

Homomesy of Order Ideals in Products of Two Chains

Tom Roby (University of Connecticut)

Describing joint research with Jim Propp

Combinatorics Seminar
University of Minnesota
Minneapolis, MN USA

6 Sept 2013

Slides for this talk are available online (or will be soon) at

<http://www.math.uconn.edu/~troby/research.html>

We consider a variety of combinatorial actions on finite sets, e.g., cyclic rotation of binary strings, promotion of Young tableaux and rowmotion on order ideals of partially ordered sets. We identify a particular phenomenon called “homomesy” appearing in many unrelated combinatorial contexts: namely that the average value of some natural statistic over each orbit is the same as the average over the entire set. Viewing these actions as products of “toggle operations” allows us to see how some of these actions are related and to extend much of this picture more broadly. In particular, we can generalize the operations of rowmotion and promotion (Striker in and Williams’s terminology) on order ideals in a poset to (1) the order polytope of a poset (the continuous piecewise-linear category), and (2) to the collection of maps from a poset P to rational functions in $|P|$ variables (the birational category).

Acknowledgments

This talk largely discusses recent work with Jim Propp, including ideas and results from Arkady Berenstein, David Einstein, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Darij Grinberg wrote invaluable Sage code to compute birational promotion. Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.

- Rotation of bit-strings;
- Unexpected averaging properties: homomesic statistics;
- Suter's dihedral symmetries in Young's lattice;
- Rowmotion and Promotion actions on antichains and order ideals of posets;
- Homomesic statistics for actions in $[a] \times [b]$;
- Generalizing to other actions on order polytopes and rational functions;

Please interrupt with questions!

Example 1: Rotation of bit-strings

Let S denote the set of length n binary strings with exactly k 1's. and set $\tau := C_R : S \rightarrow S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift). Let $\varphi(b) = \# \text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any τ -orbit \mathcal{O} we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

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0011 $\mapsto 0$	

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0110 $\mapsto 2$	AVG = $\frac{4}{2} = 2$
0011 $\mapsto 0$	
AVG = $\frac{8}{4} = 2$	

EG: $n = 6, k = 2$ gives us three orbits:

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110000	010001	010010
011000	101000	001001
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000110	001010	
000011	000101	

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011000	101000	001001
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000011	000101	

EG: $n = 6, k = 2$ gives us three orbits:

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100001 \mapsto 4	100010 \mapsto 5	100100 \mapsto 6
110000 \mapsto 8	010001 \mapsto 3	010010 \mapsto 4
011000 \mapsto 6	101000 \mapsto 7	001001 \mapsto 2
001100 \mapsto 4	010100 \mapsto 5	
000110 \mapsto 2	001010 \mapsto 3	
000011 \mapsto 0	000101 \mapsto 1	

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We know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate b (replacing a 01 somewhere in b by 10 or vice versa), or one can write the number of inversions in b as $\sum_{i < j} b_i(1 - b_j)$ and then perform algebraic manipulations.

MAIN DEF: Given an (invertible) action τ on a finite set of objects S , call a statistic $\varphi : S \rightarrow \mathbb{C}$ **homomesic** with respect to (S, τ) iff the average of φ over each τ -orbit \mathcal{O} is the same for all \mathcal{O} , i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s)$ does not depend on the choice of \mathcal{O} .

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Equivalently: the average of φ over each τ -orbit \mathcal{O} is the same as the average over the entire set S :

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

So in looking for homomesic statistics we can compute what the average should be before checking whether a statistic is homomesic.

Semi-Standard Young Tableaux

Fix a positive integer N . A **semi-standard Young tableau (SSYT)** of shape λ and ceiling N is a labeling of the cells of the Young diagram of a partition λ with numbers from $1, 2, \dots, N$ which increases **weakly** along each row and **strictly** along each column. For example,

1	1	2	4
2	3	4	
4	4		

,

1	1	1	2	2	5
4	4	4	5		
5	5	6			

but **not**

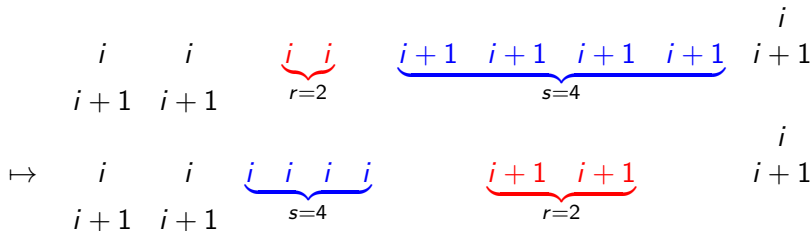
1	1	2	2	3	3
2	3	4	4		
3	4	4			

The **weight vector** $\alpha(T) = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is given by $\alpha_i := \alpha_i(T) = \# \text{occurrences of } i \text{ in } T$. EG, the weight vectors for the two tableaux above are $(2, 2, 1, 4)$ and $(3, 2, 0, 3, 4, 1)$.

We let $SSYT(\lambda, N)$ denote the set of all semi-standard Young tableaux whose entries lie within $[N] = \{1, 2, \dots, N\}$.

Bender-Knuth involutions

A standard method for proving combinatorially that Schur functions are symmetric is to use **Bender-Knuth involutions**. Given $T \in SSYT(\lambda, N)$ and $i \in [N-1]$, consider all the i 's that appear above an $i+1$ in the same column, and all the $i+1$'s that appear below an i in the same column to be **married** and the remainder **free**. Then in each row with r free i 's and s free $i+1$'s, β_i replaces these with s free i 's and r free $i+1$'s.



Define the following operator on $SSYT(\lambda, N)$:

$$\partial := \beta_{N-1} \circ \beta_{N-2} \circ \cdots \circ \beta_2 \circ \beta_1,$$

the composition of all BK involutions in order. By a result of Gansner [Gan80], this operator coincides with Schützenberger's *promotion* operator. Then for all $i \in [N]$, the weight vector coordinate $\alpha_i(T)$ is homomesic w.r.t. ∂ acting on $SSYT(\lambda, N)$.

Promotion of SSYT via Bender-Knuth involutions

Define the following operator on $SSYT(\lambda, N)$:

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EG: Let $N = 5$ and $T = \begin{smallmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\ 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 3 & 4 & 4 & 5 \end{smallmatrix}$

Then the content vectors $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_N]$ that arise as we successively apply β_i 's behave as follows, starting from $[3, 4, 6, 5, 2]$:

$[4, 3, 6, 5, 2]$	$[6, 4, 5, 2, 3]$	$[5, 6, 2, 3, 4]$	$[2, 5, 3, 4, 6]$	$[3, 2, 4, 6, 5]$
$[4, 6, 3, 5, 2]$	$[6, 5, 4, 2, 3]$	$[5, 2, 6, 3, 4]$	$[2, 3, 5, 4, 6]$	$[3, 4, 2, 6, 5]$
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An easy homomesy for Promotion

From previous slide: The content vectors $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_N]$ that arise as we successively apply β_i 's behave as follows (working down columns), starting from $[3, 4, 6, 5, 2]$:

$[4, 3, 6, 5, 2]$	$[6, 4, 5, 2, 3]$	$[5, 6, 2, 3, 4]$	$[2, 5, 3, 4, 6]$	$[3, 2, 4, 6, 5]$
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Note that each iteration of promotion (going down each full column) applies a cyclic shift of the content vector. This is enough to prove that $\alpha_i(T)$ is homomesic, even though we don't know how long the orbit is. But it must be a multiple of 5 for the content vector to return to where it started. So over any orbit, each content value visits each position equally often.

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A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942;
<http://arxiv.org/abs/1108.1172>):

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942

1927

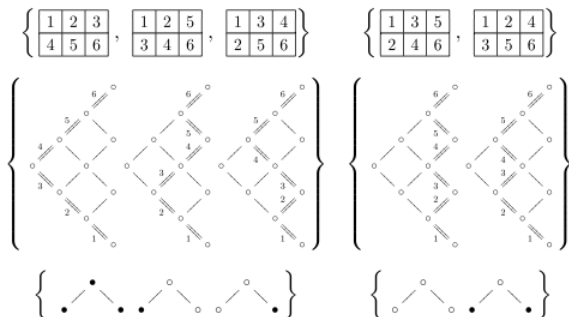


Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

A small example of promotion: centrally symmetric sums

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942

1927

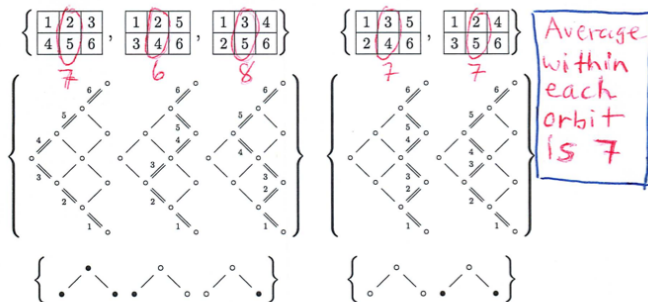


Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

Conjecture

Let S be the set of Semi-Standard Young Tableaux of rectangular shape λ and ceiling N . If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\varphi(T)$ denotes the sum of the numbers in cells c and c' , then φ is homomesic with respect to (S, ∂) with average value $N + 1$.

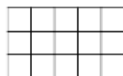
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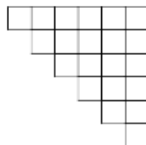
This has recently been proven by J. Bloom, O. Pechenik, & D. Saracino, <http://arxiv.org/abs/1308.0546>, using a growth diagram argument and separately by jeu de taquin. The first argument generalizes to the analogous result for “cominuscule posets”.

Cominuscule Posets

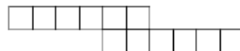
The first three pictures represent infinite classes; while the last two are sporadic. (Picture from [BPS13].)



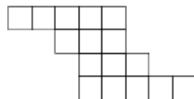
(a) Rectangle



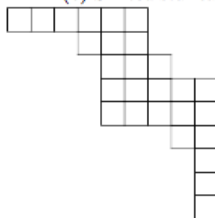
(b) Shifted staircase



(c) Propeller



(d) Cayley poset



(e) Freudenthal poset

Figure 1: The five families of cominuscule posets. The boxes are the elements of the poset and each box is covered by those immediately below it and to its right.

Example 4: Suter's symmetries

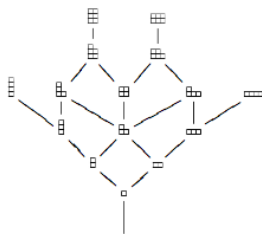
Let \mathbb{Y}_N be the set of number-partitions λ whose maximal hook lengths are strictly less than N (i.e., whose Young diagrams fit inside some rectangle that fits inside the staircase shape $(N-1, N-2, \dots, 2, 1)$).

Suter showed that the Hasse diagram of \mathbb{Y}_N has N -fold cyclic symmetry (indeed, N -fold dihedral symmetry) by exhibiting an explicit action of order N .

(taken from R. Suter, *Young's lattice and dihedral symmetries revisited: Möbius strips and metric geometry*;

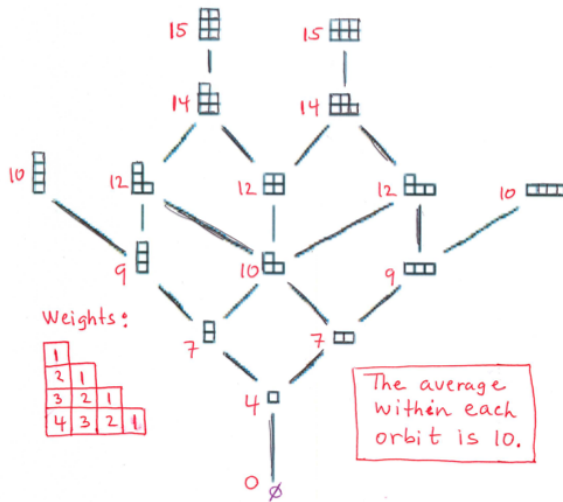
<http://arxiv.org/abs/1212.4463>):

Example (Hasse diagram of \mathbb{Y}_5 and its undirected Hasse graph)



This graph has a 5-fold (cyclic) symmetry, and its full symmetry group is a dihedral group of order 10.

Suter's action, $N = 5$: weighted sums



Assign weight 1 to the cells at the diagonal boundary of the staircase shape, weight 2 to their neighbors, ..., and weight $N - 1$ to the cell at the lower left, and for $\lambda \in \mathbb{Y}_N$ let $\varphi(\lambda)$ be the sum of the weights of all the cells in the Young diagram of λ .

Prop. (Einstein-Propp): φ is homomesic under Suter's map with average value $(N^3 - N)/12$.

More refined result: If $i + j = N$ (note: $i = j$ is permitted), and $\varphi_{i,j}(\lambda)$ is the sum of the weights of all the cells in λ with weight i plus the sum of the weights of all the cells in λ with weight j , then $\varphi_{i,j}$ is homomesic under Suter's map with average ij in all orbits.

Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the order ideal generated by A .

τ is invertible since it is a composition of three invertible operations:

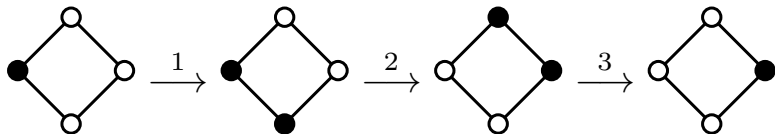
$$\text{antichains} \longleftrightarrow \text{down-sets} \longleftrightarrow \text{up-sets} \longleftrightarrow \text{antichains}$$

We can also view this as an invertible action $\bar{\tau}$ on $J(P)$, the set of order ideals of P , via the above isomorphism between $\mathcal{A}(P)$ and $J(P)$; in other words, perform the above steps in the order 2,3,1.

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams [SW12]. Following [SW12], we call this **rowmotion**.

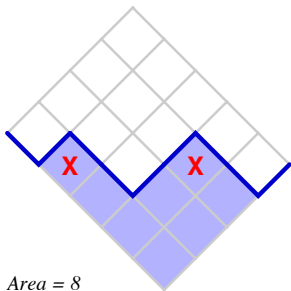
An example

1. Saturate downward
2. Complement
3. Take minimal element(s)

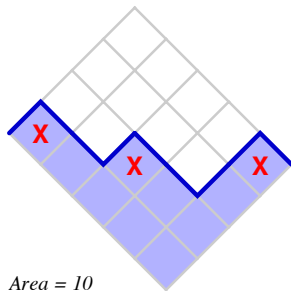


Example in lattice cell form

Viewing the elements of the poset as **squares**, we would map:



$\xrightarrow{\tau}$
 $\xrightarrow{\bar{\tau}}$



Let Δ be a reduced irreducible root system in \mathbb{R}^n . (Pictures soon!)
Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Conjecture (Conjecture 2.1(iii) in D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. 30 (2009), 586-594): Let \mathcal{O} be an arbitrary τ -orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

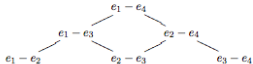
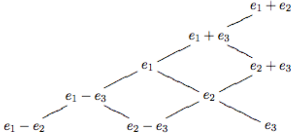
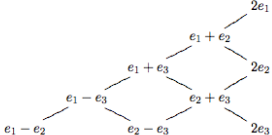
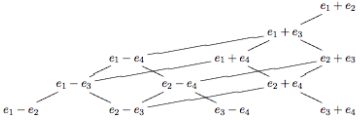
In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Panyushev's Conjecture 2.1(iii) (along with much else) was proved by Armstrong, Stump, and Thomas in their article *A uniform bijection between nonnesting and noncrossing partitions*,

<http://arxiv.org/abs/1101.1277>.

Picture of root posets

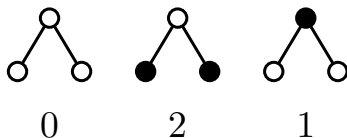
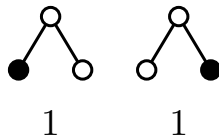
Here are the classes of posets included in Panyushev's conjecture.

$\Phi^+(A_3)$ 	$\Phi^+(B_3)$ 
$\Phi^+(C_3)$ 	$\Phi^+(D_4)$ 

(Graphic courtesy of Striker-Williams.)

Panyushev's conjecture: The A_n case, $n = 2$

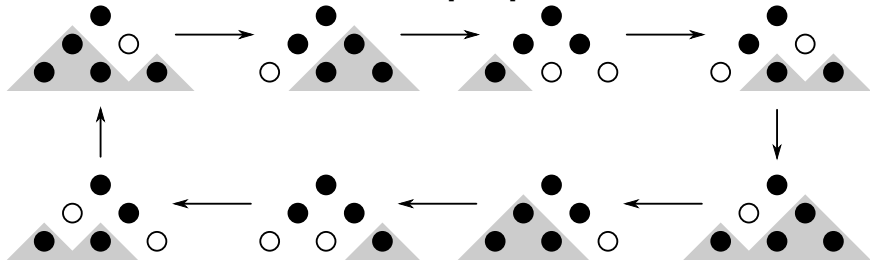
Here we have just an orbit of size 2 and an orbit of size 3:



Within each orbit, the average antichain has cardinality $n/2 = 1$.

The case A_3 .

Here's an example orbit taken from [AST] for the A_3 root poset:



For A_3 this action has three orbits (sized 2, 4, and 8), and the average cardinality of an antichain is

$$\frac{1}{8} (2 + 1 + 1 + 2 + 2 + 1 + 1 + 2) = \frac{3}{2} = \frac{n}{2}$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (where $[k]$ denotes the linear ordering of $\{1, 2, \dots, k\}$):

Theorem (Propp, R.)

Let \mathcal{O} be an arbitrary τ -orbit in $\mathcal{A}([a] \times [b])$. Then

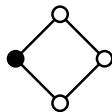
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

This is an easy consequence of unpublished work of Hugh Thomas building on earlier work of Richard Stanley: see the last paragraph of section 2 of R. Stanley, *Promotion and evacuation*,

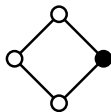
<http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i2r9> .

Antichains in $[a] \times [b]$: the case $a = b = 2$

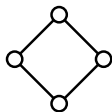
Here we have an orbit of size 2 and an orbit of size 4:



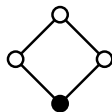
1



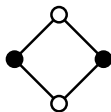
1



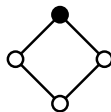
0



1



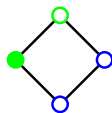
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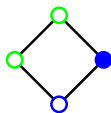
1

Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

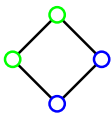
Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



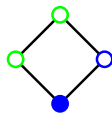
1 0



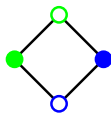
0 1



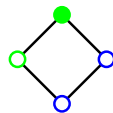
0 0



0 1



1 1



1 0

Within each orbit, the average antichain has
 $\frac{1}{2}$ a green element and $\frac{1}{2}$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i, j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i, j) .

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0, 1\}$ (the cardinality of the intersection of A with the fiber $\{(i, 1), (i, 2), \dots, (i, b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i, j ,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \quad \text{and} \quad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under τ , even though the indicator functions $1_{i,j}$ aren't.

Theorem (Propp, R.)

In any orbit, the number of A that contain (i, j) equals the number of A that contain the opposite element $(i', j') = (a + 1 - i, b + 1 - j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under τ , with average value 0 in each orbit.

Useful triviality: every linear combination of homomesies is itself homomesic.

E.g., consider the adjusted major index statistic defined by $\text{amaj}(A) = \sum_{(i,j) \in A} (i - j)$.

Propp and R. proved that amaj is homomesic under τ by writing it as a linear combination of the functions $1_{i,j} - 1_{i',j'}$. Haddadan gave a simpler proof, writing amaj as a linear combination of the functions f_i and g_j .

Question: Are there other homomesic combinations of the indicator functions $1_{i,j}$ (with $(i,j) \in [a] \times [b]$), linearly independent of the functions f_i , g_j , and $1_{i,j} - 1_{i',j'}$?

As we've seen, one can view rowmotion as acting either on *antichains* ($\mathcal{A}(P)$) or on *order ideals* ($J(P)$); we denote the latter map $\bar{\tau}$. It turns out that the cardinality of the order ideal is also homomesic with respect to rowmotion on $[a] \times [b]$.

Theorem (Propp, R.)

Let \mathcal{O} be an arbitrary $\bar{\tau}$ -orbit in $J([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

It's worth noting even though there's a strong connection between the rowmotion map on antichains and on order ideals, that the homomesy situation could be quite different.

One action, two vector spaces

The map $\bar{\tau}$ is “the same” as τ in the sense that the standard bijection from $\mathcal{A}(P)$ to $J(P)$ (downward saturation) makes the following diagram commute:

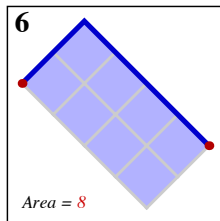
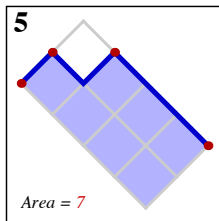
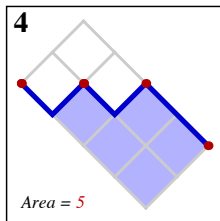
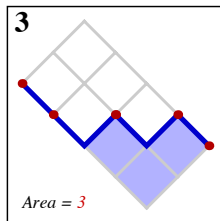
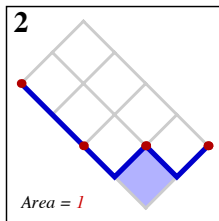
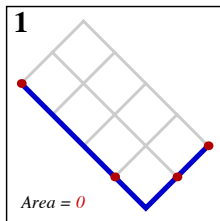
$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\tau} & \mathcal{A}(P) \\ \downarrow & & \downarrow \\ J(P) & \xrightarrow{\bar{\tau}} & J(P) \end{array}$$

However, the bijection from $\mathcal{A}(P)$ to $J(P)$ does **not** carry the vector space generated by the functions $1_{i,j}$ to the vector space generated by the functions $\bar{1}_{i,j}$ in a linear way.

So the homomesy situation for $\bar{\tau} : J(P) \rightarrow J(P)$ could be (and, as we'll see, is) different from the homomesy situation for $\tau : \mathcal{A}(P) \rightarrow \mathcal{A}(P)$.

Rowmotion on $[4] \times [2]$ **A**

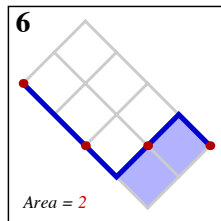
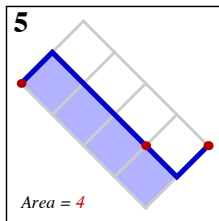
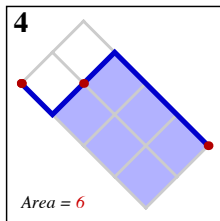
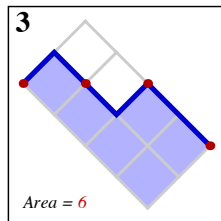
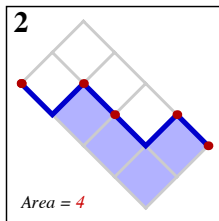
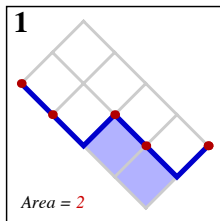
Rowmotion on $[4] \times [2]$ A



$$(0+1+3+5+7+8) / 6 = 4$$

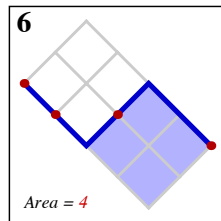
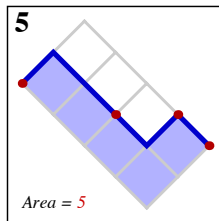
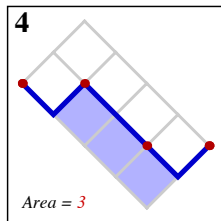
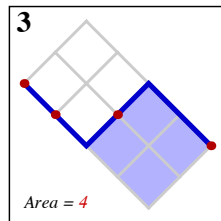
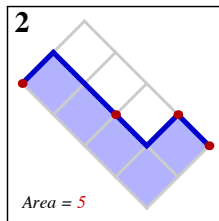
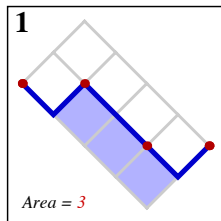
Rowmotion on $[4] \times [2]$ B

Rowmotion on $[4] \times [2]$ B



$$(2+4+6+6+4+2) / 6 = 4$$

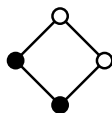
Rowmotion on $[4] \times [2]$ C



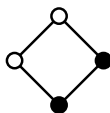
$$(3+5+4+3+5+4) / 6 = 4$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

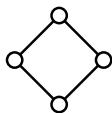
Again we have an orbit of size 2 and an orbit of size 4:



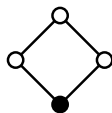
2



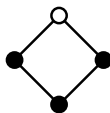
2



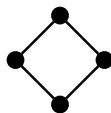
0



1



3

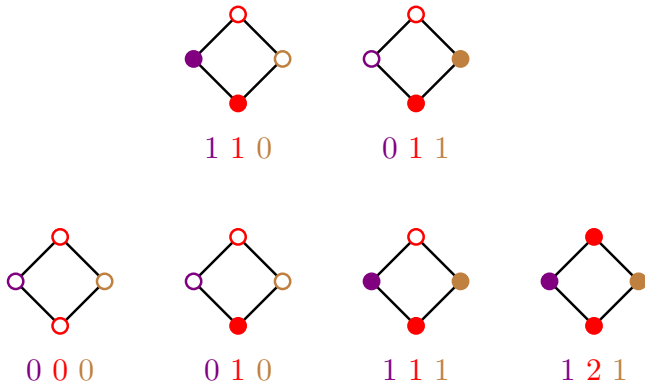


4

Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

We also have homomesies for more refined statistics than $\#I$.



Within each orbit, the average order ideal has $\frac{1}{2}$ a violet element, 1 red element, and $\frac{1}{2}$ a brown element.

For $1 - b \leq k \leq a - 1$, define the k th **file** of $[a] \times [b]$ as

$$\{(i, j) : 1 \leq i \leq a, 1 \leq j \leq b, i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the k th file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every $\bar{\tau}$ -orbit \mathcal{O} in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Given $(i, j) \in [a] \times [b]$, and I an ideal in $[a] \times [b]$, define the *indicator function* $\bar{1}_{i,j}(I)$ to be 1 or 0 according to whether or not I contains (i, j) .

Write $(i', j') = (a + 1 - i, b + 1 - j)$, the point opposite (i, j) in the poset.

Theorem (Propp, R.)

$\bar{1}_{i,j} + \bar{1}_{i',j'}$ is homomesic under $\bar{\tau}$.

The two vector spaces, compared

In the space associated with **antichains**:
 fiber-cardinalities and
 centrally symmetric **differences**
are homomesic.

In the space associated with **order ideals**:
 file-cardinalities and
 centrally symmetric **sums**
are homomesic.

The two vector spaces, compared

In the space associated with **antichains**:
 fiber-cardinalities and
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In the space associated with **order ideals**:
 file-cardinalities and
 centrally symmetric **sums**
are homomesic.

Note that the the discovery of these homomesies was driven by calculation, and the project of generalizing these results to other posets will clearly be aided by computer-assisted search.

In their 1995 article *Orbits of antichains revisited*, European J. Combin. 16 (1995), 545–554, Cameron and Fon-der-Flaass give an alternative description of $\bar{\tau}$.

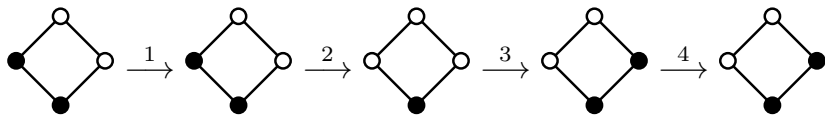
Given $I \in J(P)$ and $x \in P$, let $\tau_x(I) = I \Delta \{x\}$ (symmetric difference) provided that $I \Delta \{x\}$ is an order ideal of P ; otherwise, let $\tau_x(I) = I$.

We call the involution τ_x “toggling at x ”.

The involutions τ_x and τ_y **commute** unless x covers y or y covers x .

An example

1. Toggle the top element
2. Toggle the left element
3. Toggle the right element
4. Toggle the bottom element



Theorem ([CF95])

Let x_1, x_2, \dots, x_n be any order-preserving enumeration of the elements of the poset P . Then the action on $J(P)$ given by the composition $\tau_{x_1} \circ \tau_{x_2} \circ \dots \circ \tau_{x_n}$ coincides with the action of $\bar{\tau}$.

In the particular case $P = [a] \times [b]$, we can enumerate P rank-by-rank; that is, we can list the (i, j) 's in order of increasing $i + j$.

Note that all the involutions coming from a given rank of P commute with one another, since no two of them are in a covering relation.

Striker and Williams refer to $\bar{\tau}$ (and τ) as **rowmotion**, since for them, “row” means “rank”.

Recall that a **file** in $P = [a] \times [b]$ is the set of all $(i, j) \in P$ with $i - j$ equal to some fixed value k .

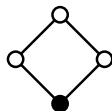
Note that all the involutions coming from a given file commute with one another, since no two of them are in a covering relation.

It follows that for any enumeration x_1, x_2, \dots, x_n of the elements of the poset $[a] \times [b]$ arranged in order of increasing $i - j$, the action on $J(P)$ given by $\tau_{x_1} \circ \tau_{x_2} \circ \dots \circ \tau_{x_n}$ doesn't depend on which enumeration was used.

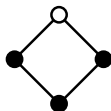
Striker and Williams call this well-defined composition **promotion**, and denote it by ∂ , since for two-rowed tableaux it can be related to Schützenberger's promotion on SYT, described earlier.

Promoting ideals in $[a] \times [b]$: the case $a = b = 2$

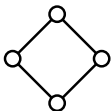
Again we have an orbit of size 2 and an orbit of size 4:



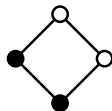
1



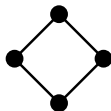
3



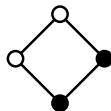
0



2



4



2

Theorem (Propp, R.)

Let \mathcal{O} be an arbitrary orbit in $J([a] \times [b])$ under the action of promotion ∂ . Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

The result about cyclic rotation of binary words discussed earlier turns out to be a special case of this.

For $1 - b \leq k \leq a - 1$, let $f_k(I)$ be the number of elements of I in the k th file of $[a] \times [b]$, so that $\#I = \sum_k f_k(I)$.

Theorem

If \mathcal{O} is any ∂ -orbit in $J([a] \times [b])$,

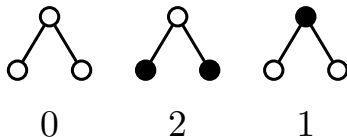
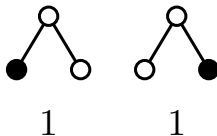
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} f_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Cardinality of antichains is not homomesic under promotion.
although the antipodal functions $1_{i,j} - 1_{i',j'}$ are.

Root posets of type A : antichains

Recall that, by the Armstrong-Stump-Thomas theorem, the cardinality of antichains is homomesic under the action of rowmotion, where the poset P is a root poset of type A_n .

E.g., for $n = 2$:

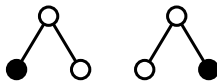


Antichain-cardinality is homomesic: in each orbit, its average is 1.

Root posets of type A : order ideals

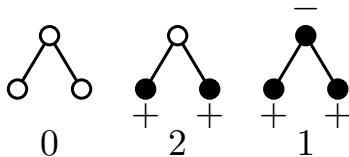
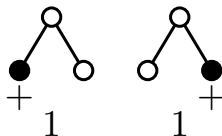
What if instead of antichains we take order ideals?

E.g., $n = 2$:



What is homomesic here?

Root posets of type A : rank-signed cardinality



Theorem (Haddadan)

Let P be the root poset of type A_n . If we assign an element $x \in P$ weight $wt(x) = (-1)^{rank(x)}$, and assign an order ideal $I \in J(P)$ weight $\varphi(I) = \sum_{x \in I} wt(x)$, then φ is homomesic under rowmotion and promotion, with average $n/2$.

The order polytope of a poset

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \rightarrow [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

We can generalize our entire setup of toggle operators and “rowmotion” to operate on these functions (the “continuous piecewise-linear (CPL) category”).

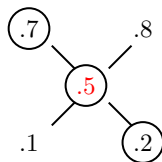
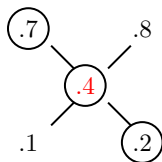
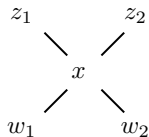
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \rightarrow \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w .

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example of flipping at a node



$$\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

If we associate each order-ideal I with the indicator function f of $P \setminus I$ (that is, the function that takes the value 0 on I and the value 1 everywhere else), then toggling I at x is tantamount to flipping f at x .

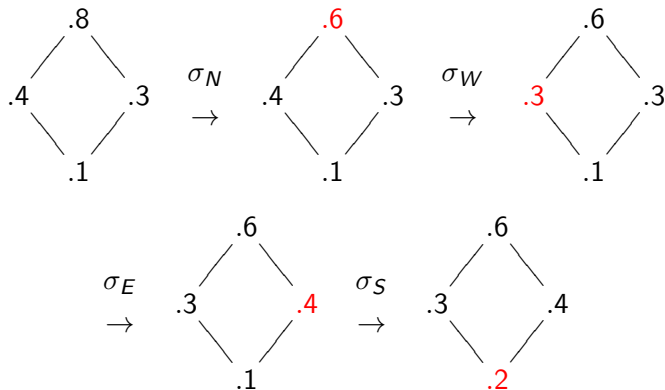
That is, we can identify $J(P)$ with the vertices of the polytope $\mathcal{O}(P)$ in such a way that toggling can be seen to be a special case of flipping.

This may be clearer if you think of $J(P)$ as being in bijection with the set of monotone 0,1-valued functions on P .

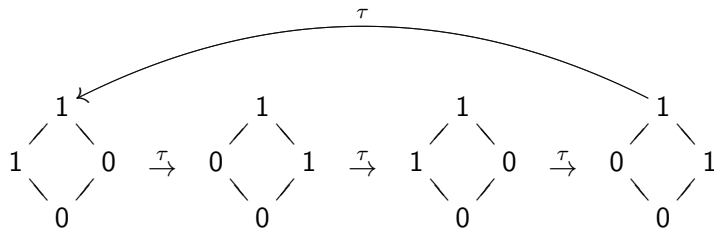
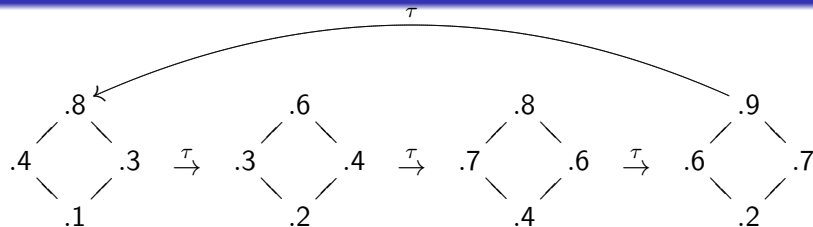
Flipping (at least in special cases) is not new, though it is not well-studied; the most worked-out example we've seen is Berenstein and Kirillov's article *Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux* (St. Petersburg Math. J. 7 (1996), 77–127); see <http://pages.uoregon.edu/arkadiy/bk1.pdf>.

Composing flips

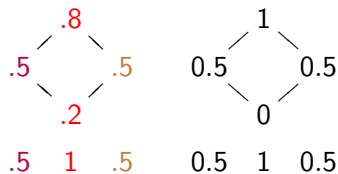
Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom (successively at the North, West, East, and South.) :



Two Examples of CPL rowmotion orbits



The average at each node across the respective orbits is shown at right, along with the **file sums**.



It appears that all of the aforementioned results on homomesy for rowmotion and promotion on $J([a] \times [b])$ lift to corresponding results in the order polytope, where instead of composing toggle-maps to obtain rowmotion and promotion we compose the corresponding flip-maps to obtain continuous piecewise-linear maps from $\mathcal{O}([a] \times [b])$ to itself.

News Flash: By lifting an argument from Propp-R. in the combinatorial category, Propp has very recently shown that promotion, and hence rowmotion, must be homomesic (in a slightly generalized sense). But we still don't have a proof that these maps have finite order.

Order of flipping affects order of the composition!

In the combinatorial category, where $\mathcal{A}(P)$ and $J(P)$ are finite, it's clear that any map defined as a product of toggles has finite order. But we can no longer take this for granted in the CPL category. Let $P = [2] \times [2]$. As we'll soon see, one can show by brute force that the CPL maps

$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,1)} \circ \sigma_{(2,2)}$ (“lifted rowmotion”) and
 $\sigma_{(2,1)} \circ \sigma_{(1,1)} \circ \sigma_{(2,2)} \circ \sigma_{(1,2)}$ (“lifted promotion”) are of order 4. However, not every composition of flips has finite order.

Proposition (Einstein)

The map $\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,2)} \circ \sigma_{(2,1)}$ (flipping values in clockwise order, as opposed to going by rows or columns of P) is of infinite order.

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the continuous piecewise-linear (CPL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \rightarrow [0, 1]$ at a point $x \in P$ with f' , where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can “detropicalize” this flip map and apply it to an assignment $f : P \rightarrow \mathbb{R}(\xi_1, \xi_2, \dots)$ of *rational functions* to the nodes of the poset (using that $\min_i(z_i) = -\max_i(-z_i)$) to get

$$f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Example of birational rowmotion

In our running example, $P = [2] \times [2]$, applying these new flip operators from top to bottom creates a new rowmotion operator. (Here we assign $f(\hat{0}) = f(\hat{1}) = 1$.)

$$\begin{array}{ccccc}
 & z & & \frac{x+y}{z} & & \frac{x+y}{z} \\
 x & & y & \mapsto & x & & y & \mapsto & \frac{w(x+y)}{xz} & & y & \mapsto & \\
 & w & & & & w & & & & & w & & \\
 & & \frac{x+y}{z} & & & & \frac{x+y}{z} & & & & \frac{x+y}{z} \\
 \frac{w(x+y)}{xz} & & & \frac{w(x+y)}{yz} & \mapsto & \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} & & & \frac{1}{z} \\
 & & w & & & & & & & &
 \end{array}$$

Example of birational rowmotion orbit

Here's an orbit of rowmotion in this category:

$$\begin{array}{ccccc}
 & z & & \frac{x+y}{z} & & \frac{w(x+y)}{xy} \\
 x & & y & \mapsto & \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} & \mapsto & \frac{1}{y} & & \frac{1}{x} & \mapsto \\
 & w & & \frac{1}{z} & & & \frac{z}{x+y} & & & & \\
 & & & \frac{1}{w} & & & z & & & & \\
 & \frac{yz}{w(x+y)} & & \frac{xz}{w(x+y)} & \mapsto & x & & y & & & \\
 & & & \frac{xy}{w(x+y)} & & & w & & & &
 \end{array}$$

Geometric Homomesy

In this category, geometric means replace arithmetic means, so let's compute the **product** of the function values at each node.

$$\begin{array}{ccccccc}
 & z & & \frac{x+y}{z} & & \frac{w(x+y)}{xy} & \\
 x & & y & \mapsto & \frac{w(x+y)}{xz} & \frac{w(x+y)}{yz} & \mapsto & \frac{1}{y} & & \frac{1}{x} & \mapsto \\
 & w & & \frac{1}{z} & & \frac{z}{x+y} & \\
 & & \frac{1}{w} & & & & & \frac{(x+y)^2}{xy} & & & \\
 \frac{yz}{w(x+y)} & & \frac{xy}{w(x+y)} & & \frac{xz}{w(x+y)} & \text{PROD} = & 1 & & 1 & & \\
 & & & & & & & \frac{xy}{(x+y)^2} & & &
 \end{array}$$

Geometric homomesy with boundary variables

If we instead generically assign variables $f(\hat{0}) = \alpha$ and $f(\hat{1}) = \omega$:

$$\begin{array}{ccccccc}
 & \omega & & & & & \\
 & z & & \frac{(x+y)\omega}{z} & & \frac{w(x+y)\omega}{xy} & \\
 x & & y & \mapsto \frac{w(x+y)\omega}{xz} & & \frac{w(x+y)\omega}{yz} & \mapsto \frac{\alpha\omega}{y} & \frac{\alpha\omega}{x} \\
 & w & & \frac{\alpha\omega}{z} & & \frac{\alpha z}{x+y} & \\
 & \alpha & & & & & \\
 & & \frac{\alpha\omega}{w} & & & \alpha\omega^3 \frac{(x+y)^2}{xy} & \\
 \frac{\alpha yz}{w(x+y)} & & \frac{\alpha xz}{w(x+y)} & \text{PROD} = & \alpha^2\omega^2 & & \alpha^2\omega^2 \\
 & \frac{\alpha xy}{w(x+y)} & & & & \alpha^3\omega \frac{xy}{(x+y)^2} &
 \end{array}$$

So the statistic “multiply opposite nodes” has *geometric mean* $\alpha\omega$ across the orbit.

For what posets does this work?

It's not hard to see that if a map such as rowmotion is homomesic with respect to some statistics in the birational category, then this implies homomesy at the CPL level, which in turn implies it in the combinatorial category.

We believe that multiplicative versions of homomesy in the birational category holds for a large class of posets, often ones that come up in representation theory. There are also simple examples of posets, e.g., the Boolean algebra B_3 for which nothing we have tried appears to hold. For example, it appears (conjecturally) that birational rowmotion has infinite order on B_3 .

- A recently identified phenomenon called *homomesy* appears to be lurking in a wide range of combinatorial situations.
- We are just beginning to develop tools for studying this, so there are many interesting open problems.
- There are intriguing conjectured generalizations to continuous piecewise-linear maps on order polytopes and to birational maps on $\{f : P \rightarrow \mathbb{R}(\xi_1, \xi_2, \dots)\}$.

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Slides for this talk are available online (or will be soon) at

<http://www.math.uconn.edu/~troby/research.html>

For more information, see:

<http://jamespropp.org/ucbcomb12.pdf>

<http://jamespropp.org/mathfest12a.pdf>

<http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf>

<http://jamespropp.org/mitcomb13a.pdf>

<http://www.math.uconn.edu/~troby/ceFPSAC.pdf>

Thanks for your attention!