Combinatorial Actions & Homomesic Orbit Averages

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Describing joint research with Jim Propp

Mathematics Colloquium
Dartmouth College
Hanover, NH, USA

23 May 2013

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

Abstract

We consider a variety of combinatorial actions on finite sets which have interesting unexpected properties. Starting with simple examples such as cyclic rotation of binary strings, we generalize to actions on Young tableaux and order ideals of other partially ordered sets. We identify a particular phenomenon called "homomesy" appearing in many unrelated combinatorial contexts: namely that the average value of some natural statistic over each orbit is the same as the average over the entire set. Viewing these actions as products of "toggle operations" allows us to see how some of these actions are related and to extend much of this picture more broadly to interesting non-combinatorial actions, such as a piecewise-linear action on the order polytope of a poset.

Acknowledgments

This colloquium largely discusses recent work with Jim Propp, including ideas and results from Arkady Berenstein, David Einstein, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Darij Grinberg, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.

Overview

- Interesting actions on subsets: rotation and winching;
- Unexpected averaging properties;
- Actions on antichains and order ideals in posets; and
- Generalizing to other categories.

Please interrupt with questions!

Subset Rotation

Set $[n] := \{1, 2, \ldots n\}$.
$\binom{[n]}{k} = \{k\text{-element subsets of } [n]\}$.
For example, $\binom{[7]}{3}$ consists of 35 subsets (dropping braces & commas): 123, 124, 125, 126, 127, 134, 135, 136, 137, 145, 146, 147, 156, 157, 167, 234,...

Consider the operation that adds 1 to each element mod n.

$$156 \mapsto 267 \mapsto 137 \mapsto 124 \mapsto 235 \mapsto 346 \mapsto 457 \mapsto 156$$

It's easy to see that the cardinality of each orbit of this action is a divisor of n.

Winching

Winching is an action on $\binom{[n]}{k} = \{k\text{-element subsets of } [n]\}$ designed by Propp and named by Winkler. The rules are: Act on each element $k \in S$ from *Left to Right*, increment k if the result is still a legitimate subset, otherwise, allow k to fall back until it runs up against the element below. More formally:

IF

$k+1 \notin S \cup \{n+1\}$, replace k by $k+1$ in S;

ELSE $k \mapsto 1 + \max\{\{0\} \cup \{i \in S, i < k\}\}$.

$$134 \mapsto 235 \mapsto 146 \mapsto 257 \mapsto 367 \mapsto 456 \mapsto 127 \mapsto 134$$

Q: Is it obvious we return to where we started?

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Q: Is it obvious we return to where we started?
Q: Well, is the action invertible?
A: Yes! But why?

Invertibility of Winching

We can think of winching as being a composition of cyclic operators: namely "winching at a location", where we increase a number within the (open) interval of its surrounding values if possible, falling back to the lowest value if not, e.g., winching at location 2 cycles among subsets of the form $1, a, 5$ as follows

$$1, 2, 5 \mapsto 1, 3, 5 \mapsto 1, 4, 5 \mapsto 1, 2, 5$$

In particular, since these operators are invertible, so is the composition.

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Q: Why should the order of winching be the same as that for cyclic rotation?

Winching Right to Left

What if we winch from *right to left* with regular gravity (so a number which cannot increment still fall back to the left)?

$$156 \mapsto 267 \mapsto 137 \mapsto 124 \mapsto 235 \mapsto 346 \mapsto 457 \mapsto 156$$

Q: Does this look familiar?

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When S avoids $n = 7$, winching R to L is cyclic rotation, since no element runs into an obstacle moving to the right. On the other hand, if $n \in S = \{a_1, a_2, \ldots, a_{k-1}, n\}$, then n falls back to $a_{k-1} + 1$, a_{k-1} falls back to $a_{k-2} + 1, \ldots$, and finally a_1 falls back to 1, which also gives the same result as cyclic rotation.

Order of LR winching

It turns out that winching from left to right is *also* an operation of order n on $\binom{[n]}{k}$, which is certainly not obvious from the definition. Proving this involves a bit more work.

Unexpected Orbit Averages

Certain statistics on these subsets exhibit an unexpected property: the sum of the first and last elements, or generally of the ith and $k+1-i$th elements, give the same average along each orbit.

$u_1 u_2 u_3$	$u_1 + u_3$	$v_1 v_2 v_3 v_4$	$v_1 + v_4$	$v_2 + v_3$
134	5	1347	8	7
235	7	2356	8	8
146	7	1457	8	9
257	9	2367	9	9
367	10	1456	7	9
456	10	2347	9	7
127	8	1256	7	7
AVG=	8	AVG=	8	8

In each case the average sums are $8 = n + 1$.

Unexpected Orbit Averages: Why?

Sketch of Proof (Haddadan): Within each table, the numbers from 1 to n can be grouped into *snakes* moving right or (cylindrically) down. It's not hard to see that each *segment* (restriction of a snake to column i) is balanced by a segment with complementary values in column $k+1-i$.

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Winching on all subsets

With a small modification, we can define a winching action W that acts on all of $2^{[n]}$. Represent $S = \{x_1 < x_2 < \cdots < x_k\} \subseteq [n]$ by the n-tuple (padding with $n - k$ initial zeroes):

$$00 \cdots 0 x_1 x_2 \cdots x_k$$

We still increment each component from L to R when possible, otherwise fall back as far as possible, *including to 0 (even if there are other zeroes)*.

EG, if $n = 5$, $S = \{2, 3, 5\}$, then the orbit begins:

$$00235 \mapsto 01245 \mapsto 00345 \mapsto 01234 \mapsto 00005 \mapsto 00012 \mapsto \cdots$$

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A: Get cycle of length 2: $00035 \mapsto 00125$.

Orbit averages for winching on $2^{[n]}$

Here's how the 2^5 subsets fall into orbits under power-set winching.

00014	00000	00004	00034
00025	00001	00015	00125
00134	00002	00023	
00235	00013	00124	
01245	00024	00035	
00345	00135	00145	
01234	00245	00234	
00005	01345	01235	
00012	02345	00045	
00003	12345	00123	

Prop (Haddadan): Each $k \in [n]$ occurs in exactly half the strings in each orbit.

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00005	01345	01235	
00012	02345	00045	
00003	12345	00123	

Prop (Haddadan): Each $k \in [n]$ occurs in exactly half the strings in each orbit.

Exercise: Use Shahrzad's snakes to prove this.

Rotation of bit-strings

We find another unexpected structure in the orbits if we consider $S = \binom{[n]}{k}$, as length n binary strings with k 1's as we cyclically shift them. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$, and count $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

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$1001 \mapsto 2$	$1010 \mapsto 3$
$1100 \mapsto 4$	$0101 \mapsto 1$
$0110 \mapsto 2$	
$0011 \mapsto 0$	

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0011	0101
$1001 \mapsto 2$	$1010 \mapsto 3$
$1100 \mapsto 4$	$0101 \mapsto 1$
$0110 \mapsto 2$	$\text{AVG} = \frac{4}{2} = 2$
$0011 \mapsto 0$	
$\text{AVG} = \frac{8}{4} = 2$	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

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000011	000101	

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$110000 \mapsto 8$	$010001 \mapsto 3$	$010010 \mapsto 4$
$011000 \mapsto 6$	$101000 \mapsto 7$	$001001 \mapsto 2$
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	

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AVG$= \frac{24}{6} = 4$	AVG$= \frac{24}{6} = 4$	AVG$= \frac{12}{3} = 4$

Note that in each of the three orbits average of the statistic φ is the same.

Main definition: Homomesic

MAIN DEF: Given an (invertible) action τ on a finite set of objects S, call a statistic $\varphi : S \to \mathbf{C}$ **homomesic** with respect to (S, τ) iff the average of φ over each τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s)$ does not depend on the choice of $\mathcal{O}$.

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Equivalently: the average of φ over each τ-orbit $\mathcal{O}$ is the same as the average over the entire set S:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

So we can compute what the average should be before checking whether a statistic is homomesic. Examples so far:

1. Bitstring rotation with the inversion statistic;
2. Winching $\binom{[n]}{k}$ with "sum of opposite coords" statistic; and
3. Winching $2^{[n]}$ with "k occurs in string" statistic.

Partitions & Tableaux

A **partition** λ **of** n is a sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots \lambda_l)$$

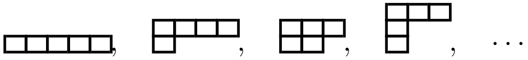
such that:

1. The terms are weakly decreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$
2. $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$

Suppressing commas, we have seven partitions of 5:

$$5, \quad 41, \quad 32, \quad 311, \quad 221, \quad 2111, \quad 11111$$

represented visually as left-justified shapes (whose squares are called **cells**):



Fix a positive integer N. A **near standard Young tableau (NYST) of shape λ and ceiling N**, is a labeling of the cells of a partition λ with distinct numbers from $1, 2, \ldots, N$ which increases along each row and column. For example,

1	4	6	10
2	8	11	
7	9		

,

1	2	3	5	7	9
4	8	10	13		
6	11	12			

but not

1	2	3	5	7	11
4	8	12	13		
6	9	10			

Such labelings are in *easy* bijection with sequences ("chains") of shapes that grow one box in each step. For example, the first tableau above corresponds to:

Promotion of Near-Standard Young Tableaux

For each $1 \leq i \leq N-1$, let s_i be the action on NSYT's with ceiling N that replaces i (if it occurs in T) by $i+1$, and vice versa, provided that this does not violate the increasing condition in the definition of Young tableaux, and let ∂ be the composition of the maps:

$$\partial T := s_{N-1} \circ s_{N-2} \circ \cdots \circ s_1 T$$

This generalizes an operation on SYT introduced by Schützenberger called **promotion**.
For example, applying s_7 transforms the following tableau as shown:

1	4	7	10
2	8	11	
6	9		

$\xrightarrow{s_7}$

1	4	8	10
2	7	11	
6	9		

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|} \hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} \end{array} ,$$

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$$\begin{array}{cccc} 1 & 3 & 6 & 10 \\ 5 & 8 & 11 & \\ 7 & 9 & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 5 & 10 \\ 6 & 8 & 11 & \\ 7 & 9 & & \end{array},$$

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$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline 5 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array},$$

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10}s_9\cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{2} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{3} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{4} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\ ,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{5} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline \color{red}{6} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} \color{red}{7} & 9 \\ \cline{1-2}\end{array}\ , \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} \color{red}{8} & 9 \\ \cline{1-2}\end{array}\ ,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} 8 & \color{red}{9} \\ \cline{1-2}\end{array}\ ,$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10}s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{2} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{3} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{4} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array},$$

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Promotion of NSYT

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$$\begin{array}{cccc} 1 & 3 & 6 & 10 \\ \color{red}{5} & 8 & 11 & \\ 7 & 9 & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 5 & 10 \\ \color{red}{6} & 8 & 11 & \\ 7 & 9 & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 5 & 10 \\ 6 & 8 & 11 & \\ \color{red}{7} & 9 & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 5 & 10 \\ 6 & 7 & 11 & \\ \color{red}{8} & 9 & & \end{array},$$

$$\begin{array}{cccc} 1 & 3 & 5 & 10 \\ 6 & 7 & 11 & \\ 8 & \color{red}{9} & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 5 & 9 \\ 6 & 7 & 11 & \\ 8 & \color{red}{10} & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 5 & 9 \\ 6 & 7 & 10 & \\ 8 & \color{red}{11} & & \end{array} = \partial T.$$

A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942; http://arxiv.org/abs/1108.1172):

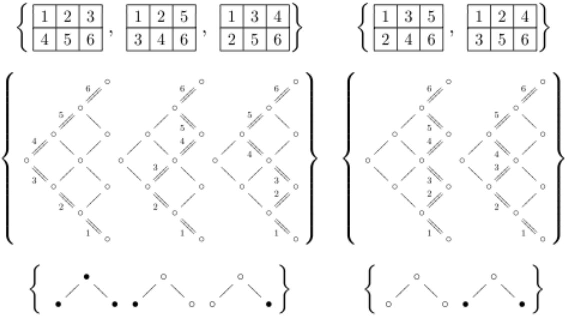


Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

A small example of promotion: centrally symmetric sums

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942 1927

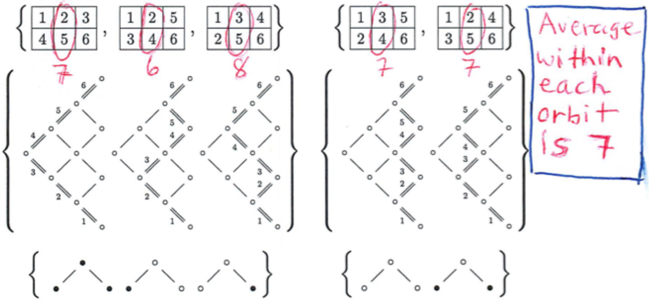


Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

Conjecture

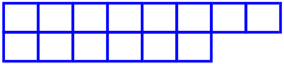
Let S be the set of Near-Standard Young Tableau of rectangular shape λ, and ceiling N. If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center, (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\varphi(T)$ denotes the sum of the numbers in cells c and c', then φ is homomesic with respect to (S, ∂) with average value $N + 1$.

Promotion of Near-Standard Young Tableaux: homomesies

Conjecture

Let S be the set of Near-Standard Young Tableau of rectangular shape λ, and ceiling N. If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center, (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\varphi(T)$ denotes the sum of the numbers in cells c and c', then φ is homomesic with respect to (S, ∂) with average value $N + 1$.

Although rectangular shapes may appear to be a very special case, they are one of the few shapes where the order of promotion on the set of SYT is small, i.e., n or $2n$. Striker & Williams point out that the order of promotion on SYT of shape $(8, 6)$ is 7,554,844,752.



Basics of partially ordered sets

A **partially ordered set** (or **poset**) is a set with a binary relation $\leq$ which satisfies the following properties for every $a, b, c \in P$:
(a) *reflexive* $(a \leq a)$,
(b) *antisymmetric* $(a \leq b$ and $b \leq a \implies a = b)$, and
(c) *transitive* $(a \leq b$ and $b \leq c \implies a \leq c)$. All our posets will be finite and can be represented by directed graphs of minimal covering relations, as I'll draw on the board.

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An **order ideal** $I \subseteq P$ is any set with the property that $z \in I$ and $x \leq z$ in $P \implies x \in I$.

Rowmotion: an invertible operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

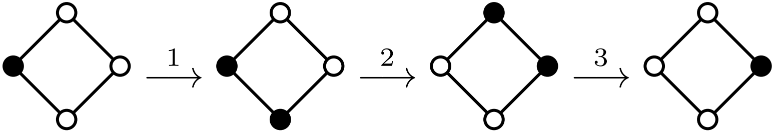
Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the downward-saturation of A.
τ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams. Following the latter we call this **rowmotion**.

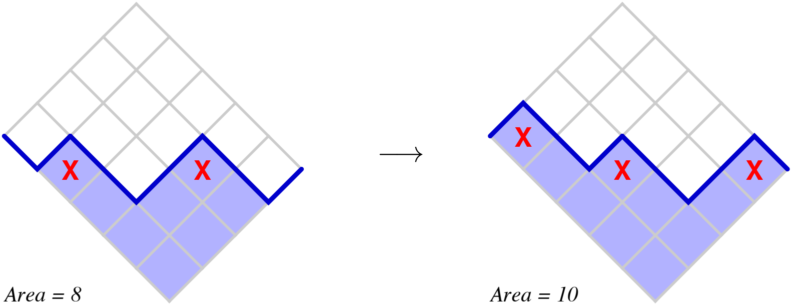
An example

1. Saturate downward
2. Complement
3. Take minimal element(s)



Example in lattice cell form

Viewing elements of the poset as **squares** below, we would map:



Panyushev's conjecture

Let Δ be a reduced irreducible root system in $\mathbf{R}^n$. (Picture coming soon!)
Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.
Conjecture (Conjecture 2.1(iii) in D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. 30 (2009), 586-594): Let $\mathcal{O}$ be an arbitrary τ-orbit. Then

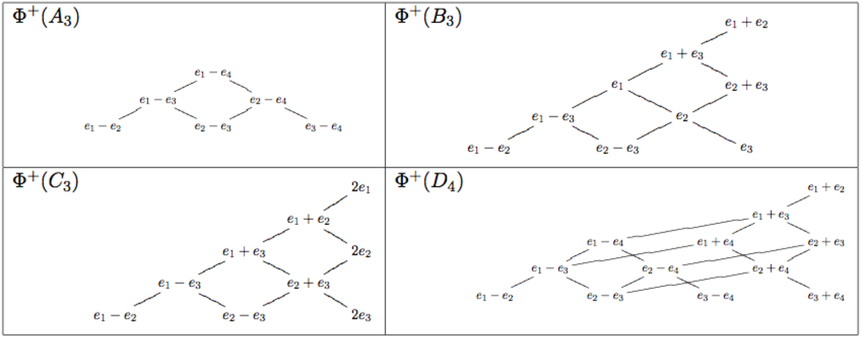
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Panyushev's Conjecture 2.1(iii) (along with much else) was proved by Armstrong, Stump, and Thomas in their article *A uniform bijection between nonnesting and noncrossing partitions*, http://arxiv.org/abs/1101.1277.

Picture of root posets

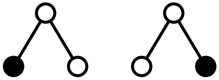
Here are the classes of posets included in Panyushev's conjecture.



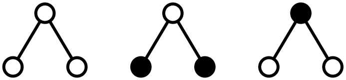
(Graphic courtesy of Striker-Williams.)

Panyushev's conjecture: The A_n case, $n = 2$

Here we have just an orbit of size 2 and an orbit of size 3:



$1 \qquad 1$

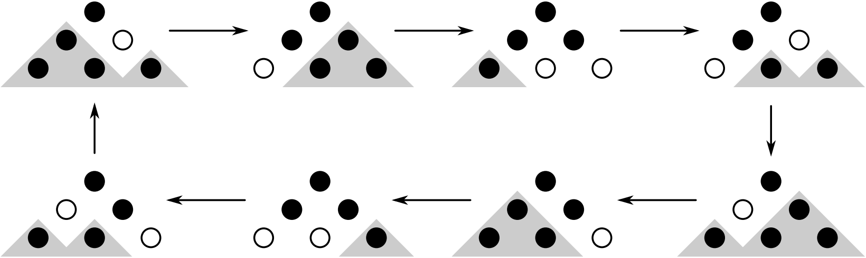


$0 \qquad 2 \qquad 1$

Within each orbit, the average antichain has cardinality $n/2 = 1$.

The case A_3.

Here's an example orbit taken from [AST] for the A_3 root poset:



For A_3 this action has three orbits (sized 2, 4, and 8), and the average cardinality of an antichain is

$$\frac{1}{8}(2+1+1+2+2+1+1+2)=\frac{3}{2}$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (where $[k]$ denotes the linear ordering of $\{1, 2, \ldots, k\}$):

Theorem (Propp, R.)

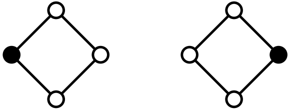
Let $\mathcal{O}$ be an arbitrary τ-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

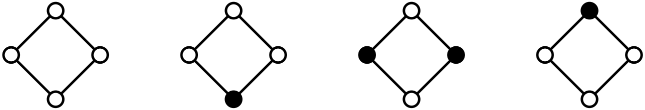
This is an easy consequence of unpublished work of Hugh Thomas building on earlier work of Richard Stanley: see the last paragraph of section 2 of R. Stanley, *Promotion and evacuation*, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i2r9 .

Antichains in $[a] \times [b]$: the case $a = b = 2$

Here we have an orbit of size 2 and an orbit of size 4:



1 1



0 1 2 1

Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

Antichains in $[a] \times [b]$: centrally symmetric homomesies

Theorem (Propp, R.)

In any orbit, the number of A that contain (i,j) equals the number of A that contain the opposite element $(i',j') = (a+1-i, b+1-j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under τ, with average value 0 in each orbit.

Ideals in $[a] \times [b]$: cardinality is homomesic

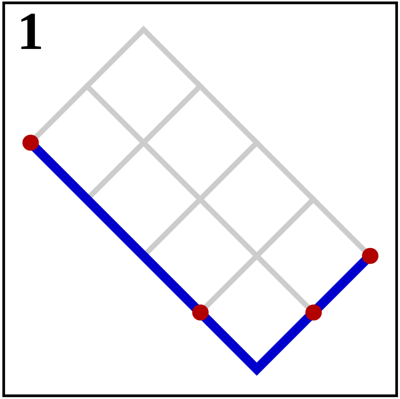
As we've seen, one can view rowmotion as acting either on *antichains* ($\mathcal{A}(P)$) or on *order ideals* ($J(P)$); we denote the latter map $\overline{\tau}$. It turns out that the cardinality of the order ideal is also homomesic w.r.t. rowmotion.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\overline{\tau}$-orbit in $J([a] \times [b])$. Then

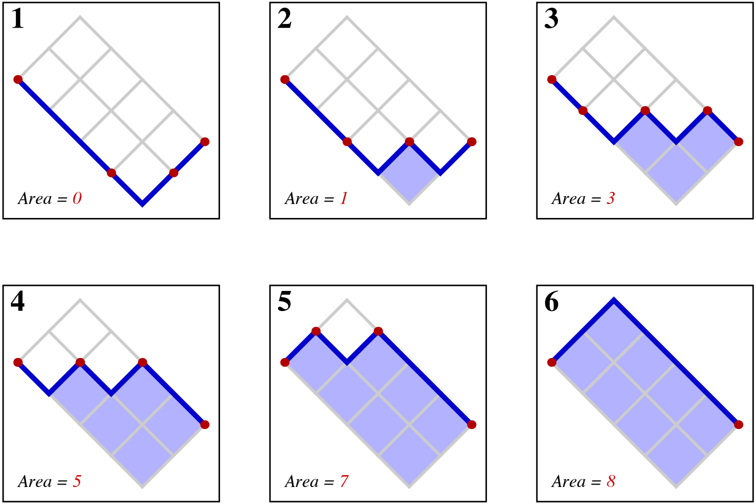
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

It's worth noting even though there's a strong connection between the rowmotion map on antichains and on order ideals, that the homomesy situation could be quite different.

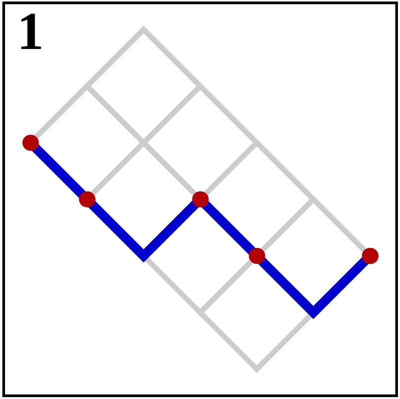


Rowmotion

Rowmotion on $[4] \times [2]$ A

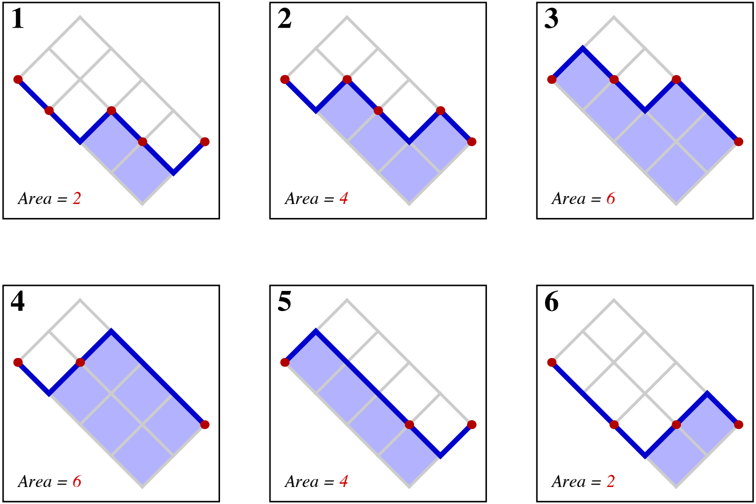


$$(0+1+3+5+7+8)\ /\ 6 = 4$$



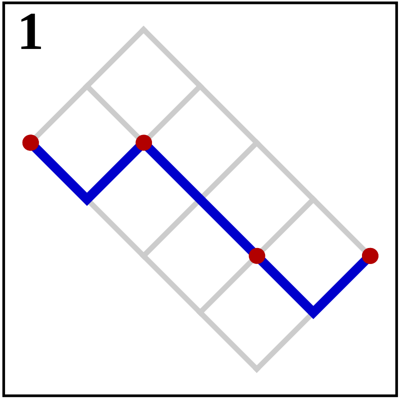
Rowmotion

Rowmotion on $[4] \times [2]$ B



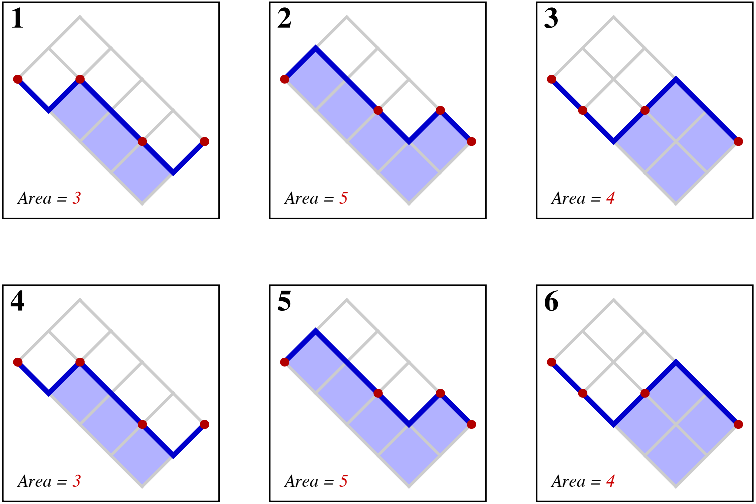
$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ C



Rowmotion

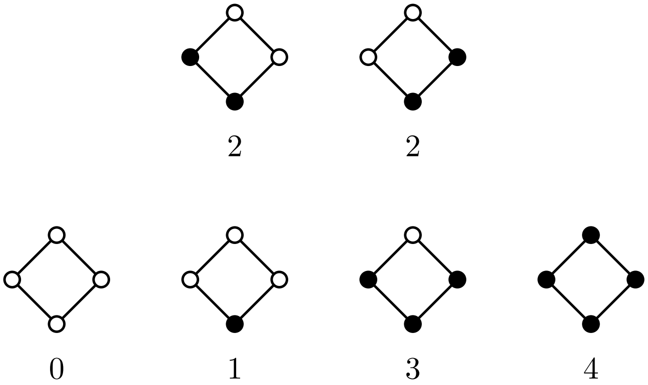
Rowmotion on $[4] \times [2]$ C



$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: centrally symmetric homomesies

We also have homomesies for more refined statistics than $\#I$. Given $(i,j) \in [a] \times [b]$, and I an ideal in $[a] \times [b]$, define the *indicator function* $\overline{1}_{i,j}(I)$ to be 1 or 0 according to whether or not I contains (i,j).

Write $(i',j') = (a+1-i, b+1-j)$, the point opposite (i,j) in the poset.

Theorem (Propp, R.)

$\overline{1}_{i,j} + \overline{1}_{i',j'}$ *is homomesic under* $\overline{\tau}$.

Toggling

In their 1995 article *Orbits of antichains revisited*, European J. Combin. 16 (1995), 545–554, Cameron and Fon-der-Flaass give an alternative description of $\overline{\tau}$.

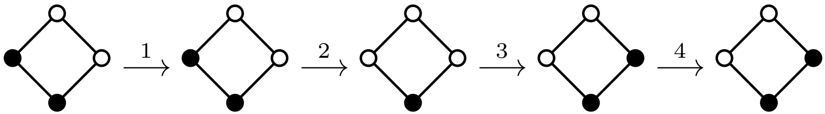
Given $I \in J(P)$ and $x \in P$, let $\tau_x(I) = I \triangle \{x\}$ (symmetric difference) provided that $I \triangle \{x\}$ is an order ideal of P; otherwise, let $\tau_x(I) = I$.

We call the involution τ_x "toggling at x".

The involutions τ_x and τ_y commute unless x covers y or y covers x.

An example

1. Toggle the top element
2. Toggle the left element
3. Toggle the right element
4. Toggle the bottom element



Theorem (Cameron and Fon-der-Flaass): Let $x_1, x_2, \ldots, x_n$ be any order-preserving enumeration of the elements of the poset P. Then the action on $J(P)$ given by the composition $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ coincides with the action of $\overline{\tau}$.

In the particular case $P = [a] \times [b]$, we can enumerate P rank-by-rank; that is, we can list the (i, j)'s in order of increasing $i + j$.

Note that all the involutions coming from a given rank of P commute with one another, since no two of them are in a covering relation.

Striker and Williams refer to $\overline{\tau}$ (and τ) as **rowmotion**, since for them, "row" means "rank".

Toggling from side to side

Define a **file** in $P = [a] \times [b]$ to be the set of all $(i, j) \in P$ with $i - j$ equal to some fixed value k.

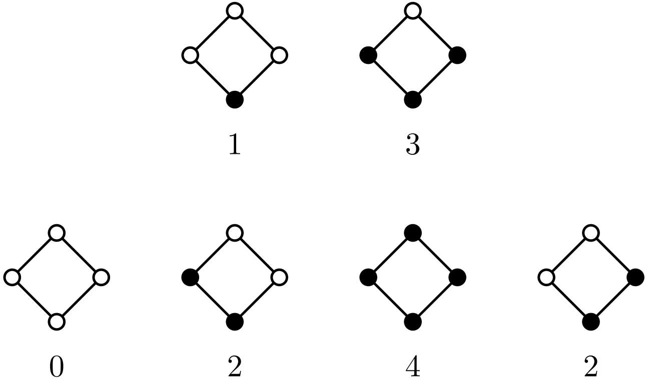
Note that all the involutions coming from a given file commute with one another, since no two of them are in a covering relation.

It follows that for any enumeration $x_1, x_2, \ldots, x_n$ of the elements of the poset $[a] \times [b]$ arranged in order of increasing $i - j$, the action on $J(P)$ given by $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ doesn't depend on which enumeration was used.

Striker and Williams call this well-defined composition **promotion**, and denote it by ∂, since for two-rowed tableaux it can be related to Schützenberger's promotion on SYT, described earlier.

Promoting ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



$J([a] \times [b])$: cardinality is homomesic under promotion

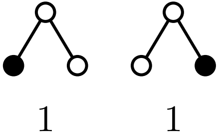
Claim (Propp, R.): Let $\mathcal{O}$ be an arbitrary orbit in $J([a] \times [b])$ under the action of promotion ∂. Then

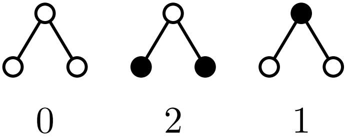
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

The result about cyclic rotation of binary words discussed earlier turns out to be a special case of this.

Root posets of type A: antichains

Recall that, by the Armstrong-Stump-Thomas theorem, the cardinality of antichains is homomesic under the action of rowmotion, where the poset P is a root poset of type A_n.
E.g., for $n = 2$:



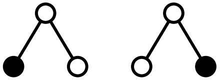


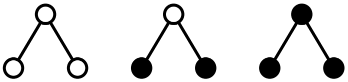
Antichain-cardinality is homomesic: in each orbit, its average is 1.

Root posets of type A: order ideals

What if instead of antichains we take order ideals?

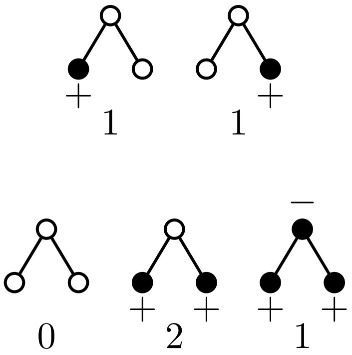
E.g., $n = 2$:





What is homomesic here?

Root posets of type A: rank-signed cardinality



Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan): Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign an order ideal $I \in J(P)$ weight $\varphi(I) = \sum_{x \in I} \mathrm{wt}(x)$, then φ is homomesic under rowmotion and promotion, with average $n/2$.

The order polytope of a poset

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.
We can generalize our entire setup of toggle operators and "rowmotion" to operate on these functions (the "continuous piecewise-linear (CPL) category").

Flipping-maps in the order polytope

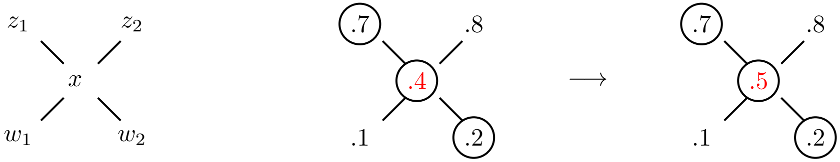
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$;
the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example of flipping at a node



$$\min_{z \cdot > x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Flipping and toggling

If we associate each order-ideal I with the indicator function f of $P \setminus I$ (that is, the function that takes the value 0 on I and the value 1 everywhere else), then toggling I at x is tantamount to flipping f at x.

That is, we can identify $J(P)$ with the vertices of the polytope $\mathcal{O}(P)$ in such a way that toggling can be seen to be a special case of flipping.

This may be clearer if you think of $J(P)$ as being in bijection with the set of monotone 0,1-valued functions on P.

Flipping

Flipping (at least in special cases) is not new, though it is not well-studied; the most worked-out example we've seen is Berenstein and Kirillov's article *Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux* (St. Petersburg Math. J. 7 (1996), 77–127); see http://pages.uoregon.edu/arkadiy/bk1.pdf.

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

	.8			.6			.6	
.4		.3	.4		.3	.3		.3
	.1			.1			.1	

	.6			.6	
.3		.4	.3		.4
	.1			.2	

(Here we successively flip values at the North, West, East, and South.)

Example of CPL rowmotion

Two orbits of CPL rowmotion (flipping values from top to bottom):

	.7			.7			.9			.9	
.2		.4	.6		.4	.6		.8	.6		.4
	.1			.3			.3			.1	

	1			1			1			1	
1		0	0		1	1		0	0		1
	0			0			0			0	

The average at each node across the respective orbits is:

	.8			1	
.5		.5	0.5		0.5
	.2			0	

It appears that all of the aforementioned results on homomesy for rowmotion and promotion on $J([a] \times [b])$ lift to corresponding results in the order polytope, where instead of composing toggle-maps to obtain rowmotion and promotion we compose the corresponding flip-maps to obtain continuous piecewise-linear maps from $\mathcal{O}([a] \times [b])$ to itself.

The first step would be to show that rowmotion and promotion on $\mathcal{O}([a] \times [b])$, defined as above, are maps of order $a + b$.

Order of flipping affects order of the composition!

In the combinatorial setting, where $\mathcal{A}(P)$ and $J(P)$ are finite, it's clear that any map defined as a product of toggles has finite order. But we can no longer take this for granted in the CPL setting.

Let $P = [2] \times [2]$. As we'll soon see, one can show by brute force that the CPL map

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,1)} \circ \sigma_{(2,2)}$$

("lifted rowmotion") is of order four, as is

$$\sigma_{(2,1)} \circ \sigma_{(1,1)} \circ \sigma_{(2,2)} \circ \sigma_{(1,2)}$$

("lifted promotion"). However, not every composition of flips has finite order.

Proposition (Einstein): The CPL map

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,2)} \circ \sigma_{(2,1)}$$

(flipping values in clockwise order, as opposed to going by rows or columns of P) is of infinite order.

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the continuous piecewise-linear (CPL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbf{R}(x)$ of *rational functions* to the nodes of the poset (using that $\min(z_i) = -\max(-z_i)$) to get

$$f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Example of birational rowmotion

In our running example, $P = [2] \times [2]$, applying these new flip operators from top to bottom creates a new rowmotion operator. (Here we assign $f(\hat{0}) = f(\hat{1}) = 1$.)

$$\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array} \mapsto \begin{array}{ccc} & \color{red}{\frac{x+y}{z}} & \\ x & & y \\ & w & \end{array} \mapsto \begin{array}{ccc} & \frac{x+y}{z} & \\ \color{red}{\frac{w(x+y)}{xz}} & & y \\ & w & \end{array} \mapsto$$

$$\begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \color{red}{\frac{w(x+y)}{yz}} \\ & w & \end{array} \mapsto \begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & \color{red}{\frac{1}{z}} & \end{array}$$

Example of birational rowmotion orbit

Here's an orbit of rowmotion in this category:

$$
\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array}
\mapsto
\begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & \frac{1}{z} & \end{array}
\mapsto
\begin{array}{ccc} & \frac{w(x+y)}{xy} & \\ \frac{1}{y} & & \frac{1}{x} \\ & \frac{z}{x+y} & \end{array}
\mapsto
$$

$$
\begin{array}{ccc} & \frac{1}{w} & \\ \frac{yz}{w(x+y)} & & \frac{xz}{w(x+y)} \\ & \frac{xy}{w(x+y)} & \end{array}
\mapsto
\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array}
$$

In this category, geometric means replace arithmetic means, so let's compute the **product** of the function values at each node.

$$\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array} \mapsto \begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & \frac{1}{z} & \end{array} \mapsto \begin{array}{ccc} & \frac{w(x+y)}{xy} & \\ \frac{1}{y} & & \frac{1}{x} \\ & \frac{z}{x+y} & \end{array} \mapsto$$

$$\begin{array}{ccc} & \frac{1}{w} & \\ \frac{yz}{w(x+y)} & & \frac{xz}{w(x+y)} \\ & \frac{xy}{w(x+y)} & \end{array} \qquad \text{PROD} = \begin{array}{ccc} & \frac{(x+y)^2}{xy} & \\ 1 & & 1 \\ & \frac{xy}{(x+y)^2} & \end{array}$$

Geometric homomesy with boundary variables

If we instead generically assign variables $f(\hat{0}) = \alpha$ and $f(\hat{1}) = \omega$:

$$
\begin{array}{ccc} & \omega & \\ & z & \\ x & & y \\ & w & \\ & \alpha & \end{array}
\mapsto
\begin{array}{ccc} & \frac{(x+y)\omega}{z} & \\ \frac{w(x+y)\omega}{xz} & & \frac{w(x+y)\omega}{yz} \\ & \frac{\alpha\omega}{z} & \end{array}
\mapsto
\begin{array}{ccc} & \frac{w(x+y)\omega}{xy} & \\ \frac{\alpha\omega}{y} & & \frac{\alpha\omega}{x} \\ & \frac{\alpha z}{x+y} & \end{array}
$$

$$
\begin{array}{ccc} & \frac{\alpha\omega}{w} & \\ \frac{\alpha yz}{w(x+y)} & & \frac{\alpha xz}{w(x+y)} \\ & \frac{\alpha xy}{w(x+y)} & \end{array}
\quad \text{PROD} =
\begin{array}{ccc} & \alpha\omega^3 \frac{(x+y)^2}{xy} & \\ \alpha^2\omega^2 & & \alpha^2\omega^2 \\ & \alpha^3\omega \frac{xy}{(x+y)^2} & \end{array}
$$

So the statistic "multiply opposite nodes" has *geometric mean* $\alpha\omega$ across the orbit.

For what posets does this work?

It's not hard to see that if a map such as rowmotion is homomesic with respect to some statistics in the birational (geometric) setting, then this implies homomesy at the CPL level, which in turn implies it in the combinatorial setting (which is the only level at which we currently have proofs).

We believe that geometrical versions of homomesy in the birational category holds for a large class of posets, often ones that come up in representation theory. There are also simple examples of posets, e.g., the Boolean algebra B_3 for which nothing we have tried appears to hold. For example, it appears (conjecturally) that birational rowmotion has infinite order on B_3.

Conclusion

- A recently identified phenomenon called *homomesy* appears to be lurking in a wide range of combinatorial settings.
- We are just beginning to develop tools for studying this, so there are many interesting open problems.
- There are intriguing conjectured generalizations to continuous piecewise-linear maps on order polytopes and to "birational" maps on $\{f : P \to \mathbf{R}(x_1, x_2, \ldots x_n)\}$.

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References 2

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The last slide of this talk

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

For more information, see:

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Thanks for your attention!