Homomesy of Order Ideals in Products of Two Chains

Tom Roby (University of Connecticut)

Describing joint research with Jim Propp

Combinatorics Seminar
Dartmouth College
Hanover, NH, USA

23 May 2013

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

Abstract

For many invertible actions τ on a finite set S of combinatorial objects, and for many natural statistics φ on S, one finds that the triple (S, τ, φ) exhibits "homomesy": the average of φ over each τ-orbit in S is the same as the average of φ over the whole set S. (Example: Let S be the set of binary sequences $s = (s_1, ..., s_n)$ containing k 1's and $n - k$ 0's, let τ be the cyclic shift, and let $\varphi(s)$ be the inversion number $\#i < j : s_i > s_j$.)

This phenomenon was first noticed by Panyushev in 2007 in the context of antichains in root posets; Armstrong, Stump, and Thomas proved Panyushev's conjecture in 2011. In this talk, describing joint work with Jim Propp and Shahrzad Haddadan, we describe a theoretical framework for results of this kind, and give a number of examples (some proved and some conjectural) from different parts of combinatorics. We also discuss in detail homomesy for the operations of rowmotion and promotion (in Striker and Williams' terminology) acting on a product of two chains.

Acknowledgments

This seminar talk discusses recent work with Jim Propp, including ideas and results from Arkady Berenstein, David Einstein, Shahrzad Haddadan, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Darij Grinberg, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.

Overview

- Main definition and introductory examples;
- Poset operations generated by toggles: “promotion” & “rowmotion”;
- Homomesy in poset operations;
- Generalizing to other categories.

Please interrupt with questions!

Main Definition

For many actions τ on a finite set S of combinatorial objects, and for many natural real-valued statistics φ on S, one finds that the ergodic average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\tau^i(x))$$

is **independent** of the starting point $x \in S$.

We say that φ is **homomesic** (from Greek: "same middle") with respect to the combinatorial dynamical system (S, τ).

I'll give a variety of examples of **homomesies** (homomesic functions), some proven and others conjectural.

Please interrupt with questions!

Introductory examples

1. Rotation of bit-strings;
2. Bulgarian solitaire;
3. Promotion of Near-Standard Young Tableaux; and
4. Suter's dihedral symmetries on Young's lattice.

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

EG: $n = 4, k = 2$ gives us two orbits:

0011	0101

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

EG: $n = 4, k = 2$ gives us two orbits:

0011	0101
1001	1010
1100	0101
0110	
0011	

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

EG: $n = 4, k = 2$ gives us two orbits:

0011	0101
$1001 \mapsto 2$	$1010 \mapsto 3$
$1100 \mapsto 4$	$0101 \mapsto 1$
$0110 \mapsto 2$	
$0011 \mapsto 0$	

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\varphi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \varphi(s).$$

EG: $n = 4, k = 2$ gives us two orbits:

0011	0101
$1001 \mapsto 2$	$1010 \mapsto 3$
$1100 \mapsto 4$	$0101 \mapsto 1$
$0110 \mapsto 2$	AVG $= \frac{4}{2} = 2$
$0011 \mapsto 0$	
AVG$= \frac{8}{4} = 2$	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

$$\underline{\quad 000011 \quad 000101 \quad 001001 \quad}$$

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
100001	100010	100100
110000	010001	010010
011000	101000	001001
001100	010100	
000110	001010	
000011	000101	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
100001	100010	100100
110000	010001	010010
011000	101000	001001
001100	010100	
000110	001010	
000011	000101	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
$100001 \mapsto 4$	$100010 \mapsto 5$	$100100 \mapsto 6$
$110000 \mapsto 8$	$010001 \mapsto 3$	$010010 \mapsto 4$
$011000 \mapsto 6$	$101000 \mapsto 7$	$001001 \mapsto 2$
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
$100001 \mapsto 4$	$100010 \mapsto 5$	$100100 \mapsto 6$
$110000 \mapsto 8$	$010001 \mapsto 3$	$010010 \mapsto 4$
$011000 \mapsto 6$	$101000 \mapsto 7$	$001001 \mapsto 2$
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	
AVG$= \frac{24}{6} = 4$	AVG$= \frac{24}{6} = 4$	AVG$= \frac{12}{3} = 4$

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
$100001 \mapsto 4$	$100010 \mapsto 5$	$100100 \mapsto 6$
$110000 \mapsto 8$	$010001 \mapsto 3$	$010010 \mapsto 4$
$011000 \mapsto 6$	$101000 \mapsto 7$	$001001 \mapsto 2$
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	
AVG$= \frac{24}{6} = 4$	AVG$= \frac{24}{6} = 4$	AVG$= \frac{12}{3} = 4$

We know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate b (replacing a 01 somewhere in b by 10 or vice versa), or one can write the number of inversions in b as $\sum_{i<j} b_i(1 - b_j)$ and then perform algebraic manipulations.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define $\tau(\lambda)$ as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).

Let $\varphi(\lambda)$ be the number of parts of λ.
In the forward orbit of $\lambda = (5, 3)$, the average value of φ is

$$(4 + 3)/2 = 7/2;$$

in the forward orbit of $\lambda = (6, 2)$, the average value of φ is

$$(3 + 4 + 4 + 3)/4 = 14/4 = 7/2.$$

Proposition

If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.)

So the number-of-parts statistic on partitions of n is homomesic under the Bulgarian solitaire map.

The same is true for the size of the largest part, the size of the second largest part, etc.

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

So an equivalent way of stating our main definition in this case is, φ is homomesic with respect to (S, τ) iff the average of φ over each periodic τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$.

In the rest of this talk, we'll restrict attention to maps τ that are invertible on S, so transience is not an issue.

Example 3: Promotion of Near-Standard Young Tableaux

Given a positive integer N, define a Near-Standard Young Tableau (NSYT) with "ceiling" N as a Young tableau T in which entries are distinct integers between 1 and N.

(When N equals the number of cells of T, this is just the definition of a Standard Young Tableau.)

For each $1 \leq i \leq N-1$, let s_i be the action on NSYT's with ceiling N that replaces i (if it occurs in T) by $i+1$, and vice versa, provided that this does not violate the increasing condition in the definition of Young tableaux, and let ∂ be the composition of the maps:

$$\partial T := s_{N-1} \circ s_{N-2} \circ \cdots \circ s_1 T$$

This generalizes an operation on SYT introduced by Schützenberger called **promotion**.
For example, applying s_7 transforms the following tableau as shown:

1	4	7	10

1	4	8	10

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|} \hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} \end{array} ,$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 3 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10}s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{2} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{3} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{4} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{2} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{3} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{4} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{5} & 8 & 11 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$
T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 & \\ \cline{1-3} 7 & 9 & & \\ \cline{1-2} \end{array}\;, \quad
\begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{2} & 8 & 11 & \\ \cline{1-3} 7 & 9 & & \\ \cline{1-2} \end{array}\;, \quad
\begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{3} & 8 & 11 & \\ \cline{1-3} 7 & 9 & & \\ \cline{1-2} \end{array}\;, \quad
\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{4} & 8 & 11 & \\ \cline{1-3} 7 & 9 & & \\ \cline{1-2} \end{array}\;,
$$

$$
\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{5} & 8 & 11 & \\ \cline{1-3} 7 & 9 & & \\ \cline{1-2} \end{array}\;, \quad
\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline \color{red}{6} & 8 & 11 & \\ \cline{1-3} 7 & 9 & & \\ \cline{1-2} \end{array}\;,
$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{ytableau} 1 & 4 & 6 & 10 \\ 2 & 8 & 11 \\ 7 & 9 \end{ytableau}, \quad \begin{ytableau} 1 & 4 & 6 & 10 \\ 2 & 8 & 11 \\ 7 & 9 \end{ytableau}, \quad \begin{ytableau} 1 & 4 & 6 & 10 \\ 3 & 8 & 11 \\ 7 & 9 \end{ytableau}, \quad \begin{ytableau} 1 & 3 & 6 & 10 \\ 4 & 8 & 11 \\ 7 & 9 \end{ytableau},$$

$$\begin{ytableau} 1 & 3 & 6 & 10 \\ 5 & 8 & 11 \\ 7 & 9 \end{ytableau}, \quad \begin{ytableau} 1 & 3 & 5 & 10 \\ 6 & 8 & 11 \\ 7 & 9 \end{ytableau}, \quad \begin{ytableau} 1 & 3 & 5 & 10 \\ 6 & 8 & 11 \\ 7 & 9 \end{ytableau},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline 3 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline 4 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline 5 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}\;, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}\;,$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \color{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{2} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \color{red}{3} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{4} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \color{red}{5} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline \color{red}{6} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} \color{red}{7} & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} \color{red}{8} & 9 \\ \cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} 8 & \color{red}{9} \\ \cline{1-2}\end{array},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10} s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \textcolor{red}{1} & 4 & 6 & 10 \\ \hline 2 & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \textcolor{red}{2} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 10 \\ \hline \textcolor{red}{3} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \textcolor{red}{4} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 10 \\ \hline \textcolor{red}{5} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline \textcolor{red}{6} & 8 & 11 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \cline{1-3} \textcolor{red}{7} & 9 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} \textcolor{red}{8} & 9 \\ \cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 6 & 7 & 11 \\ \cline{1-3} 8 & \textcolor{red}{9} \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 9 \\ \hline 6 & 7 & 11 \\ \cline{1-3} 8 & \textcolor{red}{10} \\ \cline{1-2}\end{array},$$

Promotion of NSYT

Here's a step-by-step example of promotion, where the final tableaux is $\partial T = s_{10}s_9 \cdots s_1 T$.

$$T = \begin{array}{|c|c|c|c|}\hline \textcolor{red}{1}&4&6&10\\ \hline 2&8&11\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&4&6&10\\ \hline \textcolor{red}{2}&8&11\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&4&6&10\\ \hline \textcolor{red}{3}&8&11\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&6&10\\ \hline \textcolor{red}{4}&8&11\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&6&10\\ \hline \textcolor{red}{5}&8&11\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&10\\ \hline \textcolor{red}{6}&8&11\\ \cline{1-3} 7&9\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&10\\ \hline 6&8&11\\ \cline{1-3} \textcolor{red}{7}&9\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&10\\ \hline 6&7&11\\ \cline{1-3} \textcolor{red}{8}&9\\ \cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&10\\ \hline 6&7&11\\ \cline{1-3} 8&\textcolor{red}{9}\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&9\\ \hline 6&7&11\\ \cline{1-3} 8&\textcolor{red}{10}\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&9\\ \hline 6&7&10\\ \cline{1-3} 8&\textcolor{red}{11}\\ \cline{1-2}\end{array} \quad = \partial T.$$

A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942; http://arxiv.org/abs/1108.1172):

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942 1927

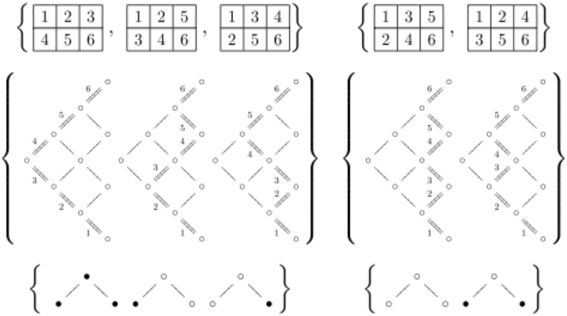


Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

A small example of promotion: centrally symmetric sums

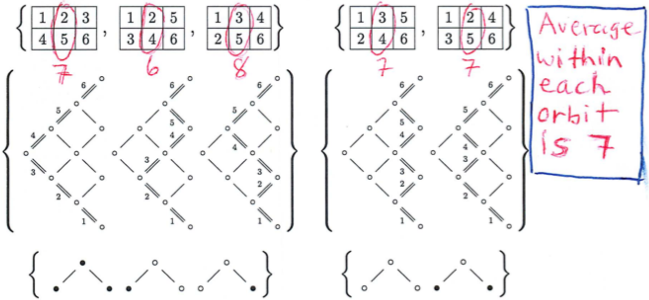


Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

Promotion of Near-Standard Young Tableaux: homomesies

Conjecture

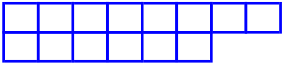
Let S be the set of Near-Standard Young Tableau of rectangular shape λ, and ceiling N. If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center, (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\varphi(T)$ denotes the sum of the numbers in cells c and c', then φ is homomesic with respect to (S, ∂) with average value $N + 1$.

Promotion of Near-Standard Young Tableaux: homomesies

Conjecture

Let S be the set of Near-Standard Young Tableau of rectangular shape λ, and ceiling N. If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center, (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\varphi(T)$ denotes the sum of the numbers in cells c and c', then φ is homomesic with respect to (S, ∂) with average value $N + 1$.

Although rectangular shapes may appear to be a very special case, they are one of the few shapes where the order of promotion on the set of SYT is small, i.e., n or $2n$. Striker & Williams point out that the order of promotion on SYT of shape $(8, 6)$ is 7,554,844,752.



Rowmotion: an invertible operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

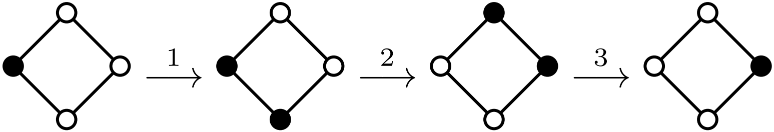
Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the downward-saturation of A.
τ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams. Following the latter we call this **rowmotion**.

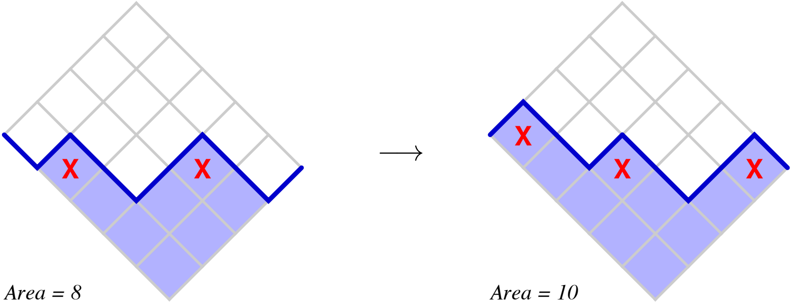
An example

1. Saturate downward
2. Complement
3. Take minimal element(s)



Example in lattice cell form

Viewing the elements of the poset as **squares** below, we would map:



Panyushev's conjecture

Let Δ be a reduced irreducible root system in $\mathbf{R}^n$. (Picture coming soon!)
Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.
Conjecture (Conjecture 2.1(iii) in D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. 30 (2009), 586-594): Let $\mathcal{O}$ be an arbitrary τ-orbit. Then

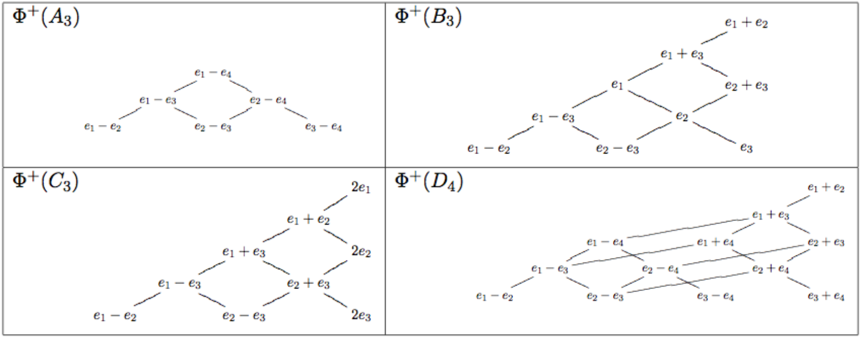
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Panyushev's Conjecture 2.1(iii) (along with much else) was proved by Armstrong, Stump, and Thomas in their article *A uniform bijection between nonnesting and noncrossing partitions*, `http://arxiv.org/abs/1101.1277`.

Picture of root posets

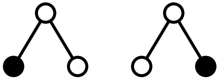
Here are the classes of posets included in Panyushev's conjecture.



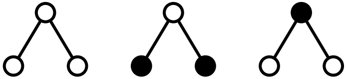
(Graphic courtesy of Striker-Williams.)

Panyushev's conjecture: The A_n case, $n = 2$

Here we have just an orbit of size 2 and an orbit of size 3:



1 1

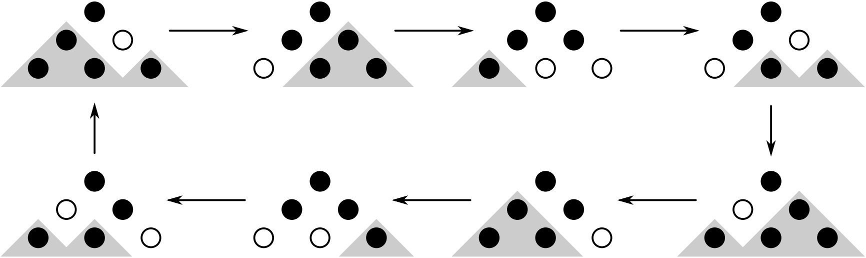


0 2 1

Within each orbit, the average antichain has cardinality $n/2 = 1$.

The case A_3.

Here's an example orbit taken from [AST] for the A_3 root poset:



For A_3 this action has three orbits (sized 2, 4, and 8), and the average cardinality of an antichain is

$$\frac{1}{8}(2+1+1+2+2+1+1+2) = \frac{3}{2}$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (where $[k]$ denotes the linear ordering of $\{1, 2, \ldots, k\}$):

Theorem (Propp, R.)

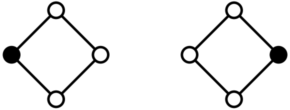
Let $\mathcal{O}$ be an arbitrary τ-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

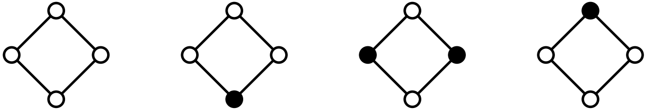
This is an easy consequence of unpublished work of Hugh Thomas building on earlier work of Richard Stanley: see the last paragraph of section 2 of R. Stanley, *Promotion and evacuation*, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i2r9 .

Antichains in $[a] \times [b]$: the case $a = b = 2$

Here we have an orbit of size 2 and an orbit of size 4:



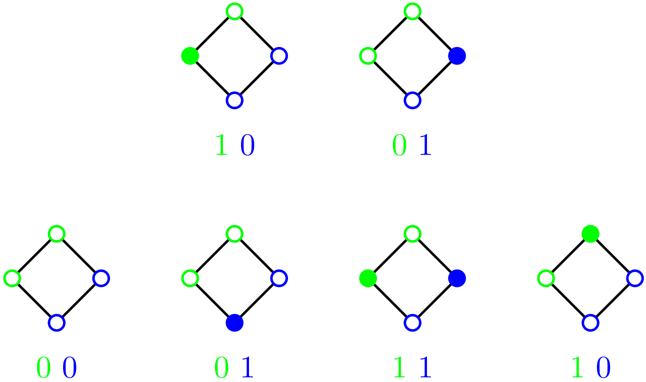
$1 \quad 1$



$0 \quad 1 \quad 2 \quad 1$

Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has
$1/2$ a green element and $1/2$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.
Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i, j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under τ, even though the indicator functions $1_{i,j}$ aren't.

Theorem (Propp, R.)

In any orbit, the number of A that contain (i, j) equals the number of A that contain the opposite element $(i', j') = (a + 1 - i, b + 1 - j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under τ, with average value 0 in each orbit.

Linearity

Useful triviality: every linear combination of homomesies is itself homomesic.

E.g., consider the adjusted major index statistic defined by $\mathrm{amaj}(A) = \sum_{(i,j)\in A}(i-j)$.

Propp and R. proved that amaj is homomesic under τ
by writing it as a linear combination of the functions $1_{i,j} - 1_{i',j'}$.
Haddadan gave a simpler proof,
writing amaj as a linear combination of the functions f_i and g_j.

Question: Are there other homomesic combinations of the indicator functions $1_{i,j}$ (with $(i,j) \in [a] \times [b]$),
linearly independent of the functions f_i, g_j, and $1_{i,j} - 1_{i',j'}$?

From antichains to order ideals

Given a poset P and an antichain A in P, let $\mathcal{I}(A)$ be the order ideal $I = \{y \in P \ : y \leq x \text{ for some } x \in A\}$ associated with A, so that for any order ideal I in P, $\mathcal{I}^{-1}(I)$ is the antichain of maximal elements of I.

As usual, we let $J(P)$ denote the set of (order) ideals of P.

We define $\overline{\tau} : J(P) \to J(P)$ by $\overline{\tau}(I) = \mathcal{I}(\tau(\mathcal{I}^{-1}(I)))$. That is, $\overline{\tau}(I)$ is the downward saturation of the set of minimal elements of the complement of I.

For $(i,j) \in P$ and $I \in J(P)$, let $\overline{1}_{i,j}(I)$ be 1 or 0 according to whether or not I contains (i,j).

One action, two vector spaces

$\overline{\tau}$ is "the same" τ in the sense that the standard bijection from $\mathcal{A}(P)$ to $J(P)$ (downward saturation) makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\tau} & \mathcal{A}(P) \\ \downarrow & & \downarrow \\ J(P) & \xrightarrow{\overline{\tau}} & J(P) \end{array}$$

However, the bijection from $\mathcal{A}(P)$ to $J(P)$ does **not** carry the vector space generated by the functions $1_{i,j}$ to the vector space generated by the functions $\overline{1}_{i,j}$ in a linear way.

So the homomesy situation for $\overline{\tau} : J(P) \to J(P)$ could be (and, as we'll see, is) different from the homomesy situation for $\tau : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{P})$.

Ideals in $[a] \times [b]$: cardinality is homomesic

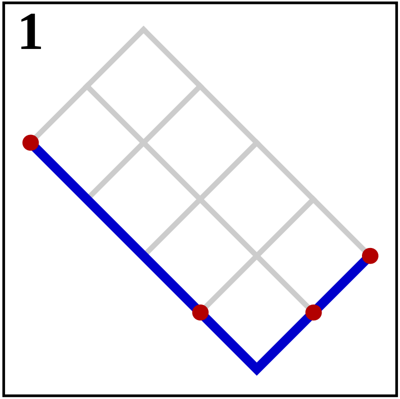
As we've seen, one can view rowmotion as acting either on *antichains* ($\mathcal{A}(P)$) or on *order ideals* ($J(P)$); we denote the latter map $\overline{\tau}$. It turns out that the cardinality of the order ideal is also homomesic w.r.t. rowmotion.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\overline{\tau}$-orbit in $J([a] \times [b])$. Then

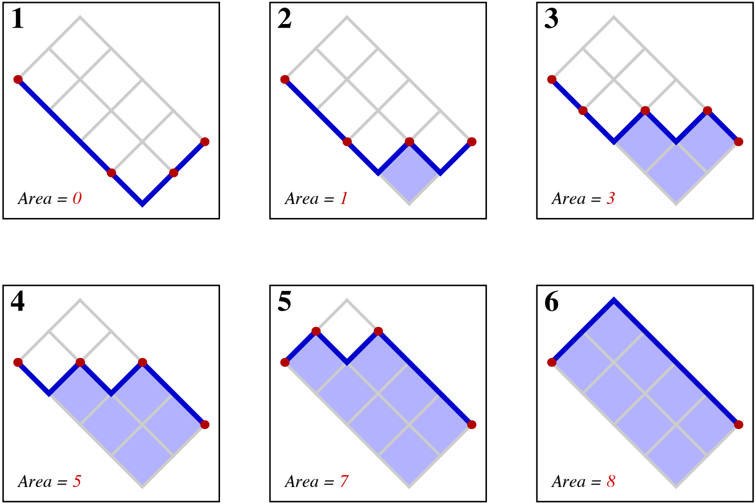
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

It's worth noting even though there's a strong connection between the rowmotion map on antichains and on order ideals, that the homomesy situation could be quite different.

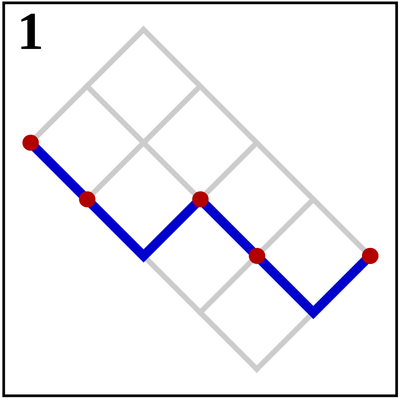


Rowmotion

Rowmotion on $[4] \times [2]$ A

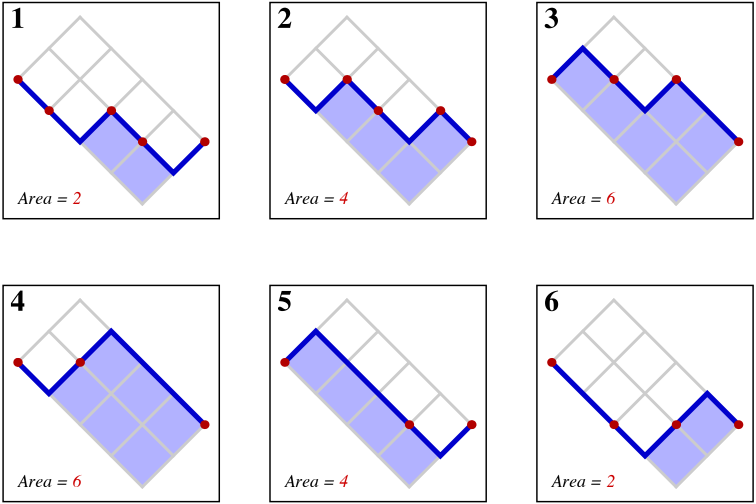


$$(0+1+3+5+7+8)\ /\ 6 = 4$$



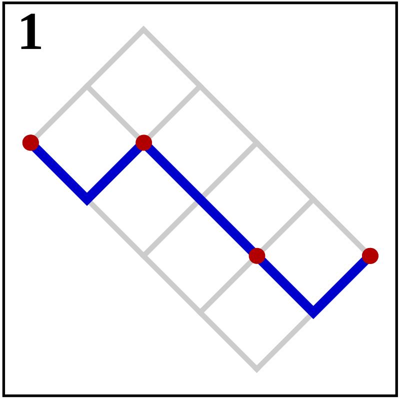
Rowmotion

Rowmotion on $[4] \times [2]$ B



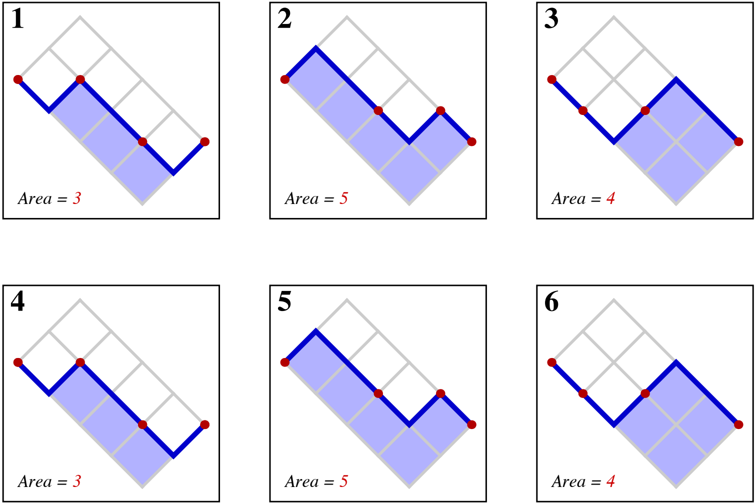
$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ C



Rowmotion

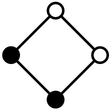
Rowmotion on $[4] \times [2]$ C



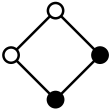
$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

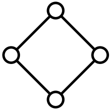
Again we have an orbit of size 2 and an orbit of size 4:



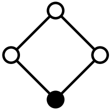
2



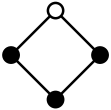
2



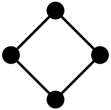
0



1



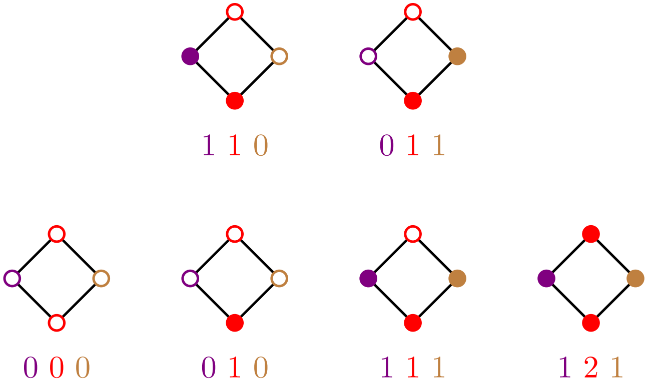
3



4

Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has
$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1-b \leq k \leq a-1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i-j=k\}.$$

For $1-b \leq k \leq a-1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every $\overline{\tau}$-orbit $\mathcal{O}$ in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Ideals in $[a] \times [b]$: centrally symmetric homomesies

Recall that for $(i, j) \in [a] \times [b]$, and I an ideal in $[a] \times [b]$, $\overline{1}_{i,j}(I)$ is 1 or 0 according to whether or not I contains (i, j).

Write $(i', j') = (a + 1 - i, b + 1 - j)$, the point opposite (i, j) in the poset.

Theorem (Propp, R.)

$\overline{1}_{i,j} + \overline{1}_{i',j'}$ *is homomesic under* $\overline{\tau}$.

Question: In addition to the functions h_k and $\overline{1}_{i,j} + \overline{1}_{i',j'}$, are there other homomesic functions in the span of the functions $\overline{1}_{i,j}$?

The two vector spaces, compared

In the space associated with antichains:

  fiber-cardinalities and

  centrally symmetric **differences**

are homomesic.

In the space associated with order ideals:

  file-cardinalities and

  centrally symmetric **sums**

are homomesic.

Toggling

In their 1995 article *Orbits of antichains revisited*, European J. Combin. 16 (1995), 545–554, Cameron and Fon-der-Flaass give an alternative description of $\overline{\tau}$.

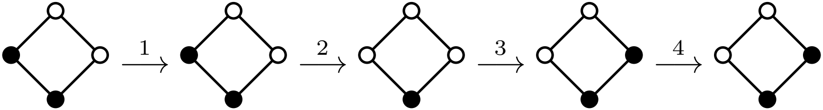
Given $I \in J(P)$ and $x \in P$, let $\tau_x(I) = I \triangle \{x\}$ provided that $I \triangle \{x\}$ is an order ideal of P; otherwise, let $\tau_x(I) = I$.

We call the involution τ_x "toggling at x".

The involutions τ_x and τ_y commute unless x covers y or y covers x.

An example

1. Toggle the top element
2. Toggle the left element
3. Toggle the right element
4. Toggle the bottom element



Toggling from top to bottom

Theorem (Cameron and Fon-der-Flaass): Let $x_1, x_2, \ldots, x_n$ be any order-preserving enumeration of the elements of the poset P. Then the action on $J(P)$ given by the composition $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ coincides with the action of $\overline{\tau}$.

In the particular case $P = [a] \times [b]$, we can enumerate P rank-by-rank; that is, we can list the (i, j)'s in order of increasing $i + j$.

Note that all the involutions coming from a given rank of P commute with one another, since no two of them are in a covering relation.

Striker and Williams refer to $\overline{\tau}$ (and τ) as **rowmotion**, since for them, "row" means "rank".

Toggling from side to side

Recall that a file in $P = [a] \times [b]$ is the set of all $(i, j) \in P$ with $i - j$ equal to some fixed value k.

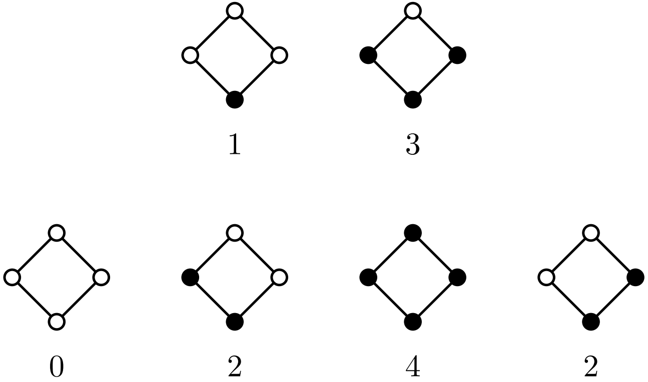
Note that all the involutions coming from a given file commute with one another, since no two of them are in a covering relation.

It follows that for any enumeration $x_1, x_2, \ldots, x_n$ of the elements of the poset $[a] \times [b]$ arranged in order of increasing $i - j$, the action on $J(P)$ given by $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ doesn't depend on which enumeration was used.

Striker and Williams call this well-defined composition **promotion**, and denote it by ∂, since it is closely related to Schützenberger's notion of promotion on linear extensions of posets.

Promoting ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



$J([a] \times [b])$: cardinality is homomesic under promotion

Claim (Propp, R.): Let $\mathcal{O}$ be an arbitrary orbit in $J([a] \times [b])$ under the action of promotion ∂. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

The result about cyclic rotation of binary words discussed earlier turns out to be a special case of this.

$J([a] \times [b])$: file-cardinality is homomesic under promotion

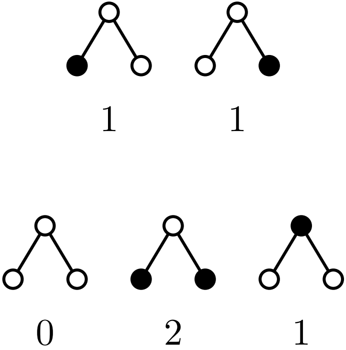
For $1-b \leq k \leq a-1$, let $f_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k f_k(I)$.

Theorem (Propp, R.): If $\mathcal{O}$ is any ∂-orbit in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} f_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Cardinality of antichains is not homomesic under promotion.
although the antipodal functions $1_{i,j} - 1_{i',j'}$ are.

Recall that, by the Armstrong-Stump-Thomas theorem, the cardinality of antichains is homomesic under the action of rowmotion, where the poset P is a root poset of type A_n.
E.g., for $n = 2$:

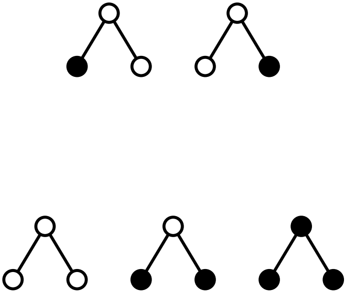


Antichain-cardinality is homomesic: in each orbit, its average is 1.

Root posets of type A: order ideals

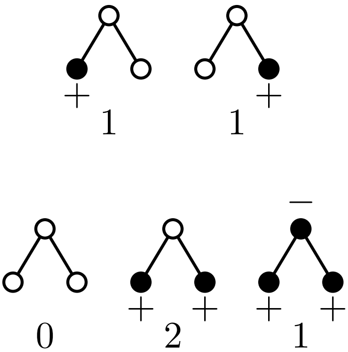
What if instead of antichains we take order ideals?

E.g., $n = 2$:



What is homomesic here?

Root posets of type A: rank-signed cardinality



Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan): Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign a order ideal $I \in J(P)$ weight $\varphi(I) = \sum_{x \in I} \mathrm{wt}(x)$, then φ is homomesic under rowmotion and promotion, with average $n/2$.

Generalizing to CPL setting: the order polytope of a poset

We can generalize this idea of composition of toggles to define a **continuous piecewise-linear (CPL)** version of rowmotion on an infinite set of functions on a poset.

Generalizing to CPL setting: the order polytope of a poset

We can generalize this idea of composition of toggles to define a **continuous piecewise-linear (CPL)** version of rowmotion on an infinite set of functions on a poset.

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.
The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

Flipping-maps in the order polytope

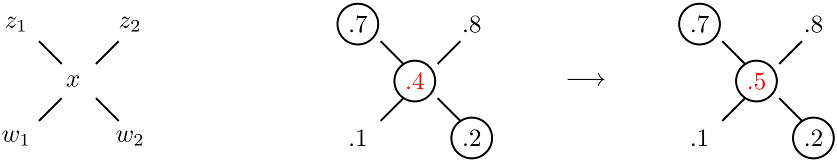
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot\, x$ means x covers w.

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot\, x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$;
the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w < \cdot\, x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example of flipping at a node



$$\min_{z \cdot > x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Flipping and toggling

If we associate each order-ideal I with the indicator function of $P \setminus I$ (that is, the function that takes the value 0 on I and the value 1 everywhere else), then toggling I at x is tantamount to flipping f at x.

That is, we can identify $J(P)$ with the vertices of the polytope $\mathcal{O}(P)$ in such a way that toggling can be seen to be a special case of flipping.

This may be clearer if you think of $J(P)$ as being in bijection with the set of monotone 0,1-valued functions on P.

Flipping

Flipping (at least in special cases) is not new, though it is not well-studied; the most worked-out example we've seen is Berenstein and Kirillov's article *Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux* (St. Petersburg Math. J. 7 (1996), 77–127); see http://pages.uoregon.edu/arkadiy/bk1.pdf.

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

$$
\begin{array}{ccc} & .8 & \\ .4 & & .3 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .6 & \\ .4 & & .3 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .6 & \\ .3 & & .3 \\ & .1 & \end{array}
$$

$$
\begin{array}{ccc} & .6 & \\ .3 & & .4 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .6 & \\ .3 & & .4 \\ & .2 & \end{array}
$$

(Here we successively flip values at the North, West, East, and South.)

Example of CPL rowmotion

Two orbits of CPL rowmotion (flipping values from top to bottom):

	.7			.7			.9			.9	
.2		.4	.6		.4	.6		.8	.6		.4
	.1			.3			.3			.1	

	1			1			1			1	
1		0	0		1	1		0	0		1
	0			0			0			0	

The average at each node across the respective orbits is:

	.8			1	
.5		.5	0.5		0.5
	.2			0	

Conjectures in the CPL category

It appears that all of the aforementioned results on homomesy for rowmotion and promotion on $J([a] \times [b])$ lift to corresponding results in the order polytope, where instead of composing toggle-maps to obtain rowmotion and promotion we compose the corresponding flip-maps to obtain continuous piecewise-linear maps from $\mathcal{O}([a] \times [b])$ to itself.

The first step would be to show that rowmotion and promotion on $\mathcal{O}([a] \times [b])$, defined as above, are maps of order $a + b$.

Order of flipping affects order of the composition!

In the combinatorial setting, where $\mathcal{A}(P)$ and $J(P)$ are finite, its clear that any map defined as a product of toggles has finite order. But we can no longer take this for granted in the CPL setting.

Let $P = [2] \times [2]$. As we'll soon see, one can show by brute force that the CPL map

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,1)} \circ \sigma_{(2,2)}$$

("CPL rowmotion") is of order four, as is

$$\sigma_{(2,1)} \circ \sigma_{(1,1)} \circ \sigma_{(2,2)} \circ \sigma_{(1,2)}$$

("CPL promotion"). However, not every composition of flips has finite order.

Proposition (Einstein): The CPL map

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,2)} \circ \sigma_{(2,1)}$$

(flipping values in clockwise order, as opposed to going by rows or columns of P) is of infinite order.

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the continuous piecewise-linear (CPL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbf{R}(x)$ of *rational functions* to the nodes of the poset (using that $\min(z_i) = -\max(-z_i)$) to get

$$f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

Example of birational rowmotion

In our running example, $P = [2] \times [2]$, applying these new flip operators from top to bottom creates a new rowmotion operator. (Here we assign $f(\hat{0}) = f(\hat{1}) = 1$.)

$$
\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array}
\mapsto
\begin{array}{ccc} & {\color{red}\frac{x+y}{z}} & \\ x & & y \\ & w & \end{array}
\mapsto
\begin{array}{ccc} & \frac{x+y}{z} & \\ {\color{red}\frac{w(x+y)}{xz}} & & y \\ & w & \end{array}
\mapsto
$$

$$
\begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & {\color{red}\frac{w(x+y)}{yz}} \\ & w & \end{array}
\mapsto
\begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & {\color{red}\frac{1}{z}} & \end{array}
$$

Example of birational rowmotion orbit

Here's an orbit of rowmotion in this category:

$$\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array} \mapsto \begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & \frac{1}{z} & \end{array} \mapsto \begin{array}{ccc} & \frac{w(x+y)}{xy} & \\ \frac{1}{y} & & \frac{1}{x} \\ & \frac{z}{x+y} & \end{array} \mapsto$$

$$\begin{array}{ccc} & \frac{1}{w} & \\ \frac{yz}{w(x+y)} & & \frac{xz}{w(x+y)} \\ & \frac{xy}{w(x+y)} & \end{array} \mapsto \begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array}$$

Geometric Homomesy

In this category, geometric means replace arithmetic means, so let's compute the **product** of the function values at each node.

$$\begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array} \mapsto \begin{array}{ccc} & \frac{x+y}{z} & \\ \frac{w(x+y)}{xz} & & \frac{w(x+y)}{yz} \\ & \frac{1}{z} & \end{array} \mapsto \begin{array}{ccc} & \frac{w(x+y)}{xy} & \\ \frac{1}{y} & & \frac{1}{x} \\ & \frac{z}{x+y} & \end{array} \mapsto$$

$$\begin{array}{ccc} & \frac{1}{w} & \\ \frac{yz}{w(x+y)} & & \frac{xz}{w(x+y)} \\ & \frac{xy}{w(x+y)} & \end{array} \qquad \text{PROD} = \begin{array}{ccc} & \frac{(x+y)^2}{xy} & \\ 1 & & 1 \\ & \frac{xy}{(x+y)^2} & \end{array}$$

Geometric homomesy with boundary variables

If we instead generically assign variables $f(\hat{0}) = \alpha$ and $f(\hat{1}) = \omega$:

$$
\begin{array}{ccc} & \omega & \\ & z & \\ x & & y \\ & w & \\ & \alpha & \end{array}
\mapsto
\begin{array}{ccc} & \frac{(x+y)\omega}{z} & \\ \frac{w(x+y)\omega}{xz} & & \frac{w(x+y)\omega}{yz} \\ & \frac{\alpha\omega}{z} & \end{array}
\mapsto
\begin{array}{ccc} & \frac{w(x+y)\omega}{xy} & \\ \frac{\alpha\omega}{y} & & \frac{\alpha\omega}{x} \\ & \frac{\alpha z}{x+y} & \end{array}
$$

$$
\begin{array}{ccc} & \frac{\alpha\omega}{w} & \\ \frac{\alpha yz}{w(x+y)} & & \frac{\alpha xz}{w(x+y)} \\ & \frac{\alpha xy}{w(x+y)} & \end{array}
\quad \mathrm{PROD} = \quad
\begin{array}{ccc} & \alpha\omega^3 \frac{(x+y)^2}{xy} & \\ \alpha^2\omega^2 & & \alpha^2\omega^2 \\ & \alpha^3\omega \frac{xy}{(x+y)^2} & \end{array}
$$

So the statistic "multiply opposite nodes" has *geometric mean* $\alpha\omega$ across the orbit.

For what posets does this work?

It's not hard to see that if a map such as rowmotion is homomesic with respect to some statistics in the birational (geometric) setting, then this implies homomesy at the CPL level, which in turn implies it in the combinatorial setting (which is the only level at which we currently have proofs).

We believe that geometrical versions of homomesy in the birational category holds for a large class of posets, often ones that come up in representation theory. There are also simple examples of posets, e.g., the Boolean algebra B_3 for which nothing we have tried appears to work.

Conclusion

- A recently identified phenomenon called *homomesy* appears to be lurking in a wide range of combinatorial settings.
- We are just beginning to develop tools for studying this, so there are many interesting open problems.
- There are intriguing conjectured generalizations to continuous piecewise-linear maps on order polytopes and to "birational" maps on $\{f : P \to \mathbf{R}(x_1, x_2, \ldots x_n)\}$.

References

- D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. **202** (2006), no. 949.
- D. Armstrong, C. Stump and H. Thomas, *A Uniform bijection between nonnesting and noncrossing partitions*, preprint, available at `arXiv:math/1101.1277v2` (2011).
- A. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math Centrum rprt ZW **24/74** (1974).
- P. Cameron and D.G. Fon-Der-Flaass, *Orbits of Antichains Revisited*, Europ. J. Comb. **16** (1995), 545–554.
- D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. **30** (2009), no. 2, 586–594.

References 2

- A. N. Kirillov, A. D. Berenstein, *Groups generated by involutions, GelfandTsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz, **7:1** (1995), 92-152.
- V. Reiner, *Non-crossing partitions for classical reflection groups*, Discrete Math. **177** (1997), 195–222.
- V. Reiner, D. Stanton, and D. White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), 17–50.
- R. Stanley, *Promotion and Evaculation*, Electronic J. Comb. **16(2)** (2009), #R9.
- J. Striker and N. Williams, *Promotion and rowmotion*, European Journal of Combinatorics **33** (2012), 1919–1942.

The last slide of this talk

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

For more information, see:

http://jamespropp.org/ucbcomb12.pdf
http://jamespropp.org/mathfest12a.pdf
http://www.math.uconn.edu/∼troby/combErg2012kizugawa.pdf
http://jamespropp.org/mitcomb13a.pdf
http://www.math.uconn.edu/~troby/ceFPSAC.pdf

Thanks for your attention!