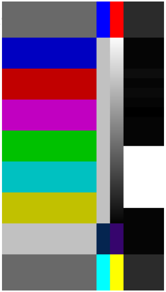
Birational Rowmotion

Tom Roby (University of Connecticut)

Describing joint research with Darij Grinberg

Workshop on Dynamical Algebraic Combinat



American Institute of Mathematics

San Jose, CA, USA

24 March 2015

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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Abstract

Earlier speakers at this workshop have described the classical (combinatorial) rowmotion map on finite posets, and its generalization to the piecewise-linear (PL) realm. We consider here the *birational* analogue, obtained by "detropicalizing" the toggle operations of which it is composed.
As in the PL case, there's no *a priori* reason for this map to have finite order; however, we show that it still has order $p+q$ on the product $[p] \times [q]$ of two chains, as well as finite (and easily boundable) order for a wide class of forest-like ("skeletal") graded posets. Our methods of proof in the $[p] \times [q]$ case are partly based on those used by Volkov to resolve the type AA (rectangular) Zamolodchikov Periodicity Conjecture.

This seminar talk discusses recent work with Darij Grinberg, including ideas and results from Arkady Berenstein, David Einstein, Jim Propp, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures. Darij Grinberg & Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, Miriam Farber, Sam Hopkins, Svante Linusson, Gregg Musiker, Alexander Postnikov, Pavlo Pylyavskyy Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.

Please feel free to interrupt with questions, comments, or general heckling.

Outline

- Review of classical and piecewise-linear rowmotion (as quick as you please);
- Detropicalizing to birational toggles and rowmotion on a finite poset P;
- Periodicity and order of birational rowmotion on P, particularly products of chains and graded forests;
- A conjectural noncommutative generalization;
- Homomesy in the birational context;
- A sketch of some proof ideas;

Classical rowmotion is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:

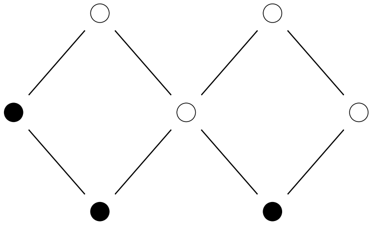
- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

Classical rowmotion: the standard definition

- Let P be a finite poset.
  Classical rowmotion is the map $\mathbf{r} : J(P) \to J(P)$ which sends every order ideal S to the order ideal obtained as follows:
  Let M be the set of minimal elements of the complement $P \setminus S$.
  Then, $\mathbf{r}(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Example:

Let S be the following order ideal ($\bullet$ = inside order ideal):



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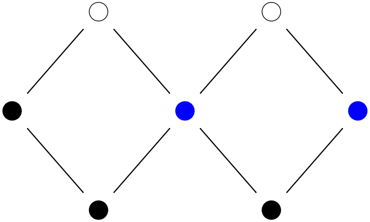
Example:
Mark M (= minimal elements of complement) blue.



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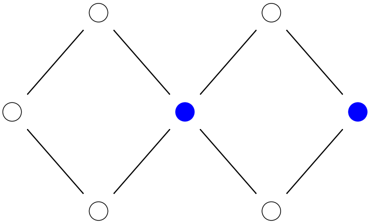
Example:
Forget about the old order ideal:



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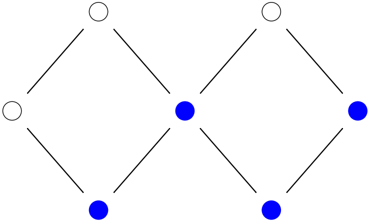
Example:
$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

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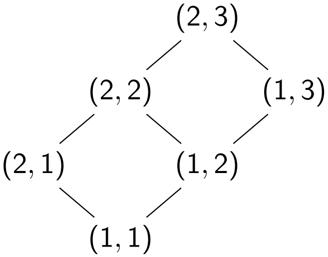
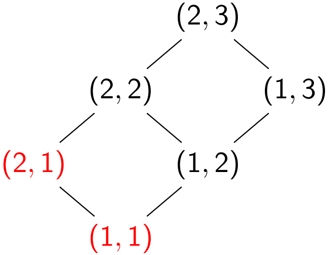
However, **for some types of** P, the order can be explicitly computed or bounded from above.
See Striker-Williams for an exposition of known results.

- If P is a $p \times q$-rectangle:



(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

Example:
Let S be the order ideal of the 2×3-rectangle given by:



Classical rowmotion: properties

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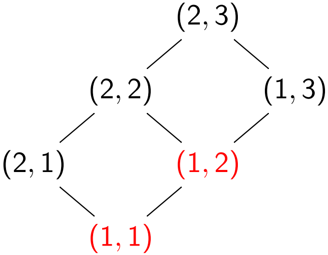
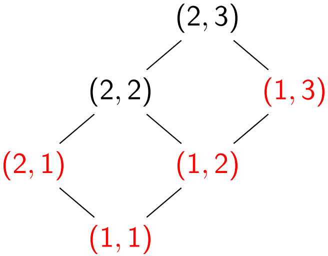
$\mathbf{r}(S)$ is



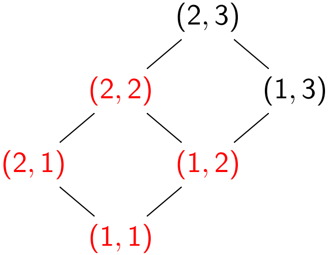
Example:

$\mathbf{r}^2(S)$ is



Classical rowmotion: properties

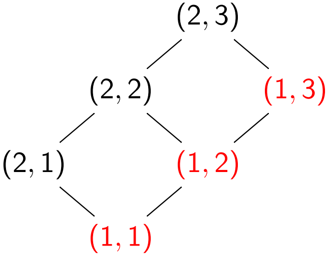
Example:
$\mathbf{r}^3(S)$ is



Classical rowmotion: properties

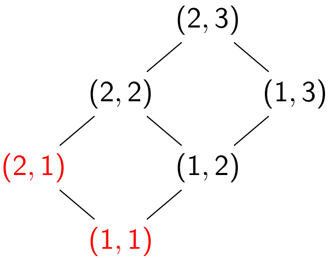
Example:

$\mathbf{r}^4(S)$ is



Example:

$\mathbf{r}^5(S)$ is



which is precisely the S we started with.

$\mathrm{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

Classical rowmotion: the toggling definition

There is an alternative definition of classical rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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  ("Try to add or remove v from S, as long as the result remains within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.

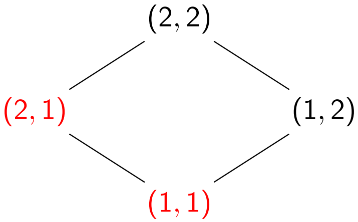
Classical rowmotion: the toggling definition

- Let $(v_1, v_2, \ldots, v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \ldots \circ \mathbf{t}_{v_n}.$$

Example:

Start with this order ideal S:



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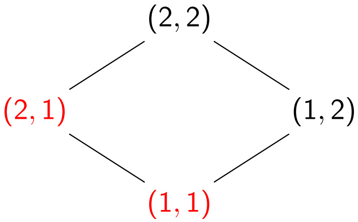
Example:
First apply $\mathbf{t}_{(2,2)}$, which changes nothing:



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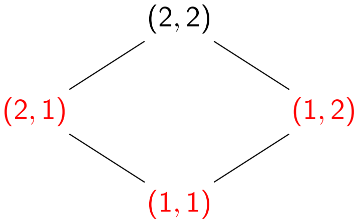
Example:
Then apply $\mathbf{t}_{(1,2)}$, which adds $(1,2)$ to the order ideal:



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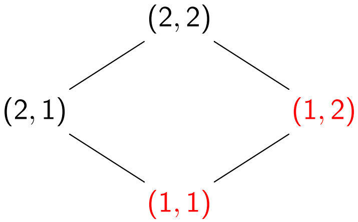
Then apply $\mathbf{t}_{(2,1)}$, which removes $(2,1)$ from the order ideal:



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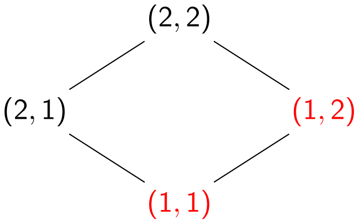
Finally apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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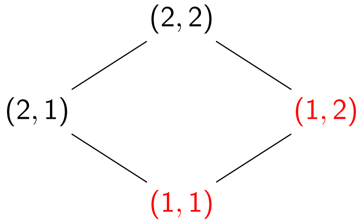
Example:
So this is $\mathbf{r}(S)$:



We can generalize this idea of composition of toggles to define a **piecewise-linear (PL)** version of rowmotion on an infinite set of functions on a poset.

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Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.
The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

Flipping-maps in the order polytope

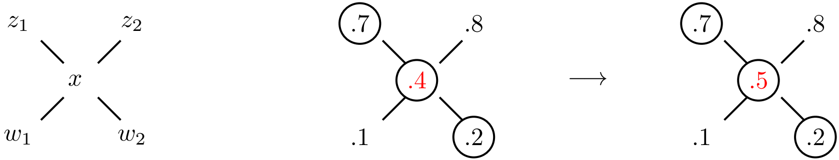
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example of flipping at a node



$$\min_{z \cdot > x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

$$
\begin{tikzcd}[column sep=small] & .8 \arrow[dl, dash] \arrow[dr, dash] & \\ .4 \arrow[dr, dash] & & .3 \arrow[dl, dash] \\ & .1 & \end{tikzcd}
\xrightarrow{\sigma_N}
\begin{tikzcd}[column sep=small] & {\color{red}.6} \arrow[dl, dash] \arrow[dr, dash] & \\ .4 \arrow[dr, dash] & & .3 \arrow[dl, dash] \\ & .1 & \end{tikzcd}
\xrightarrow{\sigma_W}
\begin{tikzcd}[column sep=small] & .6 \arrow[dl, dash] \arrow[dr, dash] & \\ {\color{red}.3} \arrow[dr, dash] & & .3 \arrow[dl, dash] \\ & .1 & \end{tikzcd}
$$

$$
\xrightarrow{\sigma_E}
\begin{tikzcd}[column sep=small] & .6 \arrow[dl, dash] \arrow[dr, dash] & \\ .3 \arrow[dr, dash] & & {\color{red}.4} \arrow[dl, dash] \\ & .1 & \end{tikzcd}
\xrightarrow{\sigma_S}
\begin{tikzcd}[column sep=small] & .6 \arrow[dl, dash] \arrow[dr, dash] & \\ .3 \arrow[dr, dash] & & .4 \arrow[dl, dash] \\ & {\color{red}.2} & \end{tikzcd}
$$

(Here we successively flip values at the North, West, East, and South.)

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0,1]$ at a point $x \in P$ with f', where

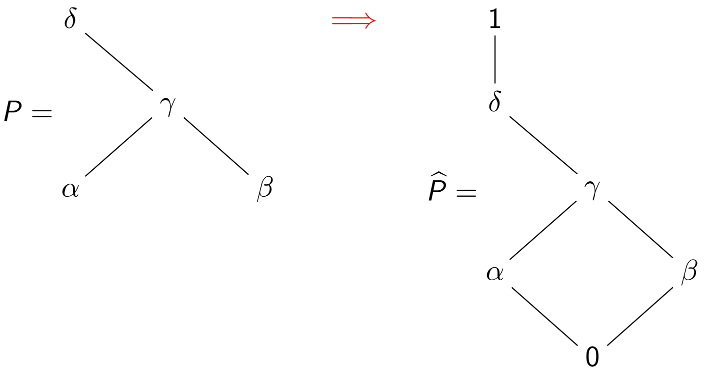
$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset (using that $\min(z_i) = -\max(-z_i)$) to get

$$f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

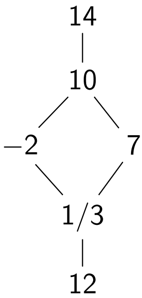
- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
  - 0 to be less than every other element, and
  - 1 to be greater than every other element.

Example:



- Let $\mathbb{K}$ be a field.
- A $\mathbb{K}$-**labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

Example: This is a $\mathbb{Q}$-labelling of the 2×2-rectangle:



Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

  for all $w \in \widehat{P}$.

- That is,
  - **invert** the label at v,
  - **multiply** by the **sum** of the labels at vertices **covered by** v,
  - **multiply** by the **parallel sum** of the labels at vertices **covering** v.

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for all $w \in \widehat{P}$.

- Notice that this is a **local change** to the label at v; all other labels stay the same.
- We have $T_v^2 = \mathrm{id}$ (on the range of T_v), and T_v is a birational map.

Birational rowmotion: definition

- We define **birational rowmotion** as the rational map

$$R := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent on the linear extension, because:

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- This is indeed independent on the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (even when they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

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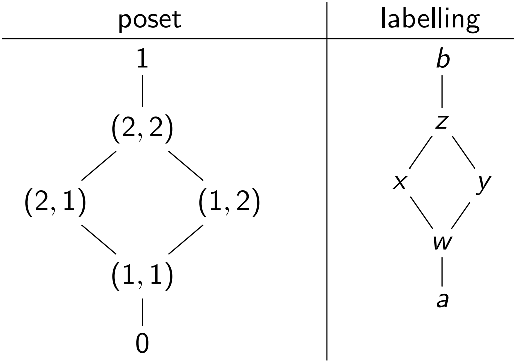
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- For more information about the lifting of rowmotion from classical to PL to birational, see, Einstein-Propp [EiPr13], where R is denoted ρ_B.

Example:
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



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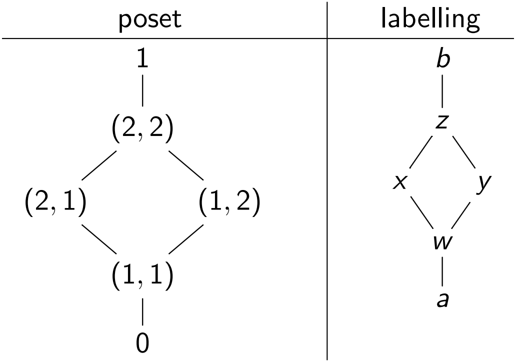

We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1), (1,2), (2,1), (2,2))$).

That is, toggle in the order "top, left, right, bottom".

Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(2,2)} f$
$\begin{tikzcd}[column sep=small] & b \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & b \arrow[d, dash] & \\ & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}$

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Birational rowmotion: example

Example:

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$$\begin{tikzcd}[column sep=small]
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 & w \arrow[d, dash] & \\
 & a &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & b \arrow[d, dash] & \\
 & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\color{red}{\frac{bw(x+y)}{xz}} \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & w \arrow[d, dash] & \\
 & a &
\end{tikzcd}$$

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ \color{red}{T_{(2,1)}} \circ T_{(2,2)}$.

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& b \arrow[d, dash] & \\
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\end{tikzcd}$$

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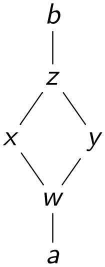
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$
$\begin{tikzcd} & b \arrow[d, no head] & \\ & z \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[dr, no head] & & y \arrow[dl, no head] \\ & w \arrow[d, no head] & \\ & a & \end{tikzcd}$	$\begin{tikzcd} & b \arrow[d, no head] & \\ & \frac{b(x+y)}{z} \arrow[dl, no head] \arrow[dr, no head] & \\ \frac{bw(x+y)}{xz} \arrow[dr, no head] & & \frac{bw(x+y)}{yz} \arrow[dl, no head] \\ & {\color{red}\frac{ab}{z}} \arrow[d, no head] & \\ & a & \end{tikzcd}$

We are using $R = {\color{red}T_{(1,1)}} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Birational rowmotion orbit on a product of chains

Example:
Iteratively apply R to a labelling of the 2×2-rectangle.
$R^0 f =$



Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

$R^1 f =$

$$
\begin{tikzcd}
 & b \arrow[d, dash] & \\
 & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{bw(x+y)}{xz} \arrow[dr, dash] & & \frac{bw(x+y)}{yz} \arrow[dl, dash] \\
 & \frac{ab}{z} \arrow[d, dash] & \\
 & a &
\end{tikzcd}
$$

Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

$R^2 f =$

$$
\begin{tikzcd}[column sep=small]
 & b \arrow[d, dash] & \\
 & \frac{bw(x+y)}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ab}{y} \arrow[dr, dash] & & \frac{ab}{x} \arrow[dl, dash] \\
 & \frac{az}{x+y} \arrow[d, dash] & \\
 & a &
\end{tikzcd}
$$

Birational rowmotion orbit on a product of chains

Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

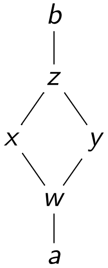
$R^3 f =$

$$
\begin{tikzcd}[arrows=dash]
 & b \arrow[d] & \\
 & \frac{ab}{w} \arrow[dl] \arrow[dr] & \\
\frac{ayz}{w(x+y)} \arrow[dr] & & \frac{axz}{w(x+y)} \arrow[dl] \\
 & \frac{xy}{aw(x+y)} \arrow[d] & \\
 & a &
\end{tikzcd}
$$

Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

$R^4 f =$



Example:
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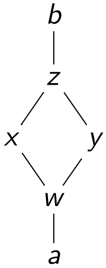

So we are back where we started.

$$\mathrm{ord}(R) = 4.$$

Birational rowmotion: order

- Let $\operatorname{ord}\phi$ denote the order of a map or rational map ϕ. This is the smallest positive integer k such that $\phi^k = \mathrm{id}$ (on the range of ϕ^k), or ∞ if no such k exists.
- A straightforward argument shows that $\operatorname{ord}(\mathbf{r}) \mid \operatorname{ord}(R)$ for every finite poset P.
- Do we have equality?

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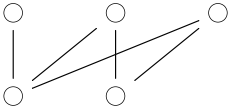
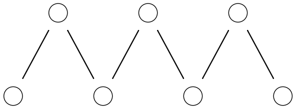
  No! Here are two posets with $\operatorname{ord}(R) = \infty$:





Birational rowmotion: order

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- A straightforward argument shows that $\operatorname{ord}(\mathbf{r}) \mid \operatorname{ord}(R)$ for every finite poset P.
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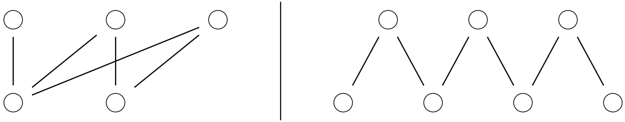
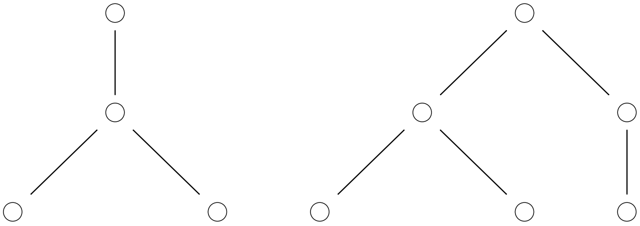


- **Nevertheless**, equality holds for many special types of P.

- **Theorem.** Assume that $n \in \mathbb{N}$, and P is a poset which is a forest (made into a poset using the "descendant" relation) having all leaves on the same level n (i.e., each maximal chain of P has n vertices). Then,

$$\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, ..., n+1).$$

Example: For P as shown, $\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, 3, 4) = 12$.

$P =$ 

Birational rowmotion: the graded forest case

- Even the $\operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, ..., n+1)$ part of this result seems to be new.
- The proof that $\operatorname{ord}(R) \mid \operatorname{lcm}(1, 2, ..., n+1)$ is essentially inductive, but with a few complications. We consider the interplay between the map $\overline{R}$, defined on *homogenous equivalence classes of labelings* and R itself.
- In fact, our proof handles the wider class of posets we call "skeletal posets". (These can be regarded as a generalization of forests where we are allowed to graft existing forests on roots on the top and on the bottom, and to use antichains instead of roots. An example is the 2×2-rectangle.)

Birational rowmotion: homogeneous equivalence

- Two $\mathbb{K}$-labellings f and g of P are said to be **homogeneously equivalent** if there is a $(\lambda_1, \lambda_2, ..., \lambda_n) \in (\mathbb{K} \setminus 0)^n$ such that

$$g(v) = \lambda_i f(v) \qquad \text{for all } i \text{ and all } v \in P_i.$$

Example: These two labellings:

$$\begin{tikzcd}[column sep=small] & a_1 \arrow[d, dash] & \\ & z_1 \arrow[dl, dash] \arrow[dr, dash] & \\ x_1 \arrow[dr, dash] & & y_1 \arrow[dl, dash] \\ & w_1 \arrow[d, dash] & \\ & b_1 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & a_2 \arrow[d, dash] & \\ & z_2 \arrow[dl, dash] \arrow[dr, dash] & \\ x_2 \arrow[dr, dash] & & y_2 \arrow[dl, dash] \\ & w_2 \arrow[d, dash] & \\ & b_2 & \end{tikzcd}$$

are homogeneously equivalent if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.

- Let $\overline{\mathbb{K}^{\widehat{P}}}$ denote the set of all $\mathbb{K}$-labellings of P (with no zero labels) modulo homogeneous equivalence.
  Let $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ be the canonical projection.
- There exists a rational map $\overline{R} : \overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ such that the diagram

$$\begin{tikzcd} \mathbb{K}^{\widehat{P}} \arrow[r, dashed, "R"] \arrow[d, dashed, "\pi"'] & \mathbb{K}^{\widehat{P}} \arrow[d, dashed, "\pi"] \\ \overline{\mathbb{K}^{\widehat{P}}} \arrow[r, dashed, "\overline{R}"'] & \overline{\mathbb{K}^{\widehat{P}}} \end{tikzcd}$$

  commutes.
- Hence $\operatorname{ord}\left(\overline{R}\right) \mid \operatorname{ord}(R)$.

- **Theorem (periodicity):** If P is the $p \times q$-rectangle (i.e., the poset $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$ with coordinatewise order), then

$$\operatorname{ord}(R) = p + q.$$

Example: For the 2×2-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

Birational rowmotion: the rectangle case

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$$\operatorname{ord}(R) = p + q.$$

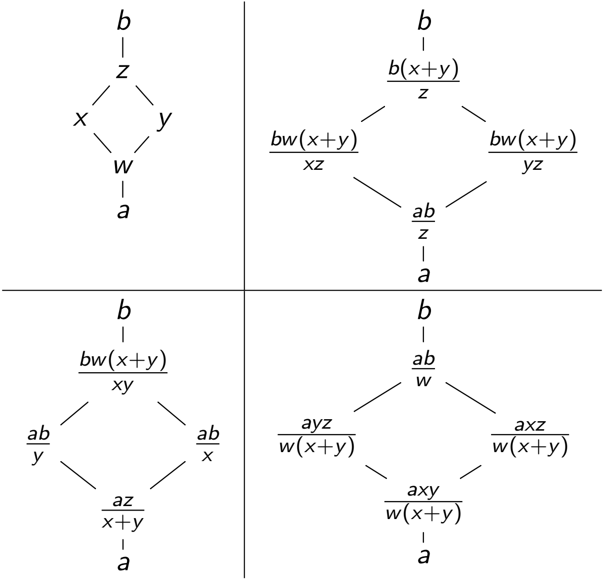
Example: For the 2×2-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

- **Theorem (reciprocity):** If P is the $p \times q$-rectangle, and $(i, k) \in P$ and $f \in \mathbb{K}^{\widehat{P}}$, then
$$f\left(\underbrace{(p+1-i, q+1-k)}_{\substack{=\text{antipode of } (i,k) \\ \text{in the rectangle}}}\right) = \frac{f(0)f(1)}{\left(R^{i+k-1}f\right)((i, k))}.$$

- These were conjectured (independently) by James Propp and R.

Birational rowmotion: the rectangle case, example

Example: Here is the generic R-orbit on the 2×2-rectangle again:



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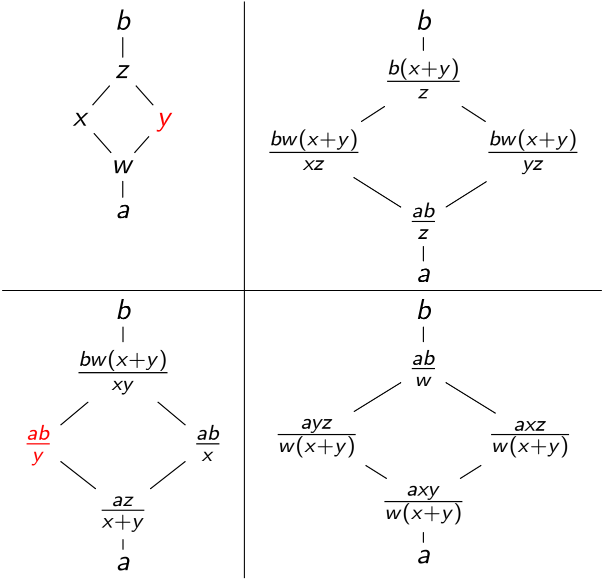

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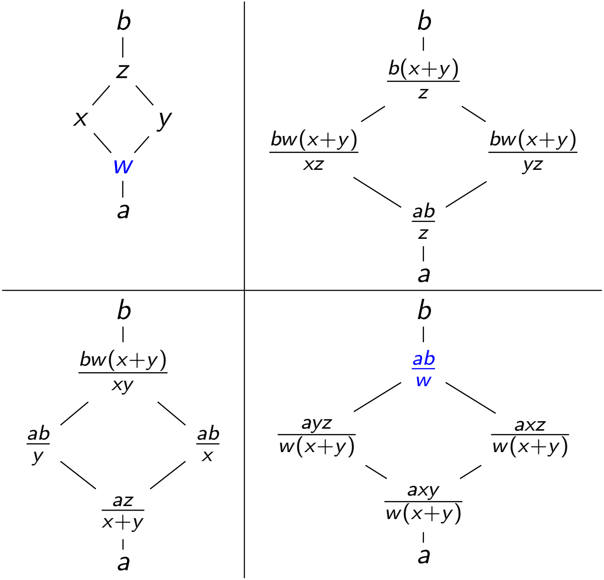

Birational rowmotion: the rectangle case, proof idea

- Inspiration: Alexandre Yu. Volkov, *On Zamolodchikov's Periodicity Conjecture*, `arXiv:hep-th/0606094`.
- We reparametrize our assignments $f : \widehat{P} \to \mathbb{K}$ through $p \times (p+q)$-matrices in such a way that birational rowmotion corresponds to "cycling" the columns of the matrix.
- This uses a 3-term Plücker relation.
- Lots of technicalities to be managed, particularly around birational maps not necessarily being defined everywhere.

Birational rowmotion: the rectangle case, proof

- Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix with p rows and $p+q$ columns.
- Let A_i be the i-th column of A. Extend to all $i \in \mathbb{Z}$ by setting

$$A_{p+q+i} = (-1)^{p-1} A_i \qquad \text{for all } i.$$

- Let $A[a : b \mid c : d]$ be the matrix whose columns are $A_a, A_{a+1}, \ldots, A_{b-1}, A_c, A_{c+1}, \ldots, A_{d-1}$ from left to right.

Birational rowmotion: the rectangle case, proof

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- Let $A[a:b \mid c:d]$ be the matrix whose columns are $A_a, A_{a+1}, \ldots, A_{b-1}, A_c, A_{c+1}, \ldots, A_{d-1}$ from left to right.
- For every $j \in \mathbb{Z}$, we define a $\mathbb{K}$-labelling $\operatorname{Grasp}_j A \in \mathbb{K}^{\widehat{P}}$ by

$$\left(\operatorname{Grasp}_j A\right)((i,k)) = \frac{\det\left(A[j+1:j+i \mid j+i+k-1:j+p+k]\right)}{\det\left(A[j:j+i \mid j+i+k:j+p+k]\right)}$$

for every $(i,k) \in P$ (this is well-defined for a Zariski-generic A) and $\left(\operatorname{Grasp}_j A\right)(0) = \left(\operatorname{Grasp}_j A\right)(1) = 1$.

- The proof of $\operatorname{ord}(R) = p + q$ now rests on four claims:
  - **Claim 1:** $\operatorname{Grasp}_j A = \operatorname{Grasp}_{p+q+j} A$ for all j and A.
  - **Claim 2:** $R\left(\operatorname{Grasp}_j A\right) = \operatorname{Grasp}_{j-1} A$ for all j and A.
  - **Claim 3:** For almost every $f \in \mathbb{K}^{\widehat{P}}$ satisfying $f(0) = f(1) = 1$, there exists a matrix $A \in \mathbb{K}^{p \times (p+q)}$ such that $\operatorname{Grasp}_0 A = f$.
  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 1 is immediate from the definitions.

Birational rowmotion: the rectangle case, proof

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.
- Claim 2 is a computation with determinants, which boils down to the three-term Plücker identities:

$$\det\left(A\left[a-1:b \mid c:d+1\right]\right) \cdot \det\left(A\left[a:b+1 \mid c-1:d\right]\right)$$
$$+ \det\left(A\left[a:b \mid c-1:d+1\right]\right) \cdot \det\left(A\left[a-1:b+1 \mid c:d\right]\right)$$
$$= \det\left(A\left[a-1:b \mid c-1:d\right]\right) \cdot \det\left(A\left[a:b+1 \mid c:d+1\right]\right).$$

for $A \in \mathbb{K}^{u \times v}$ and $a \leq b$ and $c \leq d$ and $b - a + d - c = u - 2$.

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 3 is an annoying (nonlinear) triangularity argument: With the ansatz $A = (I_p \mid B)$ for $B \in \mathbb{K}^{p\times q}$, the equation $\operatorname{Grasp}_0 A = f$ translates into a system of equations in the entries of B which can be solved by elimination.

Birational rowmotion: the rectangle case, proof

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.
- Claim 4 follows by noting that for an n-graded poset we have $\operatorname{ord}(R) = \operatorname{lcm}\left(n + 1, \operatorname{ord}\left(\overline{R}\right)\right)$.

Birational rowmotion: the rectangle case, proof

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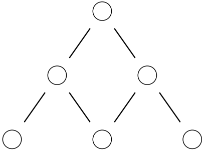
- The reciprocity statement can be proven in a similar vein.

- **Theorem (periodicity):** If P is the triangle $\Delta(p) = \{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i+k > p+1\}$ with $p > 2$, then
$$\operatorname{ord}(R) = 2p.$$

Example: The triangle $\Delta(4)$:



Birational rowmotion: the Δ-triangle case

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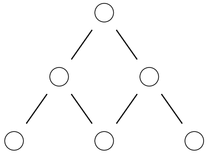

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- This is precisely the same result as for classical rowmotion.

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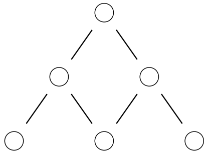


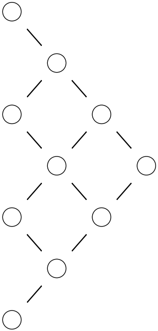
- **Theorem (reciprocity):** R^p reflects any $\mathbb{K}$-labelling across the vertical axis.
- This is precisely the same result as for classical rowmotion.
- The proofs use a "folding"-style argument to reduce this to the rectangle case.

Birational rowmotion: the $\triangleright$-triangle case

- **Theorem (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k\}$, then

$$\operatorname{ord}(R) = 2p.$$

Example: For $p = 4$, this P has the form:



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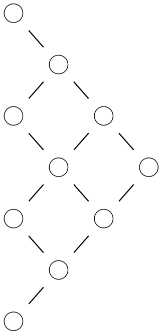

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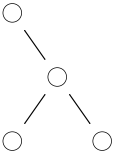
- Again this is reduced to the rectangle case.

- **Conjecture (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1\}$, then
$$\operatorname{ord}(R) = p.$$

Example: For $p = 4$, this P has the form:



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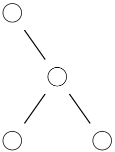
- **Conjecture (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ \ i+k > p+1\}$, then
$$\operatorname{ord}(R) = p.$$

Example: For $p = 4$, this P has the form:



.

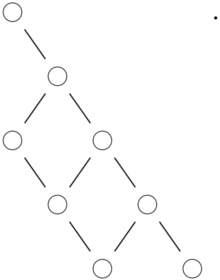
- We proved this for p odd.
- Note that for p even, this is a type-B positive root poset. Armstrong-Stump-Thomas did this for classical rowmotion.

Birational rowmotion: the trapezoid case (Nathan Williams)

- **Conjecture (periodicity):** If P is the trapezoid $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1;\ k \geq s\}$ for some $0 \leq s \leq p$, then

$$\operatorname{ord}(R) = p.$$

Example: For $p = 6$ and $s = 5$, this P has the form:



- This was observed by Nathan Williams and verified for $p \leq 7$.
- Motivation comes from Williams's "Cataland" philosophy.

- For what P is $\operatorname{ord}(R) < \infty$? This seems too hard to answer in general.

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Birational rowmotion: the root system connection (Nathan W.)

- For what P is $\operatorname{ord}(R) < \infty$? This seems too hard to answer in general.
- **Not true:** for all those P that have nice and small $\operatorname{ord}(\mathbf{r})$'s.
- **However** it seems that $\operatorname{ord}(R) < \infty$ holds if P is **the positive root poset of a coincidental-type root system** (A_n, B_n, H_3), or a **minuscule heap** (see Rush-Shi, section 6).
- But the positive root system of D_4 has $\operatorname{ord}(R) = \infty$.

Application: Promotion on SSYTs

- The following is an application of our result on rectangle-shaped posets.
- It is well known (see Striker-Williams) that **classical** rowmotion (= birational rowmotion over the boolean semiring $\{0, 1\}$) is related to promotion on **two-rowed** semistandard Young tableaux.
- Similarly, **birational** rowmotion over the tropical semiring $\operatorname{Trop}\mathbb{Z}$ relates to **arbitrary** semistandard Young tableaux.
- As an application of the periodicity theorem, we obtain the classical result that promotion done n times on a rectangular semistandard Young tableau with "ceiling" n does nothing.

QUESTIONS: What is this related to?

This line of work appears (at least superficially) to be related to several other areas of research:

- Y-systems and Zomolodchikov Periodicity?
- Cluster mutations?
- bounded octohedron recurrence?
- Kirillov-Berenstein RSK?
- Other Coxeter or Catalan combinatorics?

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Q: What orders of toggling lead to finite-order birational maps?

- This is new and unproven, and inspired by Iyudu/Shkarin, `arXiv:1305.1965v3` (Kontsevich's periodicity conjecture).
- Work in a **skew field**. Write $\overline{m}$ for m^{-1}.
- Define the v-toggle by

$$\left(T_v f\right)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

Birational rowmotion: noncommutative generalization?

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Iteratively apply R to a labelling of the 2×2-rectangle.
$R^0 f =$

$$\begin{tikzcd}
& b & \\
& z \arrow[u, no head] & \\
x \arrow[ur, no head] & & y \arrow[ul, no head] \\
& w \arrow[ul, no head] \arrow[ur, no head] & \\
& a \arrow[u, no head] &
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Iteratively apply R to a labelling of the 2×2-rectangle.

$R^1 f =$

$$\begin{tikzcd}[column sep=tiny]
& b \arrow[d, dash] & \\
& (x+y)\overline{z}b \arrow[dl, dash] \arrow[dr, dash] & \\
w\overline{x}(x+y)\overline{z}b \arrow[dr, dash] & & w\overline{y}(x+y)\overline{z}b \arrow[dl, dash] \\
& a\overline{z}b \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

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$R^2 f =$

$$\begin{tikzcd}
& b \arrow[d, dash] & \\
& w\left(\overline{x}+\overline{y}\right) b \arrow[dl, dash] \arrow[dr, dash] & \\
a \cdot \overline{x+y} \cdot x\left(\overline{x}+\overline{y}\right) b \arrow[dr, dash] & & a \cdot \overline{x+y} \cdot y\left(\overline{x}+\overline{y}\right) b \arrow[dl, dash] \\
& a\overline{b}z \cdot \overline{x+y} \cdot b \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

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$R^3 f =$

$$\begin{tikzcd}
& b \arrow[d, dash] & \\
& a\overline{w}b \arrow[dl, dash] \arrow[dr, dash] & \\
\dots \arrow[dr, dash] & & a\overline{b}z \cdot \overline{x+y} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{y} \cdot (x+y)\,\overline{w}b \arrow[dl, dash] \\
& a\overline{b} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{w}b \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

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Iteratively apply R to a labelling of the 2×2-rectangle.

$R^4 f =$

$$\begin{tikzcd}[arrows=dash]
 & b \arrow[d] & \\
 & a\overline{b}z\overline{a}b \arrow[dl] \arrow[dr] & \\
\cdots \arrow[dr] & & a\overline{b} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{x+y} \cdot y\left(\overline{x}+\overline{y}\right)\left(x+y\right)\overline{a}b \arrow[dl] \\
 & a\overline{b}w\overline{a}b \arrow[d] & \\
 & a &
\end{tikzcd}$$

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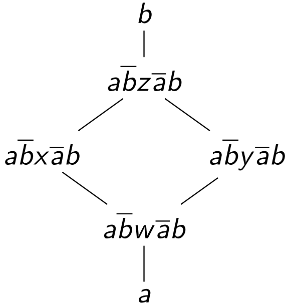

(after nontrivial simplifications).

Birational rowmotion: noncommutative generalization?

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$R^4 f =$

$$\begin{tikzcd}
& b \arrow[d, dash] & \\
& a\overline{b}z\overline{a}b \arrow[dl, dash] \arrow[dr, dash] & \\
a\overline{b}x\overline{a}b \arrow[dr, dash] & & a\overline{b}y\overline{a}b \arrow[dl, dash] \\
& a\overline{b}w\overline{a}b \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

That is, all of our labels got conjugated by $a\overline{b}$. **Is R^{p+q} always conjugation by $f(0) \cdot (f(1))^{-1}$ on a $p \times q$-rectangle?** This is similar to Kontsevich's periodicity. (Noncommutative determinants?)

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DEF: Given an (invertible) action τ on a finite set of objects S, call a statistic $\varphi : S \to \mathbb{C}$ **homomesic** [Gk., "same middle"] with respect to (S, τ) iff the average of φ over each τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s)$ does not depend on the choice of $\mathcal{O}$.

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We call the triple (S, τ, φ) a **homomesy**.

For example, the statistic $\#I$ (cardinality of the ideal) is *homomesic* with respect to rowmotion, $\mathbf{r}$, acting on $J([4] \times [2])$.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\mathbf{r}$*-orbit in* $J([p] \times [q])$*. Then*

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{pq}{2},$$

i.e., the cardinality statistic is homomesic with respect to the action of rowmotion on order ideals.

Classical rowmotion: homomesies

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It turns out that to show a similar statement for rowmotion acting on the *antichains* of P, the right tool is an equivariant bijection from Stanley's "Promotion and Evacuation" paper, as rephrased by Hugh Thomas.

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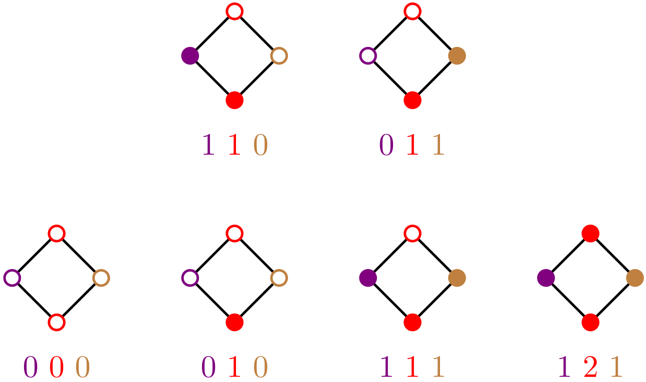
Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has
$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

We have more refined homomesies for combinatorial rowmotion on $J([p] \times [q])$.
For $1 - b \leq k \leq a - 1$, let $f_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k f_k(I)$.

Theorem (Propp, R.): If $\mathcal{O}$ is any ∂-orbit in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} f_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Homomesy for Birational rowmotion on $J([2] \times [2])$:

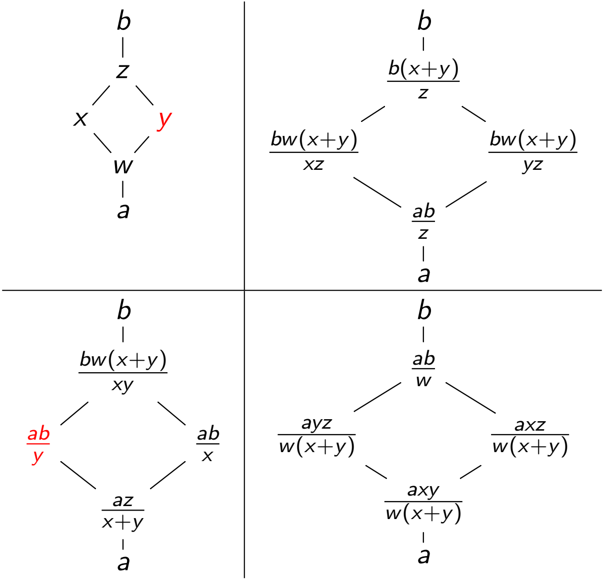
Example: Consider the **geometric** means of products in each file:

$$\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& a &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{bw(x+y)}{xz} \arrow[dr, dash] & & \frac{bw(x+y)}{yz} \arrow[dl, dash] \\
& \frac{ab}{z} \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& \frac{bw(x+y)}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ab}{y} \arrow[dr, dash] & & \frac{ab}{x} \arrow[dl, dash] \\
& \frac{az}{x+y} \arrow[d, dash] & \\
& a &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& \frac{ab}{w} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ayz}{w(x+y)} \arrow[dr, dash] & & \frac{axz}{w(x+y)} \arrow[dl, dash] \\
& \frac{axy}{w(x+y)} \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

Homomesy for Birational rowmotion on $J([2] \times [2])$:

Example: Consider the **geometric** means of products in each file:



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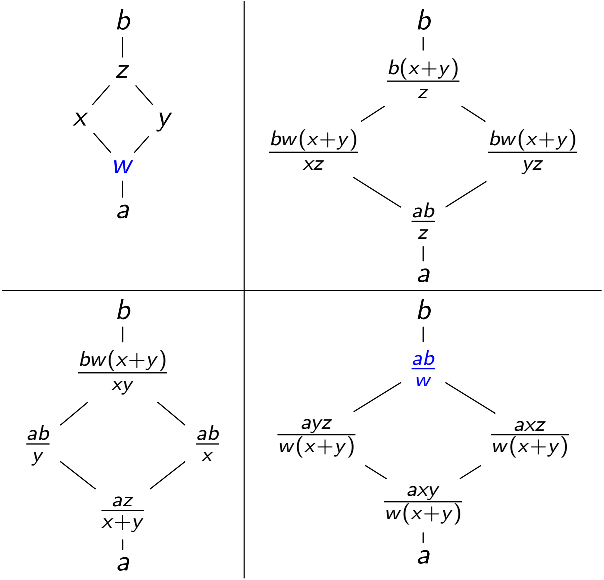

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The final slide of this talk

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Thanks very much for coming to this workshop!

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