situation that $n = k(k-1)/2$ is a triangular number (so $j = 0$), all paths lead to looping on the shape $\kappa = (k-1, k-2, \ldots, 2, 1)$.

Other statistics homomesic with respect to this action include $f_i(\lambda) := \lambda_i$, the size of the ith largest part of λ, for any $i \geq 1$. For example, when $n = 8$ one sees easily from Figure 2 that f_1 is $\frac{7}{2}$-mesic, f_2 is $\frac{5}{2}$-mesic, f_3 is $\frac{3}{2}$-mesic, and f_4 is $\frac{1}{2}$-mesic.

3 Rowmotion in posets

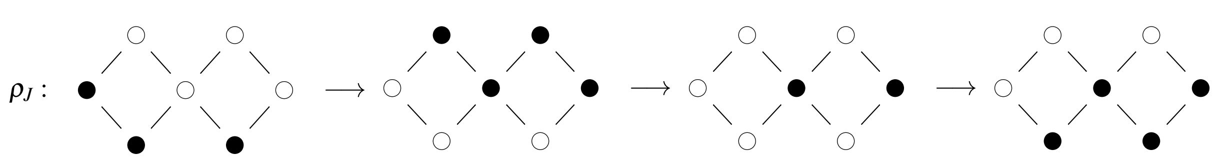
A number of actions that exhibit homomesy turn out to be defined as compositions of various bijections between sets, particularly involutions. A key example that has been well studied is the action of *rowmotion* on a finite partially ordered set. Let $\mathscr{A}(P)$ denote the set of antichains, $F(P)$ the set of order filters, and $J(P)$ the set of order ideals of a finite poset P. (See Stanley [Stan11, Ch. 3] for standard definitions and terminology about posets.) There are elementary bijections between each pair of these objects: given an order ideal I, the set of maximal elements of I forms an antichain A, while any antichain generates an order ideal by including all elements smaller or equal to some element of the antichain. A similar bijection between order filters and antichains is found by standing on one's head. Finally, complementation provides a bijection between $J(P)$ and $F(P)$, which happens also to be an involution on the collection of *all* subsets of P. Rowmotion is simply a three-step composition of these bijections.

Definition 3. Let P be a finite poset, and $I \in J(P)$. Set $\rho(I)$ to be the order ideal generated by the minimal elements of the complement of I. If A is an antichain, set $\rho(A)$ to be the set of minimal elements of the complement of the order ideal generated by A.

$$\rho_J : J(P) \to F(P) \to \mathscr{A}(P) \to J(P)$$
$$\rho_A : \mathscr{A}(P) \to J(P) \to F(P) \to \mathscr{A}(P).$$

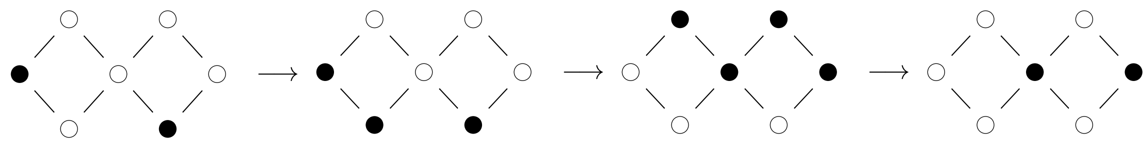
Here the subscript, which we will generally omit when confusion is unlikely, indicates whether we consider this map to act on $\mathscr{A}(P)$ or $J(P)$. We call either map **rowmotion**. Both actions appear in existing literature, often depending on whether there's a natural representation of antichains or order ideals in terms of other combinatorial objects. In Example 11, the antichains represent *nonnesting partitions*.

Example 10. **The rowmotion operator on order ideals and on antichains** The result of applying rowmotion to the 3-element order ideal I on the left is the 4-element order ideal on the right. In between we have taken the complement of I, and then the minimal elements of that result, which generate the new order ideal.



Similarly, the result of applying rowmotion to the 2-element antichain below on the left is the 2-element antichain on the right. Note that if we think about iterating either of these maps, then the resulting actions are the same up to what we consider to be "beat 1" of this 3-beat waltz.

ρ_A :



Rowmotion appears first to have been defined by in Andries Brouwer and Lex Schrijver [BrS74], as a map on antichains. It was later studied by Dmitry Fon-der-Flaass [Flaa93] and by Peter Cameron and Fon-der-Flaass [CaFl95], who also considered it as a permutation of the *monotone Boolean functions* (i.e., indicator functions of order filters of P). In all of this work, the question of the order (aka period) of this operator and the sizes of its orbits was the primary focus, and the main examples were posets that are the product of two or three chains. Jessica Striker and Nathan Williams [StWi11, § 3] give a nice summary of the early history of this map, which has been rediscovered several times, and appears in the literature under a variety of names ("Fon-Der-Flaass action", "Panyushev Complementation", etc.); conventions also differ so that some authors' maps are the inverses of the ones discussed here.

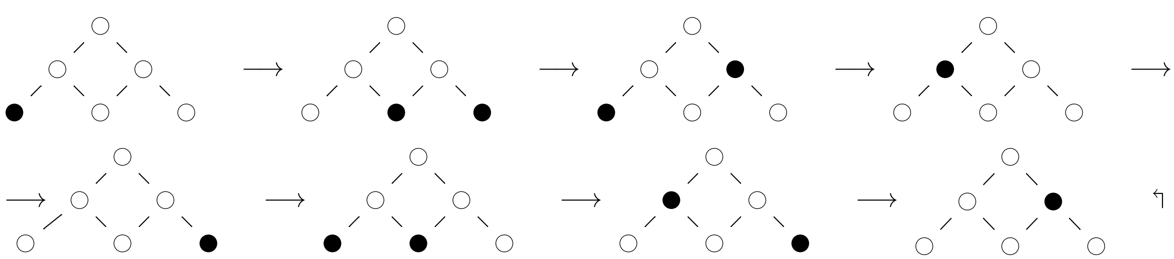
Dmitri Panyushev [Pan09] studied this map for certain graded posets associated with irreducible root systems. In particular, he made the following conjecture, which is one of the earliest statements of an action being homomesic (without using the term). It was proved by Drew Armstrong, Christian Stump, and Hugh Thomas [AST11].

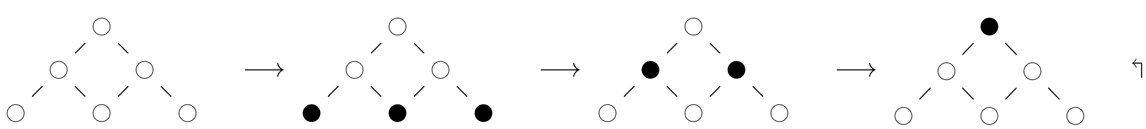
Theorem 2 (Conjectured [Pan09, Conj. 2.1(iii)] ; proved [AST11, Thm. 1.2]). *Let W be a finite Weyl group of rank r, with corresponding positive root poset $\Phi^+(W)$. Then for any orbit $\mathscr{O}$ under the action of ρ_A on $\mathscr{A}(\Phi^+(W))$, we have*

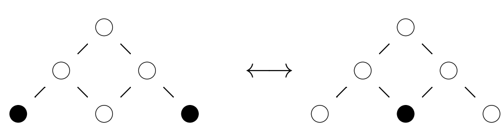
$$\frac{1}{|\mathscr{O}|}\sum_{A\in\mathscr{O}}|A| = r/2.$$

In other words, the cardinality statistic is homomesic with respect to rowmotion acting on antichains of the positive root poset, with average half the rank.

Example 11. For the root poset of type A_3 we have the following three rowmotion orbits, of sizes 8, 4, and 2:







Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2},$$

in agreement with the theorem.

For pictures of the other two classes of posets that arise in classical types, see [StWi11, Fig. 2]. Armstrong, Stump, and Thomas get this result by constructing an equivariant bijection (uniformly, though the proof is still type-by-type) between noncrossing partitions under Kreweras complementation and nonnesting partitions under rowmotion. They also show that a certain generalized q-analogue of Catalan numbers defined for any W exhibits cyclic sieving with respect to the action of rowmotion on $J(\Phi^+(W))$.

Example 12. Reconsider Example 11 as an action on $J(P)$, where each order ideal is represented by its maximal elements. Checking for homomesy of the statistics "order-ideal cardinality" we find that the averages are:

$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}, \quad \frac{0+3+5+6}{4} = \frac{7}{2}, \quad \text{and } \frac{2+1}{2} = \frac{3}{2},$$

so this statistic is **not** homomesic.

In general, it can be difficult *a priori* to guess statistics that might be homomesic with respect to a certain action, even when there may be many of them. In this situation, Shahrzad Haddadan found an interesting answer.

Theorem 3 ([Had14, Cor. 36]). *Let P denote the root poset $\Phi^+(A_n)$, and consider the action of rowmotion on $J(P)$. For $I \in J(P)$, set $f(I) := \sum_{x \in I}(-1)^{\mathrm{rk}\, x}$, where $\mathrm{rk}\, x$ is the rank of the element, i.e., f is a rank-alternating cardinality statistic. Then $(J(P), \rho_J, f)$ exhibits homomesy.*

Example 13. The averages of the rank-alternating cardinality statistic for Example 11 are

$$\frac{1+2+(3-1)+(2-1)+1+2+(3-1)+(2-1)}{8} = \frac{0+3+(3-2)+(4-2)}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

The example shows that finding interesting homomesic statistics is not always straightforward. In some cases trying to find the right weights for certain indicator functions can be fruitful. (See Section 3.4.)

3.1 Relationship with Cyclic Sieving

Comparisons naturally arise between homomesy and the *cyclic sieving phenomenon (CSP)*. The latter was identified by Victor Reiner, Dennis Stanton, and Dennis White in a seminal paper from 2004 [RSW04]; see also [RSW14] and the excellent exposition of Bruce Sagan [Sag11]. They consider an instance of the CSP to be a triple $(X, X(q), C)$, where $C = \langle c \rangle$ is a cyclic group of order n acting on X, and $X(q)$ is a generating function whose evaluation at roots of unity counts symmetry classes under the action. More specifically, for every $d \in \mathbb{Z}$, the number of elements fixed by c^d equals $X(\zeta^d)$, where $\zeta = e^{2\pi i/n}$ is a root of unity. This generalizes John Stembridge's $q = -1$ phenomenon [Ste94b], where $X(1)$ counts the total number of certain combinatorial objects and $X(-1)$ counts a symmetry class of these objects.

An equivalent definition given in [RSW04, Sec. 1] relates more clearly to orbit structures. Given C acting on X, we get a CSP $(X, X(q), C)$ exactly when

$$X(q) \equiv \sum_{i=0}^{n-1} a_i q^i \bmod (q^n - 1),$$

where a_i is the number of orbits of C on X for which the stabilizer cardinality divides i. (The *stabilizer cardinality* aka *stabilizer-order* of an orbit is defined to be the order of the stabilizer subgroup of any (each) element of the orbit.)

Example 14. Recall Example 1, where the cyclic group of order 4 acted on binary strings with exactly two 1's (in obvious bijection with 2-element subsets of [4]), giving two orbits: $(0011, 1001, 1100, 0110)$ with stabilizer cardinality 1 and $(0101, 1010)$ with stabilizer cardinality 2. Here the cyclic sieving polynomial is the Gaussian binomial coefficient

$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4 \equiv 2 + q + 2q^2 + q^3 \left(\bmod q^4 - 1 \right).$$

One reads off from this that the total number of orbits is $a_0 = 2$, and the number of *free orbits* (those whose size is equal to $n = \#C$) is $a_1 = 1$. Both stabilizers-orders divide 2, so $a_2 = 2$, etc.

The two phenomena are similar in that both involve group actions on sets and are intimately bound up with the orbit structure of the action. Interesting examples where both are present are common. And in each case the naturalness of the polynomial (CSP) or statistic (homomesy) is an important consideration.

Cyclic sieving works for cyclic groups, and in some cases for direct products of cyclic groups, i.e., finite abelian groups, but there is no straightforward way to generalize the notion to nonabelian groups. Also, if C' is a subgroup of a cyclic group C, then cyclic sieving of $(X, X(q), C')$ follows from that of $(X, X(q), C)$. Homomesy makes sense for any group action, including non-commutative ones; and by Lemma 1, if H is a subgroup of G, then homomesy for the G-action on X follows from homomesy of the H-action. So homomesies can be lifted to larger groups, and cyclic sieving specializes to smaller groups.

Until recently it was an open problem to find interesting homomesies for a nonabelian group G that are not implied by a homomesy for a commutative subgroup of G. Such homomesies have been proved in recent work of Anne Schilling, Nicolas Thiéry, Graham White, and Nathan Williams [STW^{2}15, Thm. 1.6, Thm. 5.1]. Another possible avenue is described by Sam Hopkins and Ingrid Zhang [HZ15], who define a natural statisic for oscillating tableaux, whose average value has a surprisingly simple formula. Here the missing piece of the puzzle is finding the correct dihedral action on tableaux that would explain the simplicity of their formula.

Homomesy seems to provide somewhat more flexibility in choices of statistic. There are examples of homomesy in situations where CSP appears to be unlikely, because the order of the action is very large relative to the size of X. See the discussion after Theorem 9 for a concrete example of this. (Conjectural examples were noticed earlier [PR15, § 4.2]).

3.2 Toggle Operations

Many interesting actions that exhibit homomesy can be expressed as a composition of simpler involutions on the set. For example, the action of promotion π on SSYTs (Example 3) can be expressed as the product of *Bender-Knuth involutions* β_i as follows. Given $T \in \mathrm{SSYT}_k(\lambda)$ and fixed $i \in [k-1]$, consider all entries $\begin{smallmatrix} i \\ i+1 \end{smallmatrix}$ paired within the same column to be **married**; the involution ignores these pairs. Then in a row with r unmarried i's and s unmarried $i+1$'s, β_i replaces these with s copies of i and r copies of $i+1$:

$$\begin{array}{cccccc} & & & & & i \\ i & i & \underbrace{i \;\; i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1}_{s=3} & & i+1 \\ i+1 & i+1 & & & & \end{array} \;\overset{\beta_i}{\mapsto}\; \begin{array}{cccccc} & & & & & i \\ i & i & \underbrace{i \;\; i \;\; i}_{r=3} & \underbrace{i+1 \quad i+1}_{s=2} & & i+1 \\ i+1 & i+1 & & & & \end{array}.$$

By a theorem of Gansner [Gans80, Thm. 4.1], we can write $\pi = \beta_{k-1}\beta_{k-2}\dots\beta_2\beta_1$ on $\mathrm{SSYT}_k(\lambda)$. The first step in the promotion orbit shown in Example 3 can be written as a follows. The married pairs at each step are shown as blue (since these values are frozen). Note that $\beta_i(T)$ may be T itself, and that in any case $\beta_i^2(T) = T$.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \hline \end{array} \overset{\beta_1}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \hline \end{array} \overset{\beta_2}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array} \overset{\beta_3}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 & 4 \\ \hline \end{array} \overset{\beta_4}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 5 & 5 \\ \hline \end{array}$$

Bender-Knuth involutions provide the standard method for proving combinatorially that Schur functions are symmetric [Sta99, Thm. 7.10.2]. But they also provide a useful example of *toggling*. Informally, we consider a toggle to be an *involution* on a collection of objects that changes an object minimally (possibly not at all). More formally, we define the *toggle group* $\mathscr{T}(P)$ of a finite poset as follows:

Definition 4. Let P be a poset. Given $x \in P$, we define the toggle operation $\mathbf{t}_x : J(P) \to J(P)$ ("toggling at x") via

$$\mathbf{t}_x(I) = \begin{cases} I \triangle \{x\} & \text{if } I \triangle \{x\} \in J(P); \\ I & \text{otherwise,} \end{cases}$$

where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$. The *toggle group* $\mathscr{T}(P)$ is the group of $\operatorname{Aut} J(P)$ generated by $\{\mathbf{t}_x : x \in P\}$.

Toggling an order ideal I at x means to add x to I if $x \notin I$ or take x away from I if $x \in I$, *provided* that the result is an order ideal; otherwise, do nothing. An easy argument shows the following proposition.

Proposition 4 ([CaFl95]). *Let P be a poset. (a) For every $x \in P$, $\mathbf{t}_x$ is an involution, i.e., $\mathbf{t}_x^2 = 1$.*
(b) For every $x, y \in P$ where neither x covers y nor y covers x, the toggles commute, i.e., $\mathbf{t}_x\mathbf{t}_y = \mathbf{t}_y\mathbf{t}_x$.

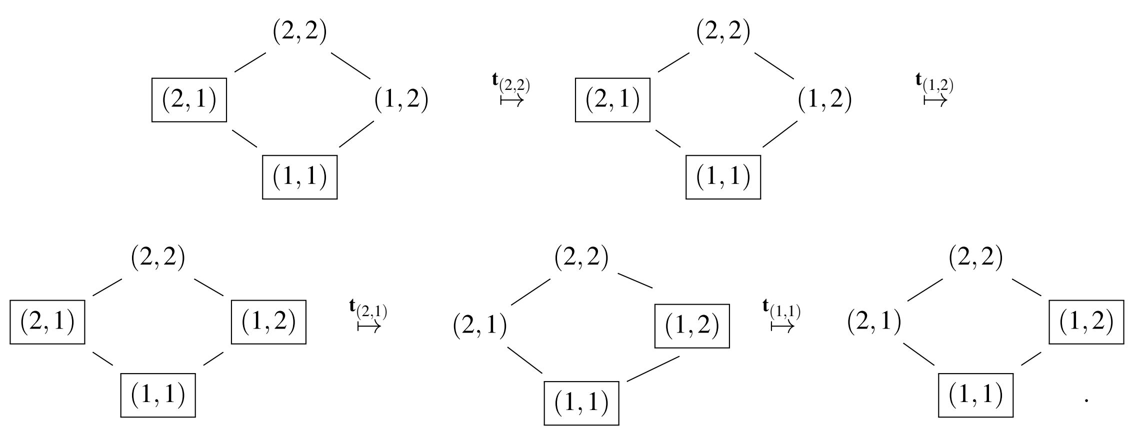
Thus, $\mathscr{T}(P)$ is the quotient of some Coxeter group. The commutativity of non-adjacent (in the Hasse diagram) toggles means composing them in different orders often leads to the same overall result.

Proposition 5 ([CaFl95]). *Let $x_1, x_2, \ldots, x_n$ be any linear extension (i.e., any order-preserving listing of the elements) of a poset P with n elements. Then the composite map $\mathbf{t}_{x_1}\mathbf{t}_{x_2}\cdots\mathbf{t}_{x_n}$ coincides with the rowmotion operation ρ_J.*

The following proposition helps explain Striker and Williams use of the term "rowmotion" for ρ_J.

Corollary 1 ([StWi11], Cor. 4.9). *Let P be a graded poset of rank r, and set $T_k := \prod_{x \text{ has rank } k} \mathbf{t}_x$, the product of all the toggles of elements of fixed rank k. (This is well-defined by Proposition 4.) Then the composition $T_0T_1T_2\cdots T_r$ coincides with ρ_J, i.e., rowmotion is the same as toggling by ranks from top to bottom.*

Example 15. Starting with the order ideal $S = \{(1,1),(2,1)\}$, we toggle successively at the top, right, left, and bottom elements of $P = [2] \times [2]$. Note that in the first step, $\mathbf{t}_{(2,2)}$ leaves the order ideal unchanged, since the set $S' = \{(1,1),(2,1),(2,2)\} \notin J(P)$.



So $\rho(S) = \{(1,1),(1,2)\}$. The reader can easily check that one gets the same result via Definition 3.

Beyond rowmotion, there are several other examples of combinatorial objects with interesting actions that can be realized as some product of toggles acting on linear extensions (e.g., [AKS12]) or on order ideals in a graded poset. It often appears fruitful to look at products where every toggle is used exactly once, which we will call *Coxeter elements* of the toggle group. Striker and Williams [StWi11] provide a number of examples, including ones involving set partitions and alternating sign matrices (ASMs). Many graded posets have a natural notion of *files* (called *columns* in [StWi11]), which can be used to define an operator on $J(P)$ by toggling at each element in files from left to right. (Such posets used to be called *rc-posets*, but now it is understood that any finite graded poset has an **rc-embedding**, which allows for a well-defined notion of files.) The operator, denoted here by ∂, is known as *promotion* (because in some cases it can be related to the Schützenberger promotion operator π). A third operation of this type is called *gyration*, denoted γ, which toggles first at all the even ranks, then at all the odd ranks. Striker and Williams are able to realize Wieland's gyration action on (objects in bijection with) ASMs as toggle-group gyration on the ASM poset. They also prove a general theorem that gives conditions under which various Coxeter elements will be conjugate elements in $\mathscr{T}(P)$, which means that they will have the same orbit structure when acting on $J(P)$. Striker and Williams exploit this to reduce the problem of showing that cyclic sieving holds for *rowmotion* on $J(P)$ to the more easily understood action of *promotion*, obtaining another proof of the cyclic sieving phenomenon for rowmotion acting on $J(\Phi^+(W))$, conjectured by David Bessis and Victor Reiner and proved in [AST11].

Although results on the order/periodicity of actions and CSP which only depend on the orbit structure will be the same for promotion and rowmotion, this is not true for results on homomesy. There are statistics that are homomesic for rowmotion on a product of two chains, but not for promotion; see the remark after Theorem 4.

3.3 Homomesies in products of two chains

In the papers where they first isolated the notation of homomesy [PR15, PR13b], Propp and the author studied cardinality homomesies for rowmotion ρ and promotion ∂ acting on a product of two chains.

Example 16. Figures 3 and 4 show the three orbits for the homomesic triple $(J([4]\times[2]),\rho,\mathrm{card})$, where card denotes the cardinality of an order ideal. Note that here, as opposed to other pictures in this article, we are using *squares* to represent elements of the poset, so the cardinality of the order ideal corresponds to the *area* of the order ideal, which is cut out by a lattice path. (See [PR15, Fig. 4–5] for more explanation about these different representations.) The homomesic average $c=\frac{4\cdot 2}{2}=4$ (half the area of the rectangle) as claimed in Theorem 4.

Figures 3 and 4 are the final pictures taken from excellent animation by Michael LaCroix. The pictures were created in raw postscript, with the interpreter taking as input a binary string that indicates the lattice path cutting out an order ideal. The code takes care of doing all the computations as well as the drawings. The LaTeX `animate` package takes advantage of a "feature" in Adobe Acrobat Reader to display these. Full animations are available at the author's research webpage, http://www.math.uconn.edu/~troby/research.html.

Theorem 4 ([PR15]). *Let $P=[p]\times[q]$ be the poset which is the product of two chains. Then the following statistics are c-mesic with respect to the corresponding actions on the set of order ideals, $J(P)$, or the set of antichains $\mathscr{A}(P)$:*

1. *The cardinality of $I\in J(P)$ with respect to rowmotion on $J(P)$, where $c=pq/2$;*
2. *The cardinality of $A\in\mathscr{A}(P)$ with respect to rowmotion on $\mathscr{A}(P)$, where $c=pq/(p+q)$;*
3. *The cardinality of $I\in J(P)$ with respect to promotion on $J(P)$, where $c=pq/2$.*

The cardinality of $A\in\mathscr{A}([p]\times[q])$ with respect to promotion on $\mathscr{A}([p]\times[q])$ is not generally homomesic [PR15, Example 21], even though the orbit structures for ρ and ∂ are the same.

3.4 Refined homomesies and indicator functions

As we saw earlier in Example 6, one can often think of a homomesy as being built up from simpler or more refined homomesies. For example, define a **file** F of $P=[p]\times[q]$ to be a set of (k,ℓ) with constant $\ell-k$, i.e., a "vertical slice"

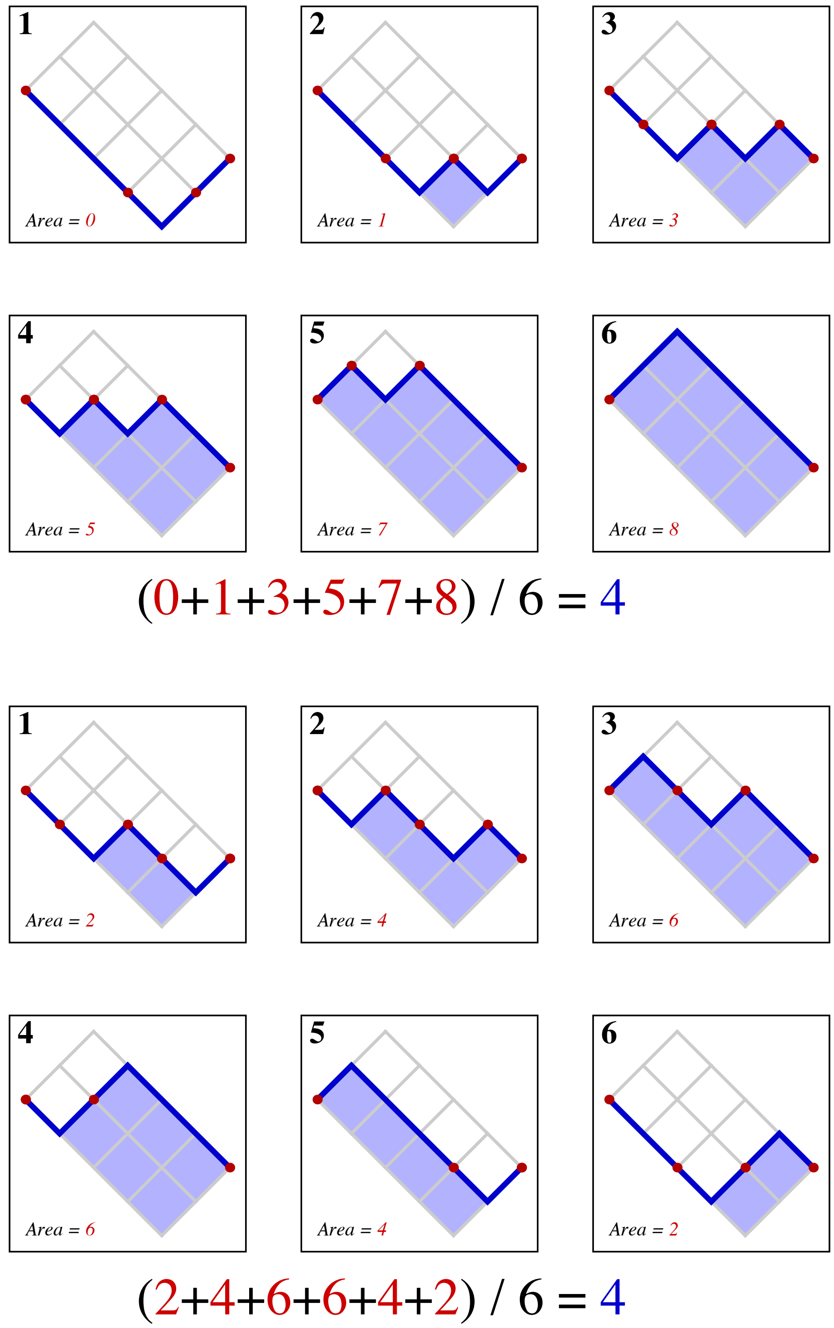


Fig. 3: Two of the three rowmotion orbits in $J([4]\times[2])$. Here the elements of the poset are represented by the filled in squares and order ideals by the shaded regions.

of a poset, consisting of elements in the standard rc-embedding (as shown in all the pictures). If one restricts attention to a particular file F of $P=[p]\times[q]$, then the cardinality function $f := \#(I\cap F)$ for $I\in J(P)$ is homomesic with respect to ρ.

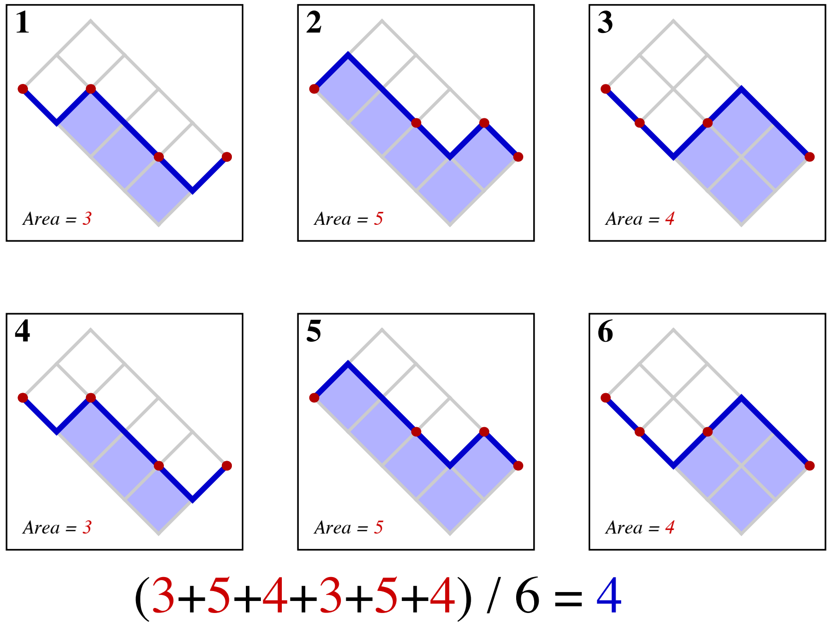


Fig. 4: The third of the three rowmotion orbits in $J([4]\times[2])$, shown as a superorbit, repeating twice the orbit of size three.

Example 17. **File cardinalities under rowmotion acting on a product of two chains** Consider the 2-element file F_4 that contains the smallest element within $P=[4]\times[2]$ (adjacent to the rightmost file). Looking at Figures 3 and 4, we can read off the following orbit averages for f_4, all equal to $4/3$:

$$\frac{0+1+1+2+2+2}{6}, \qquad \frac{1+1+2+2+1+1}{6}, \qquad \text{and} \qquad \frac{1+1+2}{3}.$$

The reader can easily check in this example that for each $i\in[5]$, the statistic f_i counting the number of elements in the intersection of an order ideal and the ith file is homomesic. Clearly $\mathrm{card}=f_1+\cdots+f_5$.

It is worth mentioning that in general if f and g are homomesic statistics, then so are $f+g$ and kf, for any $k\in\mathbb{K}$. In other words, given a fixed set and action, the set of homomesic statistics forms a vector space. This naturally leads to questions of dimension, which in general are nontrivial to answer. In the particular case of ρ acting on $\mathscr{S}=J(P)$, we can look for homomesies within the space spanned by the *indicator functions*

$$\mathbb{1}_x(I) := \begin{cases} 1 & \text{if } x\in I \\ 0 & \text{if } x\notin I \end{cases}$$

for every $x\in P$. Then the above example is stating that for any file F (in the obvious rc-embedding of $P=[4]\times[2]$), the statistic $f_F := \sum_{x\in F}\mathbb{1}_x$ is homomesic. It is also true that $\mathbb{1}_x+\mathbb{1}_{x'}$ is homomesic whenever x' is obtained from x by rotating the poset 180 degrees (cf. Example 3). It is possible (though not easy) to show that these homomesies generate the full subspace of homomesic statistics within the vector space $\mathbb{K}^P$ generated by $\{\mathbb{1}_x : x\in P\}$ when $P=[p]\times[q]$. Similarly, for promotion acting on order ideals, the subspace of homomesic statistics is generated by sums of indicator functions along rows and along columns, along with $\mathbb{1}_x+\mathbb{1}_{x'}$ for each $x\in P$. For more information see the discussions at [PR15, Sec .4.1 and Proof of Thm. 19]. (A later version of [EiPr13] should also contain details of these results.) Theorem 3 is an example of successfully finding suitable weights on the indicator functions to give a homomesic statistic.

The situation for ρ_A acting on $\mathscr{A}(P)$ is somewhat different. Define a **positive fiber** of $[p]\times[q]$ to be a set $(k,\ell)\in[p]\times[q]$ with k constant, and a **negative fiber** to be a set with ℓ fixed [PR15, DEf. 15]. (These are just the natural rows and columns of $[p]\times[q]$, but we use this terminology to avoid confusion with the "rows" and "columns" in the sense of "rc-embedding".) Within the vector space generated by the indicator functions that check whether $x\in P$ belongs to an antichain (defined analogously to those above), the homomesic statistics are generated by the following: $f_F := \sum_{x\in F}\mathbb{1}_x$, where F is any fiber of P, and $\mathbb{1}_x - \mathbb{1}_{x'}$, with x' obtained from x by rotating the poset 180 degrees.

A probabilistic generalization that fiber-cardinalities are homomesic for ρ_A on $\mathscr{A}(P)$ was unearthed by Melody Chan, Shahrzad Haddadan, Sam Hopkins, and Luca Moci [CH^{2}M], whose collaboration began at the aforementioned AIM workshop. They define the *jaggedness* of $I\in J(P)$ to be the number of maximal elements in I plus the number of minimal elements of P not in I. A probability distribution on $J(P)$ is called *toggle-symmetric* if for every $x\in P$, the probability that x is maximal in I equals the probability that x is minimal not in I. They give a formula for the expected jaggedness of an order ideal of P under any toggle-symmetric probability distribution when P is the poset of boxes in a skew Young diagram. Their results depend heavily on results of Striker that a certain "toggleability statistic" is 0-mesic for the actions of rowmotion ρ_J or gyration γ on any finite ranked poset [Str15, Lemma 6.2, Thm. 6.7].

As corollaries they rederive that fiber-cardinality statistics are homomesic for ρ_A acting on $\mathscr{A}([p]\times[q])$, and also prove that these statistics are homomesic for *gyration* γ acting on antichains. This is noteworthy, since these statistics are not generally homomesic for the promotion operator ∂ on the same space, so the homomesy is not merely a consequence of having the action defined as a Coxeter element of toggles.

3.5 Refined homomesies for minuscule posets

The product of two chains is an example of a *minuscule poset* aka *minuscule heaps*, a class that arises naturally in the representation theory of complex semisimple Lie algebras [Pro84, Ste94, Ste01, Gr13]. These are classified into three infinite families (types A, B, and D, with subcases depending on the choice of minuscule element), and a few exceptional types (E_6 and E_7). For a complete classification of these and more pictures, see Stembridge [Ste94, Appendix] or Rush and Shi [RuSh12, End of § 1]. Readers unfamiliar with the underlying representation theory can view them simply as a special class of posets with a structure that can be described purely combinatorially, though we do not take the space to do so here. Example 18 should convey the main point of this section.

Following up on an idea of Stanley [Sta09], Striker and Williams were able to give elegant proofs via equivariant bijections that the action of rowmotion on minuscule posets in types A and B exhibits cyclic sieving [StWi11, Cor. 6.3]. Rush and Shi then gave a uniform proof that cyclic sieving holds for rowmotion acting on minuscule posets in all types [RuSh12].

Since the type A minuscule posets are always products of chains, it is natural to wonder whether the homomesy results extend to other types. This has recently been accomplished by David Rush and Kelvin Wang [RuWa15+]. Recall that for products of chains, not only is the cardinality of an antichain homomesic, but so are all the statistics defined by counting intersections of the order ideals with any particular file (Example 17). What is exceptionally nice about their work is that they correctly identify what generalizes the notion of files, namely subsets of elements that are labeled by the same simple root within the minuscule poset.

Example 18. **Cardinality of weight-labeled elements under rowmotion acting on a minuscule posets** Consider the Dynkin diagram D_4 and minuscule heap P_{α_1} shown in Figure 5. Note that the Dynkin diagram is embedded in the bottom part of the heap. Although this particular poset happens to be isomorphic to a heap of type B_4, the labeling is different, and this is just a coincidence for n so small. For larger values of n, heaps of type D_n labeled with the weight α_1 (assumed to be a leaf of the trivalent vertex) are not isomorphic to heaps of type B_n.

The two rowmotion orbits on P_{α_1} are shown below.

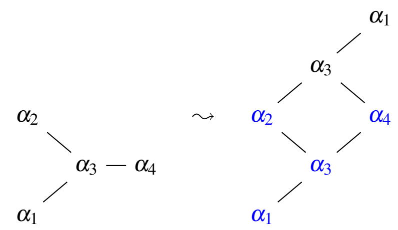
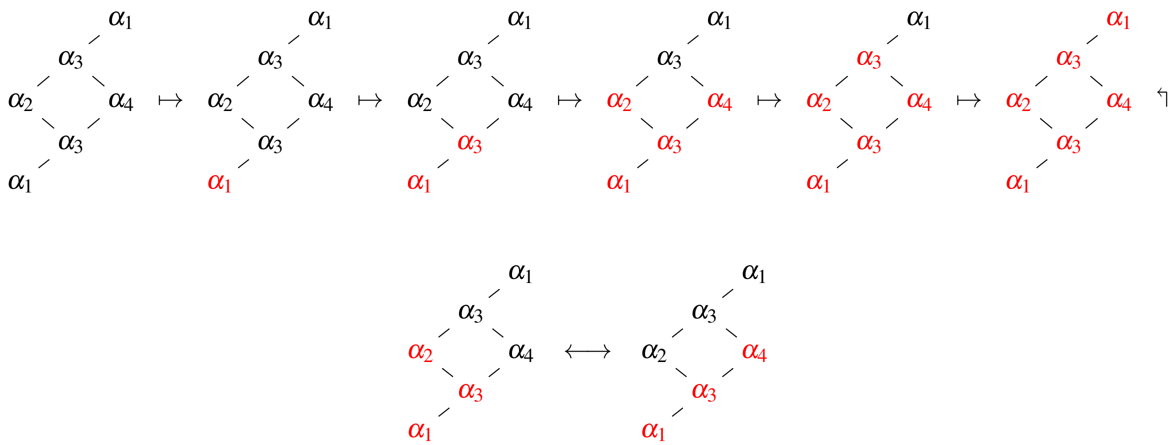


Fig. 5: The Dynkin diagram of type D_4 and the minuscule heap corresponding to the weight α_1.



In the first orbit, the average number of times an element labeled α_1 occurs in the order ideal is $\frac{0+1+1+1+1+2}{6} = 1$, while in the second orbit it is $\frac{1+1}{2} = 1$. The same is true for any weight α_i.

The general situation is as follows.

Theorem 5 ([RuWa15+]). *Let V be a minuscule $\mathfrak{g}$-representation with minuscule weight λ and minuscule heap P_λ. Let α be a simple root of $\mathfrak{g}$ with corresponding fundamental weight ω, and let $P_\lambda^\alpha \subseteq P_\lambda$ be the set of elements of P_λ labeled by α. Let $f^\alpha : J(P_\lambda) \to \mathbb{R}$ be the statistic $|I \cap P_\lambda^\alpha|$ that counts the number of elements in the order ideal labeled by α. Then with respect to the action of rowmotion on P_λ, the statistic f^α is c-mesic with $c = 2\frac{(\lambda,\omega)}{(\alpha,\alpha)}$. Hence, the statistic $f = \mathrm{card}\, I$ is also homomesic.*

Rush and Wang also give explicit formulae for the homomesic average in terms of standard Lie-theoretic parameters.

4 Piecewise-linear and birational rowmotion

Another advantage of the toggling approach to rowmotion is its flexibility. By defining appropriate generalized toggles, we can lift rowmotion and similar maps to the piecewise-linear and birational categories. We outline here the process for doing this, which is explained in detail in a preprint of Einstein and Propp [EiPr13, EiPr14]. Note that in these settings, following standard practice, f will be used to denote a poset labeling, i.e., an object upon which rowmotion acts, rather than a statistic, as earlier in this paper. Context should prevent any serious confusion.

4.1 The order polytope and piecewise-linear toggling

Definition 5. Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining to P a new global minimum element $\widehat{0}$ and new global maximum element $\widehat{1}$. The **order polytope** $\mathscr{O}(P)$ (introduced by R. Stanley [Stan86]) is the set of functions $f : \widehat{P} \to [0,1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_{\widehat{P}} y$.

Informally, we think of an element of $\mathscr{O}(P)$ as being a labeling of the nodes of the Hasse diagram of $\widehat{P}$ with real numbers that respects the partial order. In this sense they naturally generalize order ideals, which are represented by $\{0,1\}$-labelings, with 0's labeling elements of the order ideal, and 1's labeling the complementary order filter. In fact, $J(P)$ naturally corresponds to the set of vertices of $\mathscr{O}(P)$.

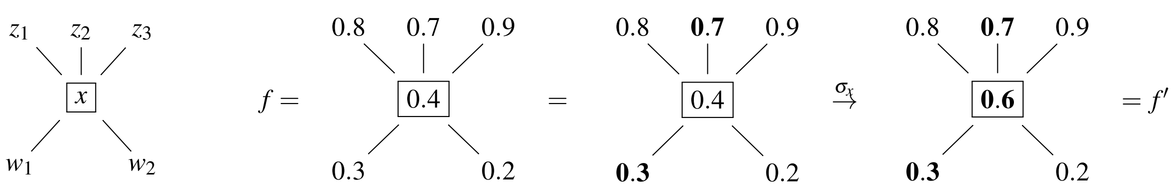
Definition 6. For each $x \in P$, define the **piecewise-linear toggle** $\sigma_x : \mathscr{O}(P) \to \mathscr{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x) & \text{if } y = x, \end{cases} \tag{3}$$

where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

Note that the interval $[\max_{w \lessdot x} f(w), \min_{z \gtrdot x} f(z)]$ is precisely the set of values that $f'(x)$ could possibly take on while satisfying the order-preserving condition (as long as $f'(y) = f(y)$ for all $y \neq x$), and the map that sends $f(x)$ to $\min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints of this interval.

Example 19. Consider the snippet of a poset shown at the left below, where the element x is covered by exactly three elements z_1, z_2, z_3 and covers two elements, w_1, w_2. In order to toggle the labeling f at element x, we first identify the minimum label assigned to an element covering x, and the maximum label below. Then σ_x changes the labeling *only* at the element x:



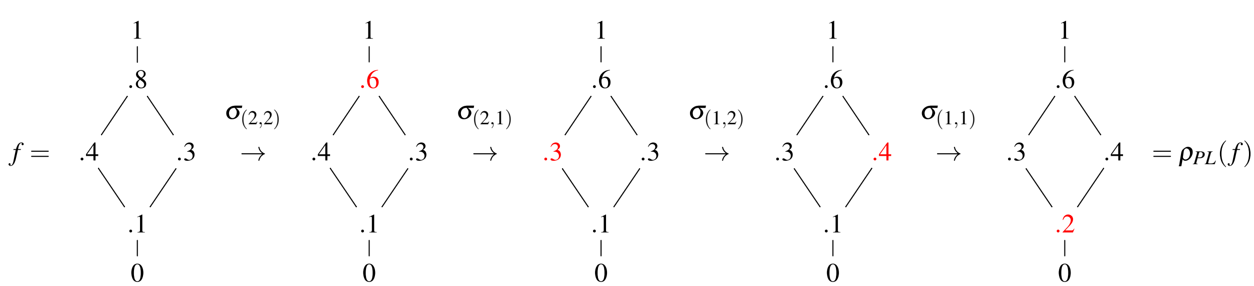
Note that here we have

$$\min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) = 0.7 + 0.3 = 1.0 = 0.4 + 0.6 = f(x) + f'(x)$$

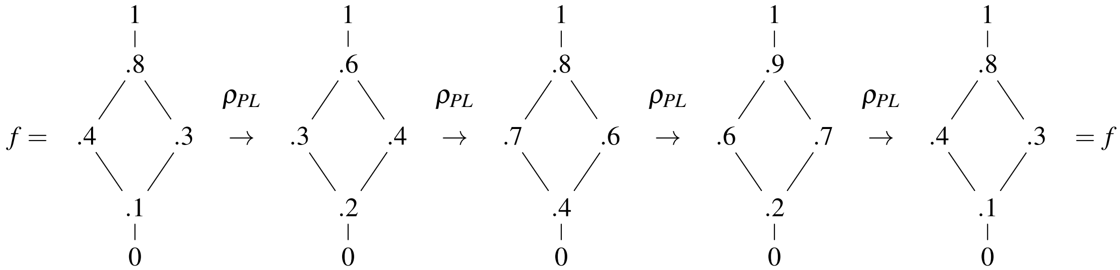
It is straightforward to show that an analogue of Proposition 4 holds for the piecewise-linear toggles, so as before, we can define **piecewise-linear rowmotion** via any linear extension $x_1, \ldots, x_n$ of P to be

$$\rho_{PL} := \sigma_{x_1} \sigma_{x_2} \ldots \sigma_{x_n} : \mathscr{O}(P) \to \mathscr{O}(P).$$

Example 20. Consider the poset $P = [2] \times [2]$, labeled as in Example 15. We show the step-by-step process of toggling at each element of P by rows from top to bottom to obtain $\rho_{PL}(f)$, where f is the element of $\mathscr{O}(P)$ shown at the left.



What happens if we apply rowmotion repeatedly? *A priori*, there is no reason to expect the orbits to even be finite since rowmotion is acting on an infinite set. For the f above, we find:



(4)

This periodicity turns out to hold in general: the order of the map ρ_{PL} on $\mathscr{O}([p]\times[q])$ is $p+q$, which is the same as the order of ρ on $J([p]\times[q])$. The only proof currently known of this result relies on results from the birational setting. It is an interesting open problem to prove this more directly.

4.2 The birational setting

Much work has been done in the *tropical semiring*, where the usual ring operations $+$ and $\cdot$ are replaced by the tropical operations max and $+$, respectively (see e.g., [Kiri00]). The nature of the formula for piecewise-linear toggles allows them to be *detropicalized* to **birational toggles** as follows. First rewrite each occurrence of min in (3) in terms of max using $\min(z_i) = -\max(-z_i)$; then replace each instance of max with $+$, each instance of $+$ with $\cdot$, and each negation with taking reciprocals. This yields the following definition.

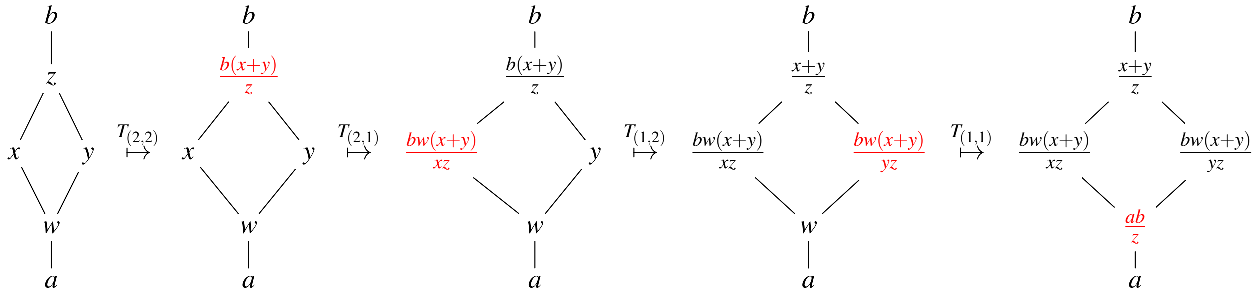
Definition 7. Let $\mathbb{K}$ be any field, and $f \in \mathbb{K}^{\widehat{P}}$ any labeling of the elements of $\widehat{P}$ by elements of $\mathbb{K}$. We define the **birational toggle** $T_x : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ at $x \in P$ by

$$(T_x f)(y) = \begin{cases} f(y), & \text{if } y \neq x; \\ \dfrac{1}{f(x)} \cdot \dfrac{\sum\limits_{\substack{w\in\widehat{P};\\ w\lessdot x}} f(w)}{\sum\limits_{\substack{z\in\widehat{P};\\ z\gtrdot x}} \dfrac{1}{f(z)}}, & \text{if } y = x \end{cases}$$

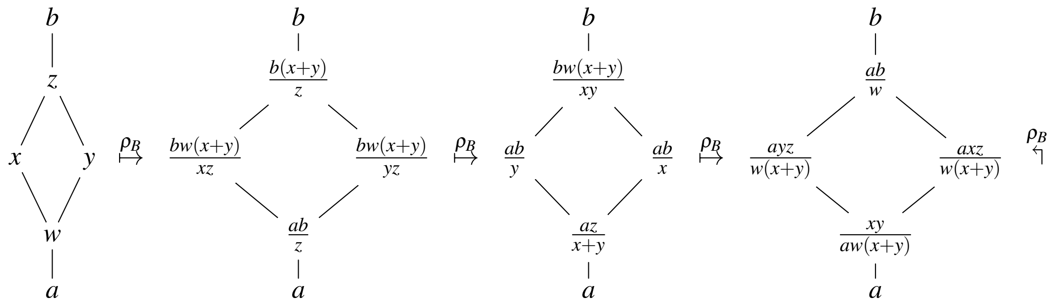
for all $y \in \widehat{P}$. Note that this rational map T_x is well-defined, because the right-hand side of is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$. Finally, define birational rowmotion by $\rho_B := T_{x_1} T_{x_2} \ldots T_{x_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, where $x_1, x_2, \ldots, x_n$ is any linear extention of P.

In words, toggling at x changes only the label at x and does this by (a) *inverting* the label at x, (b) multiplying by the *sum* of the labels at vertices *covered by* x, and (c) multiplying by the *parallel sum* of the labels at vertices *covering* x. (By **parallel sum** here, we mean the associative operation $\parallel$ defined by $a \parallel b := \frac{1}{\frac{1}{a}+\frac{1}{b}}$.)

Example 21. Let $P = [2]\times[2]$, our running example. One iteration of birational rowmotion acting on $\mathbb{K}^{\widehat{P}}$ is shown, toggling step-by-step along the usual linear extension.



And here is a (generic) orbit of birational rowmotion:



(5)

It seems almost miraculous that the orbit of birational rowmotion ρ_B is finite, let alone the same size as for rowmotion ρ_J on order ideals, which we now dub **combinatorial rowmotion**. This appears to be true only for relatively rare types of posets. Indeed for many simple posets such as the three shown in Figure 6, the order of ρ_B can be proven to be infinite [GRarX, § 20].

Einstein and Propp [EiPr13, EiPr14] were able to generalize homomesy results for $P = [p] \times [q]$ to the actions of ρ_{PL} and ρ_B by lifting techniques used for combinatorial rowmotion to the birational case, then specializing to the piecewise-linear case. However, their work did not show that the order of these maps on P was finite. The periodicity question loomed large, particularly since they preferred to avoid using some sort of asymptotic definition of homomesy in this situation, where the orbits in fact always appeared to be finite. They describe how periodicity results for ρ_B imply the same results for ρ_{PL}, but not vice-versa. This led Darij Grinberg and the author to tackle the periodicity question for birational rowmotion.

Theorem 6 ([GrRo16, GrRo15, GRarX]). *The action of ρ_B on generic $\mathbb{K}$-labelings of a finite poset P has finite order when:*

a) $P = [p] \times [q]$, is the product of two chains of lengths p and q;

b) $P = \Phi^+(A_n)$, the positive root poset of type A_n (cf. Example 11);

c) P is a graded rooted forest, with every component oriented towards its root and having the same rank;

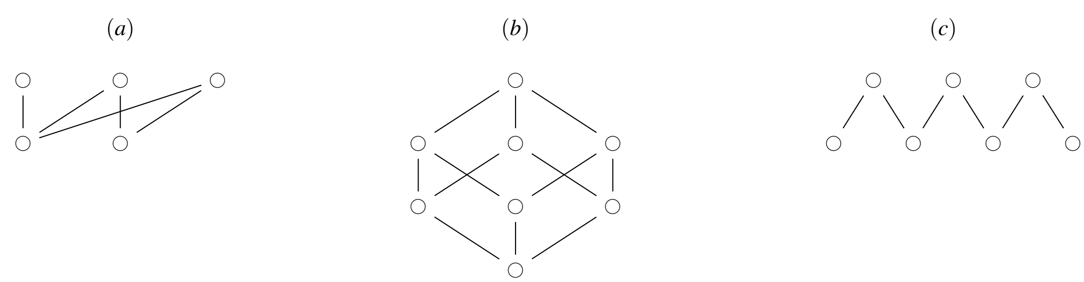


Fig. 6: Three posets for which birational rowmotion ρ_B is **not** periodic

d) and more generally, P is a skeletal *poset (a class generalizing graded forests), built up inductively by grafting antichains of various sizes.*

There are a few other posets for which the order of ρ_B is proven to be finite or conjecturally finite, mostly ones coming from Lie theory (root posets and minuscule posets) or built up inductively from simple structures. But getting a complete classification currently seems unlikely.

4.3 Homomesy for piecewise-linear and birational rowmotion

The notion of homomesy is easily lifted to the piecewise-linear and birational settings. In the latter, we need roughly to replace arithmetic means with geometric means. But branching issues with the function $z \mapsto z^{1/n}$ make it more convenient to just compare products across superorbits of the same length, rather than geometric means.

For P a product of two chains, Einstein and Propp are able to lift the homomesy results of [PR15] as follows.

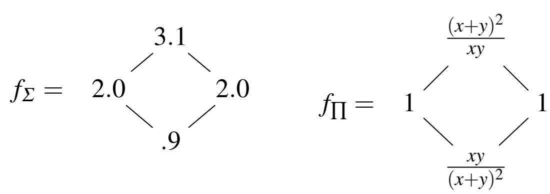
Theorem 7 (Einstein-Propp [EiPr13, Thm. 2, Thm. 4]). *Let $P = [p] \times [q]$ be a product of two chains. Then*

1. *The statistic $G(f) := \sum_{x \in P} f(x)$ is c-mesic with respect to the action of ρ_{PL} on $\mathscr{O}(P)$ with $c = pq/2$.*
2. *For any $\mathbb{K}$-labeling f of $\widehat{P}$ with $f(0) = f(1) = 1$ for which iterations of birational rowmotion $\rho_B \colon \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ are well-defined, we have*

$$\prod_{k=0}^{p+q-1} \prod_{x \in P} \rho_B^k(f)(x) = 1$$

In fact, even the more refined homomesies seem to lift to these settings. For example, for any *file* F_i in P, the statistic $G(f) := \sum_{x \in F_i} f(x)$ is claimed to be homomesic with respect to the action of ρ_{PL} on $\mathscr{O}(P)$ [EiPr13].

Example 22. Consider the orbit of ρ_{PL} shown in (4) and the orbit of ρ_B in 5. For each element $x \in P$, we write the *sum* of the ρ_{PL}-labels at x in the left poset, and the *product* of the ρ_B-labels at x in the right one.



If we start with a different point in the order polytope and construct the ρ_{PL}-orbit, we might get different sums (other than 3.1 and 0.9) across a ρ_{PL}-orbit at the top and bottom elements, but the sum of these sums will always be 4, just as the sums at the left and right elements will always be 2. And the average value across all elements will be $8/4 = 2 = \frac{2 \cdot 2}{2}$. For the birational orbit, the product of the orbit-products in the middle files is 1, as is the product of all labels at all elements.

4.4 Connection with Y-systems

The proof that ρ_B is periodic on the posets $[p] \times [q]$ and $\Phi^+(A_n)$ (Theorem 6) used a parameterization of $\mathbb{K}$-labelings of P as a ratio of two determinants and a simple three-term Plücker relation. This was modeled on Volkov's approach to proving the type $A_m \times A_n$ Zamolodchikov periodicity conjecture [Volk06], leading a number of people to suspect a possible connection between Y-systems and birational rowmotion. It was only at the aforementioned AIM workshop in March 2015 that an explicit connection was uncovered by Arkady Berenstein, Max Glick, Darij Grinberg, Gregg Musiker and Hugh Thomas. This gives another path to proving the periodicity of birational rowmotion on these posets and connects it with the theory of cluster algebras. Readers unfamiliar with Y-systems may wish to read this section lightly or seek necessary background and historical information in the excellent survey by Lauren Williams [Will14].

Informally, a Y-system is a dynamical system of rational functions defined on a graph coming from root systems. The setup is as follows. Let Δ,Δ' be *Dynkin diagrams* (of any type A,B,C,D,E,F or G) on vertex sets I,I' and let C,C' be the corresponding *Cartan matrices*. Set the graph $\Gamma := I\times I'$.

Define $A=(a_{i,j}) := 2J_{\#I}-C$ and $A'=(a'_{i',j'}) := 2J_{\#I'}-C'$, where J_n denotes the $n\times n$ identity matrix. The matrices A and A' control how this dynamical system updates.

The only example we consider here is Type A, where, for example,

$$\Delta = A_4 \implies C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \implies A = \begin{bmatrix} 0&1&0&0 \\ 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&0 \end{bmatrix}.$$

Definition 8. Let h,h' denote the *Coxeter numbers* of Δ,Δ' (i.e., the order of the product of all the simple reflections in any order). The $\Delta\times\Delta'$ **Y-system** is then the collection $\{Y_{i,i',t}\colon (i,i')\in I\times I', t\in\mathbb{Z}\}$ satisfying the relations

$$Y_{i,i',t+1}Y_{i,i',t-1} = \frac{\prod_{j\in I}(1+Y_{j,i',t})^{a_{i,j}}}{\prod_{j'\in I'}(1+Y_{i,j',t}^{-1})^{a_{i',j'}}}.$$

(We can think of the $Y_{i,i',t}$ as positive real numbers or rational functions.)

Simplifying a more complex history, the main conjecture was that this system was periodic, with period given in terms of the Coxeter numbers.

Theorem 8 (Zamolodchikov periodicity conjecture). *With the setup as above, the type (Δ,Δ') Y-system is periodic with period $2(h+h')$, i.e.,*

$$Y_{i,i',t+2(h+h')} = Y_{i,i',t}.$$

Special cases were proved by a number of authors, including Volkov as described above using determinants and cross ratios, Fomin and Zelevinsky using cluster algebras. The theorem in full generality was finally resolved by Bernard Keller [Kel12], using cluster algebra theory and categorification. See [Will14, § 4] for more details. In type A_n, $h=n+1$, so the $A_{p-1}\times A_{q-1}$ Y-system has order $2(p+q)$.

The connection between Y-systems and birational rowmotion is then given as follows.

Proposition 6 (Berenstein-Glick-Grinberg-Musiker-Thomas). *Let f be a $\mathbb{K}$-labeling of $\widehat{P}$, where $P=[p]\times[q]$, for which iterations of birational rowmotion $\rho_B:\mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ are well-defined. Assume $f(0)=f(1)=1$. If we set*

$$Y_{i,i',i+i'-2k} = \frac{\rho_B^k f(i,i'+1)}{\rho_B^k f(i+1,i')},$$

then the collection $\{Y_{i,i',j}\}$ form a type $A_{p-1}\times A_{q-1}$ Y-system. Furthermore, almost all type $A_m\times A_n$ Y-systems can be obtained in this way.

Although it takes a certain amount of work, this pathway eventually allows one to obtain periodicity of ρ_B on a product of two chains from periodicity of the corresponding Y-system, and vice-versa. More importantly, it shows a very strong connection between these two phenomena which is yet to be fully explored. In particular, perhaps homomesy results for ρ_B might translate into some interesting property of Y-systems.

5 Homomesy in other toggling situations

A useful abstract generalization of the toggle group of a poset has recently been proposed by Jessica Striker [Str15+]. In this framework, the poset P is replaced with any ground set E, and $J(P)$ with any fixed subset $\mathscr{L}$ of 2^E. Then one defines the **toggle** $t_e : \mathscr{L} \to \mathscr{L}$ by

$$t_e(X) := \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in \mathscr{L} \\ X \backslash \{e\} & \text{if } e \in X \text{ and } X \backslash \{e\} \in \mathscr{L} \\ X & \text{otherwise} \end{cases} .$$

It is easy to see that $t_e^2 = 1$ for all $e \in E$, allowing one to define a **generalized toggle group** $T(\mathscr{L})$ to be the subgroup of the symmetric group $\mathfrak{S}_{\mathscr{L}}$ generated by $\{t_e : e \in E\}$. This definition significantly broadens the range of examples to which the ideas of toggling can be applied. Striker gives a few general structure theorems, and describes a number of examples where this approach appears to be fruitful.

One particularly interesting example came out of a working group at the aforementioned AIM workshop in March 2015.

Definition 9. A *noncrossing partition* is a partition $\{B_1, \ldots, B_s\}$ of $[n] := \{1, \ldots, n\}$ such that if $a < b < c < d$ with both $a, c \in B_i$ and $b, d \in B_j$, then $i = j$. Let $\mathrm{NC}(n)$ denote the set of all noncrossing partitions of $[n]$.

A noncrossing partition can be represented pictorially by a collection of arcs connecting nodes labeled by $[n]$. For each block B_i, arrange the $|B_i|$ vertices in increasing order and include $|B_i| - 1$ arcs, one for each adjacent pair. Even though the arcs are undirected, we always describe one as an ordered pair (i, j) with $i < j$. Within the set of all possible such graphs on $[n]$ given by connecting any two distinct vertices, the ones representing noncrossing partitions are characterized by the following disallowed types (shown in Figure 7). Two distinct arcs $(i, j), (k, \ell)$ in a noncrossing partition must not satisfy:

1. $i < k < j < \ell$ (crossing),
2. $i = k$ (left half-nesting), or
3. $j = \ell$ (right half-nesting).

Figure 8 shows the noncrossing partition $\{\{1,6,7\},\{2\},\{3,5\},\{4\},\{8,9\}\}$ of size 9, drawn using arcs. Note that nesting arcs are allowed, but by construction, they must describe vertices in different blocks.

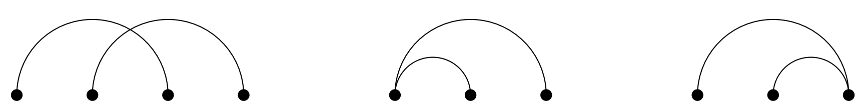


Fig. 7: Disallowed pairs of arcs in a noncrossing partition: crossing, left half-nesting, and right half-nesting, respectively.

Now in Striker's framework, take $E = \{(i, j) \in [n] \times [n] : i < j\}$, and consider $\mathrm{NC}(n)$ as a subset of 2^E. The toggle $t_{(i,j)}$ applied to $\nu \in \mathrm{NC}(n)$ simply adds or removes the arc (i, j) whenever the result still lies in $NC(n)$, and otherwise does nothing.

The original problem proposed by James Propp was to show that under the action of certain "Coxeter elements" on $\mathrm{NC}(n)$, the *arc count statistic* $\alpha :=$ the number of pairs $(i, j) \in \nu$ is $\frac{n-1}{2}$-mesic. He noted that toggling in the "nice" order:

$$\tau_{\text{nice}} = (t_{(n-1,n)})(t_{(n-2,n)}t_{(n-2,n-1)})\ldots(t_{(2,n)}t_{(2,n-1)}\ldots t_{(2,4)}t_{(2,3)})(t_{(1,n)}\ldots t_{(1,3)}t_{(1,2)})$$

appears to be the inverse of Kreweras complementation on $NC(n)$; hence, this map has order $2n$ and the homomesy follows easily. On the other hand, consider toggling each element of E in the "naughty" order, where all toggles of

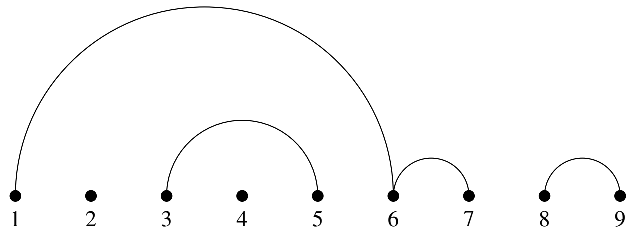


Fig. 8: The noncrossing partition $\{(1,6),(3,5),(6,7),(8,9)\} \in \mathrm{NC}(9)$.

nodes one apart are applied first, followed by toggles of nodes two apart, etc. In this situation the orbit struture is unpredictable and the period of the map is much greater, but the same statistic appeared to be homomesic. The group working on this problem in fact showed a much more general result.

Theorem 9 (EFGJMPR [E+15+]). *Let τ be any product of toggles $t_{(i,j)}$ (where $i < j$) that contains no toggle more than once (a "partial Coxeter element") and that contains $t_{(i,i+1)}$ for every $i \in [n-1]$. Then the arc count statistics α is $\frac{n-1}{2}$-mesic.*

This theorem provides a good example of homomesy where cyclic sieving seems to be unlikely. For $n = 7$ consider the Coxeter element

$$\tau_{\text{naughty}} = (t_{(1,7)})(t_{(2,7)}t_{(1,6)})\dots(t_{(4,7)}t_{(3,6)}t_{(2,5)}t_{(1,4)})(t_{(5,7)}t_{(4,6)}t_{(3,5)}t_{(2,4)}t_{(1,3)})(t_{(6,7)}\dots t_{(3,4)}t_{(2,3)}t_{(1,2)})$$

Theorem 9 applies, so we get homomesy. But the orbit sizes are 3, 3, 3, 3, 3, 6, 9, 9, 11, 13, 15, 31, 39, 39, 109, 133, with least common multiple 5783868090, which is the order of τ. So an interesting example of cyclic sieving seems to be unlikely for this action, since the size of the cyclic group is so large relative to the number of objects, which in this case is the Catalan number $C_7 = 429$.

References

AST11. Drew Armstrong, Christian Stump, and Hugh Thomas, *A uniform bijection between nonnesting and noncrossing partitions*, Trans. Amer. Math. Soc. **365** (2013), no. 8, 4121–4151. Also available as arXiv:1101.1277v2.

AKS12. Arvind Ayyer, Steven Klee, Anne Schilling, *Combinatorial Markov chains on linear extensions*, J. Alg. Combin., **39(4)** (2014), 853–881. Also available at arXiv:1205.7074v3.

BR15+. Roger Behrend and Tom Roby, *Dihedral homomesy for Alternating Sign Matrices*, in preparation.

BPS13. Jonathan Bloom, Oliver Pechenik, and Dan Saracino, *Proofs and generalizations of a homomesy conjecture of Propp and Roby*, Discrete Math., **339** (2016), 194–206.

BrS74. Andries E. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math. Centr. report ZW24, Amsterdam (Jun. 1974), http://www.win.tue.nl/~aeb/preprints/zw24.pdf.

CaFl95. Peter J. Cameron and Dmitry G. Fon-der-Flaass, *Orbits of Antichains Revisited*, Europ. J. of Combin., **16(6)**, (1995), 545–554, http://www.sciencedirect.com/science/article/pii/0195669895900365.

CH²M. Melody Chan, Shahrzad Haddadan, Sam Hopkins, and Luca Moci, *The expected jaggedness of order ideals* (2015), arXiv:1507.00249v1.

E+15+. David Einstein, Miriam Farber, Emily Gunawan, Michael Joseph, Matthew Macauley, James Propp, and Simon Rubinstein-salzedo, *Noncrossing partitions, toggles, and homomesies*, arXiv:1510.06362v1.

EiPr13. David Einstein and James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains* (2013), arXiv:1310.5294.

EiPr14. David Einstein and James Propp, *Piecewise-linear and birational toggling (Extended abstract)*, DMTCS proc. FPSAC 2014, http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAT0145/4518. Also available at arXiv:1404.3455v1.

Flaa93. Dmitry G. Fon-der-Flaass, *Orbits of Antichains in Ranked Posets*, Europ. J. Combin., **14(1)**, (1993), 17–22, http://www.sciencedirect.com/science/article/pii/S0195669883710036.

Ful97. William Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry*, Cambridge: Cambridge University Press, 1997.

Gans80. Emden Gansner, *On the equality of two plane partition correspondences*, Discrete Math. **30** (1980), 121–132.

Gard83. Martin Gardner, *Mathematical Games: Tasks you cannot help finishing no matter how hard you try to block finishing them*, Scientific American **249** (1983), 12–21.

Gr13. Richard M. Green, *Combinatorics of Minuscule Representations*, Cambridge Univ. Press, 2013.

GrRo14. Darij Grinberg and Tom Roby, *The order of birational rowmotion (Extended abstract)*, DMTCS proc. FPSAC 2014, http://www.dmtcs.org/pdfpapers/dmAT0165.pdf.

GrRo16. Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. of Combin. **23**(1), #P1.33 (2016). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i1p33

GrRo15. Darij Grinberg, Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. of Combin. **22**(3), #P3.40 (2015). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p40.

GRarX. Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion*, 2014, arXiv:1402.6178v4. (Full version of above papers with detailed proofs.)

Had14. Shahrzad Haddadan, *Some Instances of Homomesy Among Ideals of Posets* (2014), arXiv:1410.4819v2.

HP10. Alexander E. Holroyd and James Propp, *Rotor walks and Markov chains*, in *Algorithmic Probability and Combinatorics*, American Mathematical Society (2010), 105–126.

H+08. Alexander E. Holroyd, Lionel Levine, Karola Meszaros, Yuval Peres, James Propp, David B. Wilson, *Chip-Firing and Rotor-Routing on Directed Graphs*, Progress in Probability, **60** (2008), 331–364. Updated version at arXiv:0801.3306v4.

Hop12. Brian Hopkins, *30 years of Bulgarian solitaire*, College Math J., **43**, #2, March 2012, 135–140.

HZ15. Sam Hopkins and Ingrid Zhang, *A Note on Statistical Averages for Oscillating Tableaux*, Electron. J. Combin. **22(2)** (2015), #P2.48. http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i2p48. Also at arXiv:1408.6183v1

Kel12. Bernard Keller, *The periodicity conjecture for pairs of Dynkin diagrams*, Ann. of Math. **177(1)** (2013), 111–170,

Kiri00. Anatol N. Kirillov, *Introduction to tropical combinatorics*, Physics and combinatorics: Proceedings of the Nagoya 2000 International Workshop, held 21 - 26 August 2000 in Nagoya University. Edited by Anatol N. Kirillov (Nagoya University) & Nadejda Liskova. Published by World Scientific Publishing Co. Pte. Ltd., Singapore, 2001. ISBN #9789812810007, 82-150.

KiBe95. Anatol N. Kirillov and Arkady D. Berenstein, *Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz, **7(1)** (1995), 92–152. Preprint available at http://math.uoregon.edu/~arkadiy/bk1.pdf.

Pan09. Dmitri I. Panyushev, *On orbits of antichains of positive roots*, Europ. J. Combin. **30(2)** (2009), 586–594. Also available at arXiv:0711.3353v2.

Pro84. Robert A. Proctor, *Bruhat lattices, plane partition generating functions, and minuscule representations*, Europ. J. Combin. **5** (1984), 331–350.

PR15. James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22(3)** (2015), #P3.4, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4. Also available at arXiv:1310.5201v5.

PR13b. James Propp and Tom Roby, *Homomesy in products of two chains*, (extended abstract) DMTCS proc. FPSAC 2013, http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAS0180/4290.

AIM15. James Propp, Tom Roby, Jessica Striker, and Nathan Williams (organizers), Sam Hopkins (notetaker), *Notes from the AIM workshop on dynamical algebraic combinatorics*, American Institute of Math., San Jose, CA, 23–27 March 2015, http://aimath.org/pastworkshops/dynalgcomb.html, http://mit.edu/~shopkins/docs/aim_dyn_alg_comb_notes.pdf.

RSW04. Victor Reiner, Dennis Stanton, and Dennis White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), 17–50.

RSW14. Victor Reiner, Dennis Stanton, and Dennis White, *What is... cyclic sieving?*, AMS Notices **61**, #2 (2014), 169–171.

Rho10. Brendon Rhoades, *Cyclic sieving, promotion, and representation theory*, J. Combin. Theory Ser. A, vol. 117, no. 1, (2010), 38–76. http://www.sciencedirect.com/science/article/pii/S0097316509000703
Also available at arXiv:1005.2568v1.

RuSh12. David B. Rush and XiaoLin Shi, *On Orbits of Order Ideals of Minuscule Posets*, J. of Algebraic Combin., May 2013, Volume 37, Issue 3, 545–569. Also available at arXiv:1108.5245v2.

RuWa15+. David B. Rush and Kelvin Wang, *On orbits of order ideals of minuscule posets II: Homomesy*, arXiv:1509.08047v1.

Sag11. Bruce E. Sagan, *The cyclic sieving phenomenon: A survey*, London Math. Soc. Lecture Note Ser. **392**, Cambridge Univ. Press, 2011. Also available at arXiv:1008.0790v3.

Sage14. William A. Stein et. al., *Sage Mathematics Software (Version 6.2.beta2)*, The Sage Development Team (2014), http://www.sagemath.org.

Sage08. The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics* (2008).

STW^{2}15. Anne Schilling, Nicolas M. Thiéry, Graham White, Nathan Williams, *Braid moves in commutation classes of the symmetric group* (2015), arXiv:1507.00656v2.

Sch72. Schützenberger Marcel-Paul, *Promotion des morphisms d'ensemble ordonnes*, Disc. Math. **2** (1972), 73–94.

Stan86. Richard P. Stanley, *Two Poset Polytopes*, Discrete & Computational Geometry (1986), Volume 1, Issue 1, 9–23.

Stan11. Richard P. Stanley, *Enumerative Combinatorics, volume 1, 2nd edition*, no. 49 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2011. Also available at http://math.mit.edu/~rstan/ec/ec1/.

Sta99. Richard P. Stanley, *Enumerative Combinatorics Volume 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.

Sta09. Richard P. Stanley, *Promotion and Evacuation*, Electron. J. Combin. **16(2)** (2009), #R9, http://www.combinatorics.org/ojs/index.php/eljc/article/download/v16i2r9/pdf.

Ste01. John R. Stembridge, *Minuscule elements of Weyl groups*, J. Algebra **235** (2001), 722–743.

Ste94. John R. Stembridge, *On minuscule representations, plane partitions and involutions in complex lie groups*, Duke Math. J. **73(2)** (1994), 469–490.

Ste94b. John R. Stembridge, *Some hidden relations involving the ten symmetry classes of plane partitions*, J. Combin. Theory Ser. A **68** (1994), 372–409.

Str15. Jessica Striker, *The toggle group, homomesy, and the Razumov-Stroganov correspondence*, Electron. J. Combin. **22(2)** (2015), #P2.57, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i2p57/3248. Also available at arXiv:1503.08898v1.

Str15+. Jessica Striker, *Rowmotion and generalized toggle groups* (2016), arXiv:1601.03710.

StWi11. Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942, http://www.sciencedirect.com/science/article/pii/S0195669812000972. Also available at arXiv:1108.1172v3.

Su02. Ruedi Suter, *Young's Lattice and Dihedral Symmetries*, Europ. J. Combin. **23** (2002), 233–238.

ThoYo07. Hugh Thomas and Alexander Yong, *Cominuscule tableau combinatorics*, (2007), arXiv:0701215v2.

Too81. Andrei Toom, *Problem solution M655*, Kvant (Quantum), **7** (1981), 28–30, http://kvant.mccme.ru/1981/07/p28.htm.

Volk06. Alexandre Yu. Volkov, *On the Periodicity Conjecture for Y-systems*, Commun. Math. Phys. 276 (2007), 509–517. Preprint also available, under the name *On Zamolodchikov's Periodicity Conjecture*, at arXiv:hep-th/0606094v1.

Will14. Lauren Williams, *Cluster Algebras: An Introduction*, Bull. AMS **51(1)** (2014), 1–26, http://www.ams.org/journals/bull/2014-51-01/S0273-0979-2013-01417-4/S0273-0979-2013-01417-4.pdf.

Will13. Nathan Williams, *Cataland*, dissertation, University of Minnesota, August 2013, https://conservancy.umn.edu/bitstream/159973/1/Williams_umn_0130E_14358.pdf.