The Eleven Clocks Problem

Tom Roby (UConn)
Joint work with Jim Propp (UMass Lowell)

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Presented at G4G 11

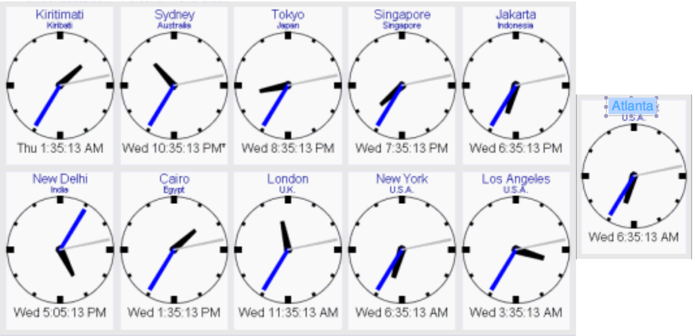
Slides available at:
http://www.math.uconn.edu/~troby/research.html
[or google "Tom Roby" for links from my website]

Who is Tom Roby?

- Mathematician and educator, based at UConn
- Interests in Combinatorics, Algebra, & Math Ed;
- Worked intensively with programs for K–12 teachers, high-ability HS students, and ugrads.
- Other interests include: folkdancing, Japanese culture & linguistics, Bulgarian singing, . . .
- More at `http://www.math.uconn.edu/~troby`

The Eleven Clocks Problem

- Imagine eleven clocks, each showing the time in a particular city.
- On the hour, you compute the median (middle value of the list) of the times shown on all the clocks.
- Repeat this eleven more times at one-hour intervals. What is the mean of the medians?



An Example 1

- On the hour, compute the median time shown on the clocks.
- Repeat this eleven more times at one-hour intervals. What is the mean of the medians?

1st hr	3	1	4	11	5	9	2	6	5	3	5
2nd hr	4	2	5	12	6	10	3	7	6	4	6
3rd hr	5	3	6	1	7	11	4	8	7	5	7
4th hr	6	4	7	2	8	12	5	9	8	6	8
5th hr	7	5	8	3	9	1	6	10	9	7	9
6th hr	8	6	9	4	10	2	7	11	10	8	10
7th hr	9	7	10	5	11	3	8	12	11	9	11
8th hr	10	8	11	6	12	4	9	1	12	10	12
9th hr	11	9	12	7	1	5	10	2	1	11	1
10th hr	12	10	1	8	2	5	11	3	2	12	2
11th hr	1	11	2	9	3	6	12	4	3	1	3
12th hr	2	12	3	10	4	7	1	5	4	2	4

An Example 2

- Reorganize the data so that the first hour is in increasing order.

1st hr	1	2	3	3	4	5	5	5	6	9	11
2nd hr	2	3	4	4	5	6	6	6	7	10	12
3rd hr	3	4	5	5	6	7	7	7	8	11	1
4th hr	4	5	6	6	7	8	8	8	9	12	2
5th hr	5	6	7	7	8	9	9	9	10	1	3
6th hr	6	7	8	8	9	10	10	10	11	2	4
7th hr	7	8	9	9	10	11	11	11	12	3	5
8th hr	8	9	10	10	11	12	12	12	1	4	6
9th hr	9	10	11	11	12	1	1	1	2	5	7
10th hr	10	11	12	12	1	2	2	2	3	6	8
11th hr	11	12	1	1	2	3	3	3	4	7	9
12th hr	12	1	2	2	3	4	4	4	5	8	10

An Example 3

- Identify in red the median for each hour. Their mean is 6.5.

1st hr	1	2	3	3	4	5	5	5	6	9	11
2nd hr	2	3	4	4	5	6	6	6	7	10	12
3rd hr	3	4	5	5	6	7	7	7	8	11	1
4th hr	4	5	6	6	7	8	8	8	9	12	2
5th hr	5	6	7	7	8	9	9	9	10	1	3
6th hr	6	7	8	8	9	10	10	10	11	2	4
7th hr	7	8	9	9	10	11	11	11	12	3	5
8th hr	8	9	10	10	11	12	12	12	1	4	6
9th hr	9	10	11	11	12	1	1	1	2	5	7
10th hr	10	11	12	12	1	2	2	2	3	6	8
11th hr	11	12	1	1	2	3	3	3	4	7	9
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This is still open.

Bulgarian solitaire on variable-sized decks

Bulgarian Solitaire was a game popularized by Martin Gardner in a 1983 article in *Scientific American*. Here we play the game on a deck of any size.

Bulgarian Solitaire

Start with the cards in any collection of heaps. Remove one card from each heap and use them to make a new heap.

For convenience, we reorder the heaps to be decreasing by size. A heap disappears if it has only one card which gets removed.

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E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).

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The mean number of heaps in each cycle will be the same from any starting point.

Actions with Interesting Averages

The more general phenomenon Jim Propp & I and several others are studying is this:

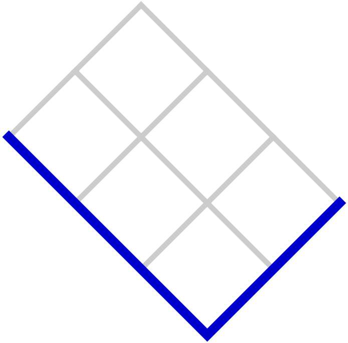
Homomesy

Find actions on sets of objects which preserve certain statistics **on average**, i.e., across any orbit of the action, the statistic should have the same value. We call this **homomesy**.

For example, in the problem above, the objects are "lists of times", the action is "increment by one hour", and the statistic is "median".

But there are many other examples. Some of my favorites involve actions on certain subsets (called order ideals) of rectangular graphs (posets). These have been coded up by Mike La Croix into some wonderful applets.

Promotion on $[3] \times [2]$ A



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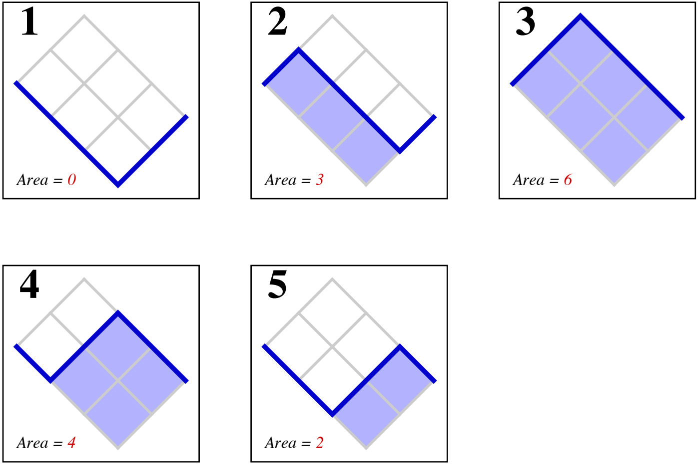
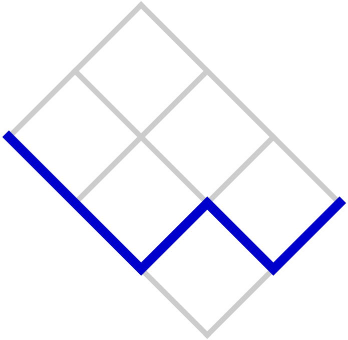


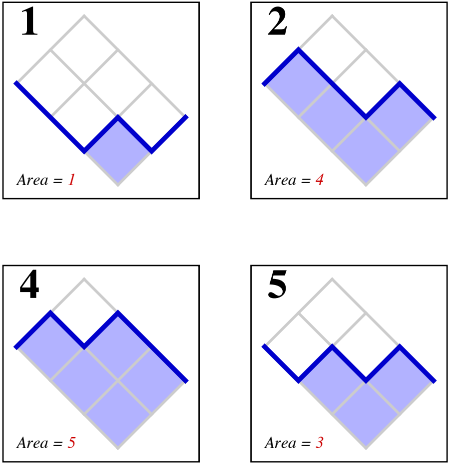
$$(0+3+6+4+2)\ /\ 5 = 3$$

Promotion on $[3] \times [2]$ B



Promotion

Promotion on $[3] \times [2]$ B



$$(1+4+2+5+3)\,/\,5 = 3$$

For more information

- The community of people working on this kind of dynamical combinatorics has identified many examples of homomesy within combinatorics;
- More details at `http://arxiv.org/abs/1310.5201`;
- Better yet, google "Tom Roby" for links from my website (including this talk);
- Please let us know if you encounter this phenomenon in your own work with puzzles, etc!

`http://www.math.uconn.edu/~troby`