Lifting Rowmotion to higher realms and noncommutative periodicity

Tom Roby (UConn)

Combinatorics Seminar
Dartmouth College
Hanover, NH USA
AND virtually over Zoom

23 May 2023

Slides for this talk are available online (or will be soon) at

Google "Tom Roby".

Abstract

Abstract: Within dynamical algebraic combinatorics one well-studied map is *rowmotion*, which permutes the order ideals (or the antichains) of a finite poset. On many posets, the orbit structure is interesting, periodicity occurs surprisingly quickly, and many natural statistics satisfy the *homomesy* (constant average for each orbit) property.

This entire story can be lifted to three higher levels: (a) the piecewise-linear realm of order/chain polytopes of a poset; (b) the birational realm of poset labelings by rational expressions; and (c) the noncommutative realm, with partial maps on poset labelings by elements of any ring. Antichains and order ideals provide two parallel liftings to each realm which can be directly related to each other. While some properties generalized surprisingly straightforwardly, others were more challenging. In particular, periodicity in the noncommutative realm for rectangular posets was only settled fairly recently in joint work with Darij Grinberg.

Acknowledgments

This talks discusses the work of several authors, including joint work with Darij Grinberg, Mike Joseph, Gregg Musiker, and Jim Propp.

I'm grateful to Darij Grinberg, Mike Joseph and Soichi Okada for sharing source code for slides from their talks, which I shamelessly cannibalized.

Thanks also to Drew Armstrong, Arkady Berenstein, Anders Björner, Karen Edwards, Robert Edwards, David Einstein, Sergi Elizalde, Max Glick, Shahrzad Haddadan, Sam Hopkins, Maxim Kontsevich, Joe Johnson, Mike La Croix, Svante Linusson, Ricky Liu Gregg Musiker, Soichi Okada, Pavlo Pylyavskyy, Vic Reiner, Ralf Schiffler, David Speyer, Jessica Striker, Richard Stanley, Hugh Thomas, Nathan Williams, Pete Winkler, and Ben Young.

Mathematisches Forschungsinstitut Oberwolfach provided hospitality in July/August 2021, when we found the tools to resolve noncommutative periodicity.

Please feel free to interject questions and comments in person or in the chat, and the moderator will convey them with appropriate timing and finesse. Or someone else may answer them!

Outline

In this talk we have two types of rowmotion, which we lift in parallel, four realms for each:

1. Combinatorial rowmotion on the set of antichains of a poset P, $\rho_{\mathcal{A}}$;
2. Piecewise-linear rowmotion on the chain polytope of P, $\rho_{\mathcal{C}}$;
3. Birational Antichain Rowmotion (BAR-motion) on $\mathbb{K}$-labelings of P, BAR;
4. Noncommutative Antichain Rowmotion (NAR-motion) on $\mathbb{K}$-labelings of P, NAR;
5. Combinatorial rowmotion on order filters/ideals of P, $\rho_{\mathcal{J}}$;
6. Piecewise-linear rowmotion on the order polytope of P, $\rho_{\mathcal{O}}$;
7. Birational Order Rowmotion (BOR-motion) on $\mathbb{K}$-labelings of P, BOR;
8. Noncommutative Order Rowmotion (NOR-motion) on $\mathbb{K}$-labelings of P, NOR;

THEMES in DAC:

1. Periodicity/order and orbit structure;
2. Homomesy: statistics with the same average over every orbit;
3. Equivariant bijections: often give nice proofs;

Antichain Rowmotion on Posets

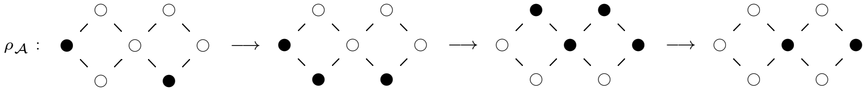
Rowmotion: an invertible operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the set of minimal elements of the complement of the *downward-saturation* of A (the smallest order ideal containing A).

$\rho_{\mathcal{A}}$ is invertible since it is a composition of three invertible operations:

antichains $\longleftrightarrow$ order ideals $\longleftrightarrow$ order filters $\longleftrightarrow$ antichains



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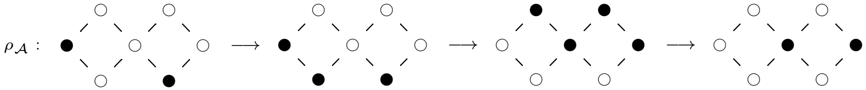
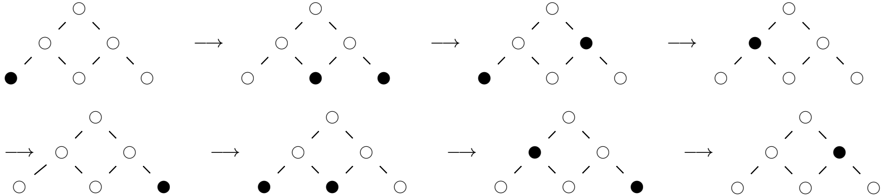
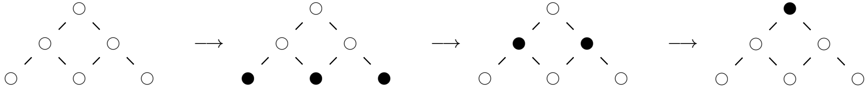


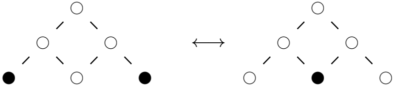
This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Example of antichain rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:







Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbb{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong–Stump–Thomas [AST11], Conj. [Pan09])

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language: the cardinality statistic is *homomesic* with respect to the action of rowmotion on antichains in root posets.

Picture of root posets

Here are the main classes of posets included in Panyushev's conjecture.

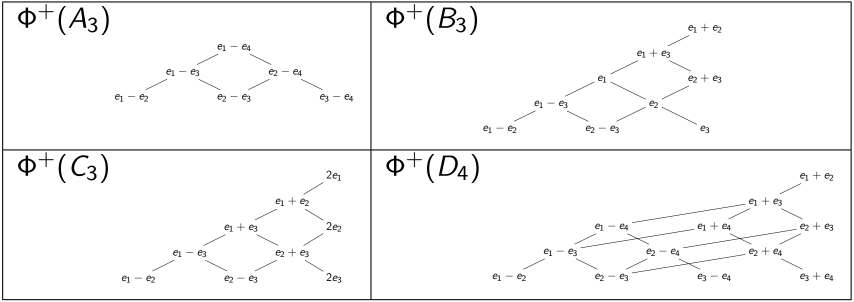


Figure: The positive root posets A_3, B_3, C_3, and D_4.

(Graphic courtesy of Striker–Williams.)

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

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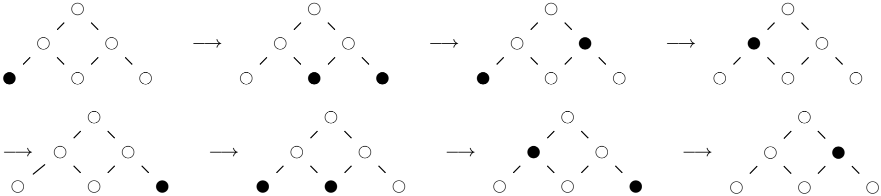
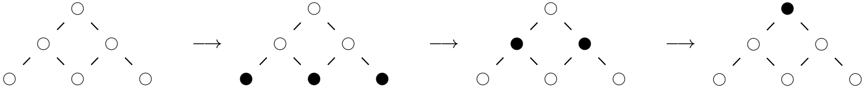
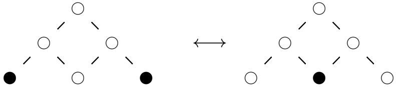


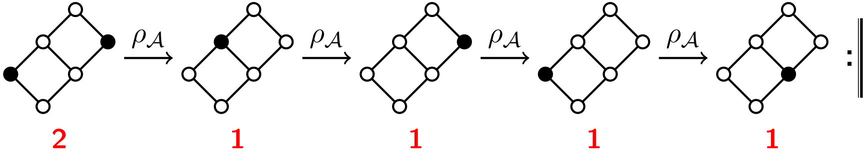




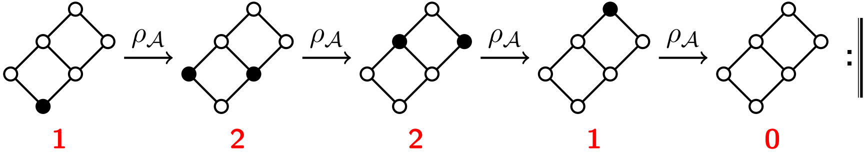
Checking the average cardinality for each orbit we find that

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Orbits of rowmotion on antichains of $[2] \times [3]$



Average cardinality: 6/5



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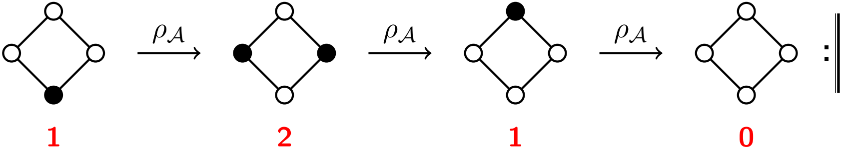

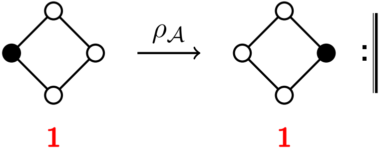

Average cardinality: 1

For antichain rowmotion on this poset, periodicity has been known for a long time:

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

Antichain rowmotion on $[a] \times [b]$: cardinality is homomesic

For rectangular posets $[a] \times [b]$ (the type A *minuscule* poset, where $[k] = \{1, 2, \ldots, k\}$), the cardinality homomesy is easier to show than for root posets.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}.$$

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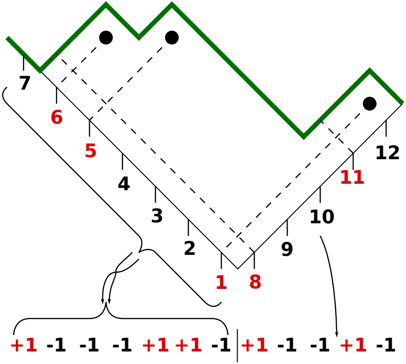

(Graphic courtesy of Ben Young.)

The simplest proof uses an non-obvious equivariant bijection (the "Stanley–Thomas" word [Sta09, §2]) between antichains in $[a] \times [b]$ and binary strings, which carries the $\rho_{\mathcal{A}}$ map to cyclic rotation of bitstrings.

The figure shows the ST-word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

Antichain rowmotion on $[a] \times [b]$: cardinality is homomesic

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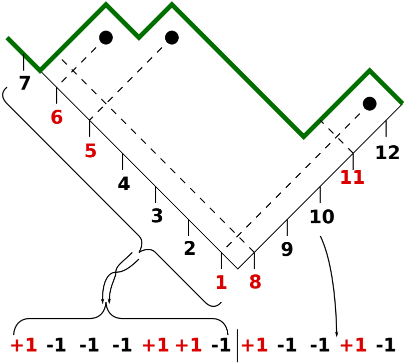

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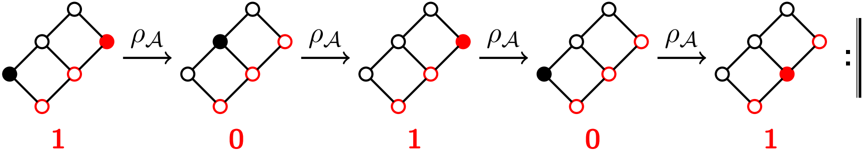
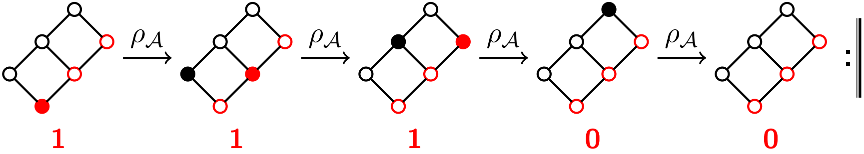
This bijection also allowed Propp–R. to derive refined homomesy results for fibers and antipodal points in $[a] \times [b]$.

Orbits of rowmotion on antichains of $[2] \times [3]$: Refined homomesies

Look at the cardinalities across a **positive fiber** such as the one highlighted in red.



Average: 3/5



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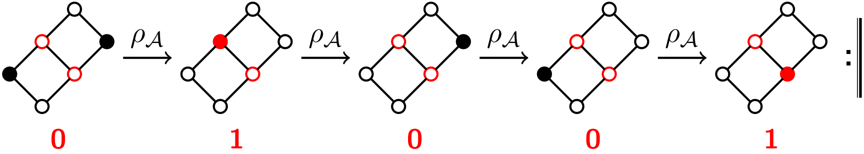
Orbits of rowmotion on antichains of $[2] \times [3]$: Refined homomesies

How about across a **negative fiber** such as the one highlighted in red.



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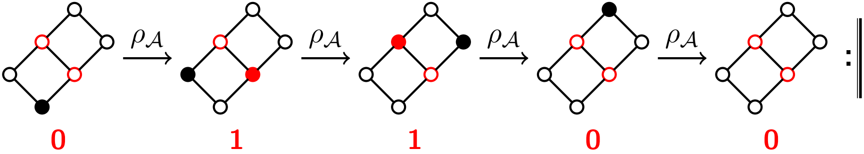


Average: 2/5

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $\mathbb{1}_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} \mathbb{1}_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} \mathbb{1}_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem ([PrRo15])

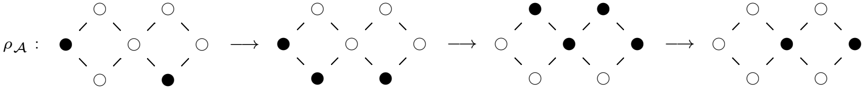
For all i,j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

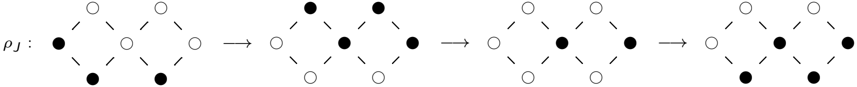
The indicator functions f_i and g_j are homomesic under $\rho_{\mathcal{A}}$, even though the indicator functions $\mathbb{1}_{i,j}$ aren't.

Rowmotion on order ideals and order filters

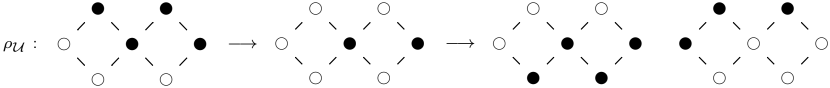
We've already seen examples of Rowmotion on antichains $\rho_{\mathcal{A}}$:



We can also define it as an operator ρ on $J(P)$, the set of order ideals (down-sets) of a poset P, by shifting the waltz beat by 1:



Or as an operator on the *order filters (up-sets)* $\mathcal{U}(P)$, of P:



Rowmotion via Toggling (Rowmotion in Slow motion)

Toggling order filters

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order filters* (equivalently *order ideals*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{U}(P)$ be the set of order filters of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{U}(P) \to \mathcal{U}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{U}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{U}(P), \\ U & \text{otherwise.} \end{cases}$$

Theorem (Cameron and Fon-Der-Flaass 1995)

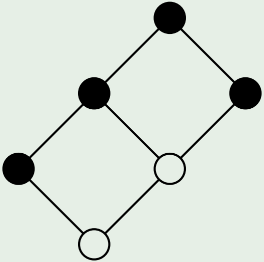
Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order filters of P.

Toggling Order filters and order rowmotion

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order filters of P.

Example



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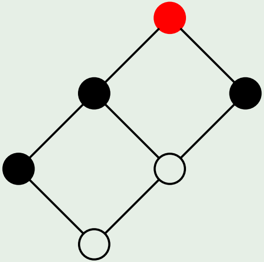

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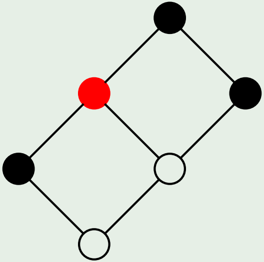

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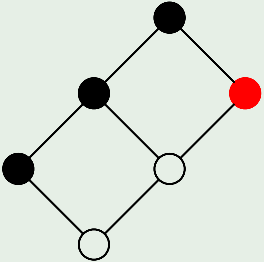

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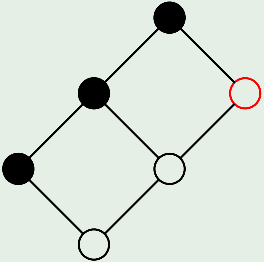

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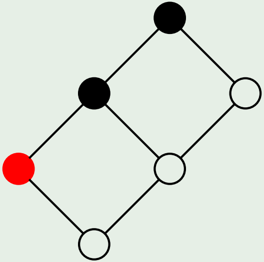

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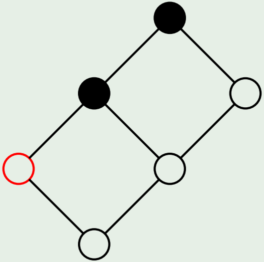

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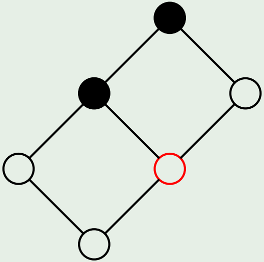

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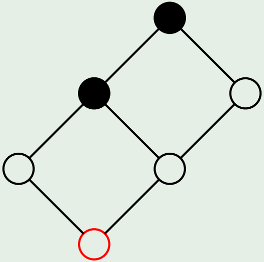

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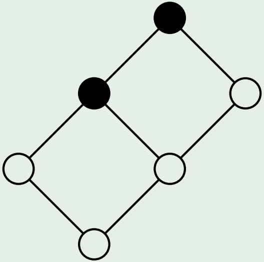


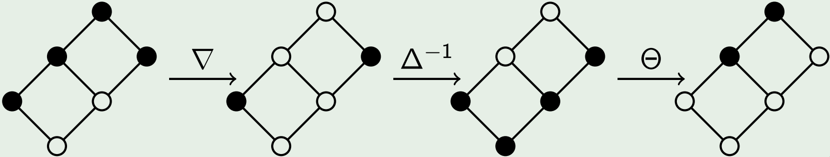
Rowmotion

This step-by-step toggling process gives the same result as the three-step one mentioned earlier:

Start with an order filter, and

1. ∇: Take the minimal elements (giving an antichain)
2. Δ^{-1}: Saturate downward (giving a order ideal)
3. Θ: Take the complement (giving an order filter)

Example



Antichain toggling and rowmotion

Striker has generalized the notion of toggles relative to any class of "allowed" subsets, not necessarily order filters.

Definition

Let $e \in P$. Then the **antichain toggle** corresponding to e is the map $\tau_e : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_e(A) = \begin{cases} A \cup \{e\} & \text{if } e \notin A \text{ and } A \cup \{e\} \in \mathcal{A}(P), \\ A \setminus \{e\} & \text{if } e \in A, \\ A & \text{otherwise.} \end{cases}$$

Let $\mathrm{Tog}_{\mathcal{A}}(P)$ denote the **toggle group** of $\mathcal{A}(P)$ generated by the toggles $\{\tau_e \mid e \in P\}$.

Theorem (Joseph 2017)

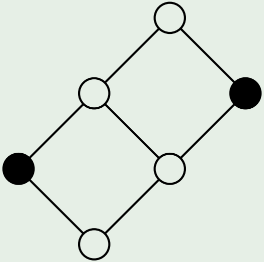
Applying the antichain toggles τ_e from bottom to top along a linear extension of P gives $\rho_{\mathcal{A}}$, rowmotion on antichains of P.

Antichain toggling and rowmotion

Theorem (Joseph 2017)

Applying the antichain toggles τ_e *from bottom to top on* P *gives* $\rho_{\mathcal{A}}$*, rowmotion on antichains of* P*.*

Example



Antichain toggling and rowmotion

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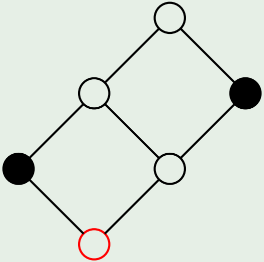

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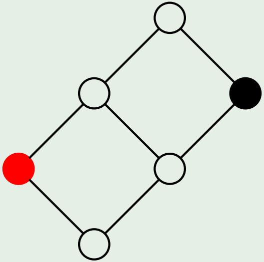

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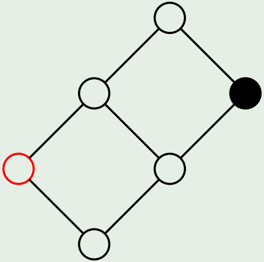

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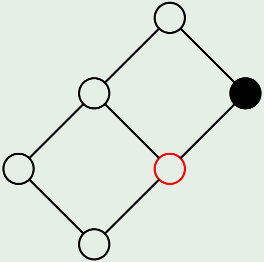

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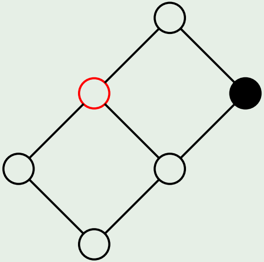

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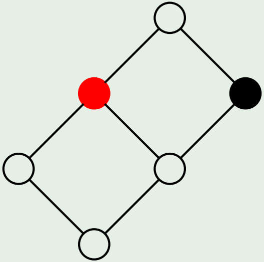

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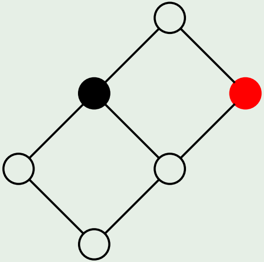

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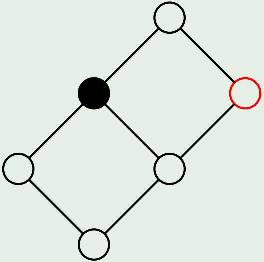

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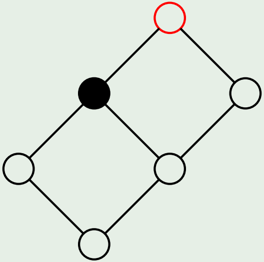

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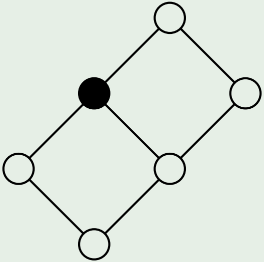
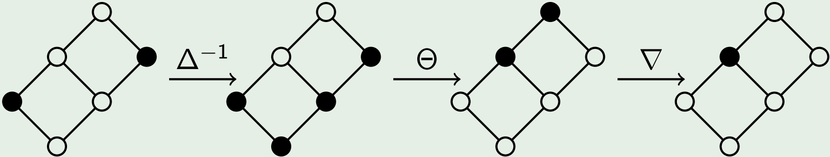


This gives the same result as the 3-step process

1. Δ^{-1}: Saturate downward (giving a order ideal)
2. Θ: Take the complement (giving an order filter)
3. ∇: Take the minimal elements (giving an antichain)

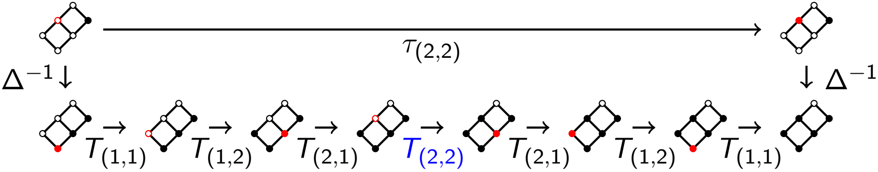
Example



Toggle Group Isomorphisms

Let $\mathrm{Tog}_{\mathcal{J}}(P) := \langle T_v : v \in P \rangle$, the **order toggle group**. Let $\mathrm{Tog}_{\mathcal{A}}(P) := \langle \tau_v : v \in P \rangle$, the **antichain toggle group**. M. Joseph constructed an explicit isomorphism between these: Set $\eta_e := T_{x_1} T_{x_2} \cdots T_{x_k}$, where $(x_1, x_2, \ldots, x_k)$ is a linear extension of the subposet $\{x \in P : \ x < e\}$. Then $\tau_e^* := \eta_e T_e \eta_e^{-1}$ mimics the action of τ_e.

$$\begin{tikzcd}
\mathcal{A}(P) \arrow[r, "\tau_e"] \arrow[d, "\Delta^{-1}"'] & \mathcal{A}(P) \arrow[d, "\Delta^{-1}"] \\
\mathcal{J}(P) \arrow[r, "\tau_e^*"'] & \mathcal{J}(P)
\end{tikzcd}$$

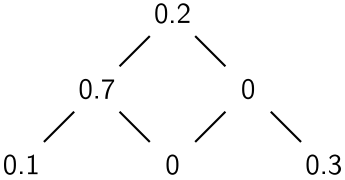


The piecewise-linear realm (Chain and Order Polytopes)

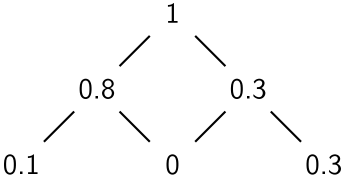
Generalization to the piecewise-linear realm

Stanley defined some polytopes associated with posets [Sta86].

- $\mathcal{C}(P)$ is the **chain polytope** of P, the set of $f \in [0,1]^P$ such that $\sum_{i=1}^{n} f(x_i) \leq 1$ for all chains $x_1 < x_2 < \cdots < x_n$.
- $\mathcal{O}(P)$ is the **order polytope** of P, the set of all order-preserving labelings $f \in [0,1]^P$. Saying f is order-preserving means $f(x) \leq f(y)$ when $x \leq y$ in P.



$\in \mathcal{C}(P)$;



$\in \mathcal{O}(P)$

Generalizing toggling to the piecewise-linear realm

Definition (Einstein–Propp)

Set $\widehat{P} := P \cup \{\widehat{0}, \widehat{1}\}$. The **piecewise-linear order toggle** $T_v : \mathcal{O}(P) \to \mathcal{O}(P)$ is (where $f\left(\widehat{0}\right) = 0$ and $f\left(\widehat{1}\right) = 1$ are fixed)

$$\left(T_v(f)\right)(x) = \begin{cases} f(x) & \text{if } x \neq v \\ \max\limits_{y \lessdot v} f(y) + \min\limits_{y \gtrdot v} f(y) - f(v) & \text{if } x = v \end{cases}$$

"Midpoint reflection of $f(v)$ in allowable interval $\left[\max\limits_{y \lessdot v} f(y), \min\limits_{y \gtrdot v} f(y)\right]$."

Definition (M. Joseph)

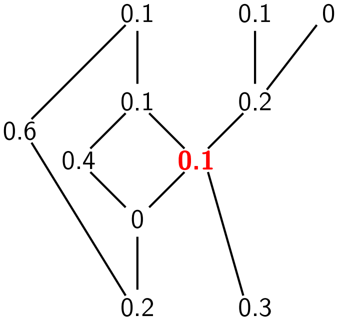
For $v \in P$, let $\mathrm{MC}_v(P)$ denote the set of all maximal chains of P through v. The **piecewise-linear antichain toggle** (or **chain polytope toggle**) $\tau_v : \mathcal{C}(P) \to \mathcal{C}(P)$ is

$$\left(\tau_v(g)\right)(x) = \begin{cases} 1 - \max\left\{ \sum\limits_{i=1}^{k} g(y_i) \,\middle|\, (y_1, \ldots, y_k) \in \mathrm{MC}_v(P) \right\} & \text{if } x = v \\ g(x) & \text{if } x \neq v \end{cases}.$$

Toggles on the chain polytope $\mathcal{C}(P)$

As usual, To define $\tau_e : \mathcal{C}(P) \to \mathcal{C}(P)$, given $g \in \mathcal{C}(P)$ and $e \in P$, $\tau_e(g)$ can only differ from g at the value of e.

$$\big(\tau_e(g)\big)(e) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c}(y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e\end{array} \right\}$$



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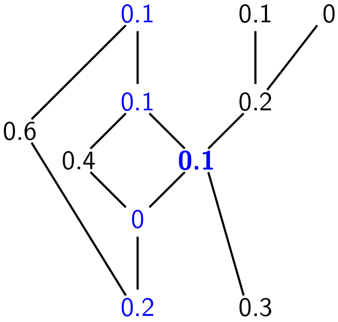

$$0.2 + 0 + 0.1 + 0.1 + 0.1 = 0.5$$

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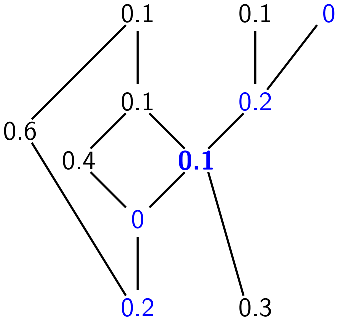

$$0.2 + 0 + 0.1 + 0.2 + 0 = 0.5$$

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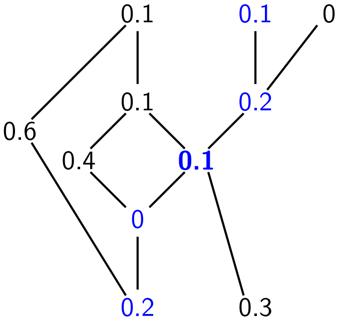

$$0.2 + 0 + 0.1 + 0.2 + 0.1 = 0.6$$

Toggles on the chain polytope $\mathcal{C}(P)$

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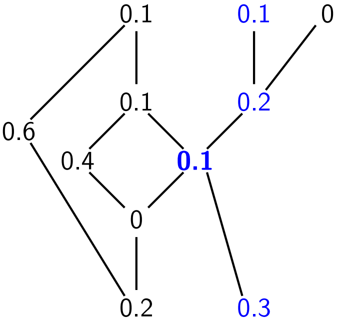

$$0.3 + 0.1 + 0.2 + 0.1 = 0.7$$

Toggles on the chain polytope $\mathcal{C}(P)$

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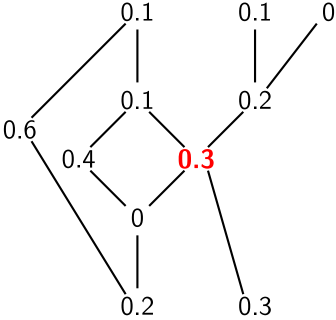
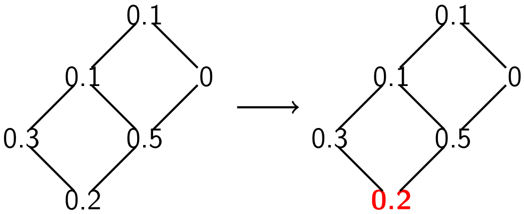


0.7 is max and $1 - 0.7 = 0.3$

Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



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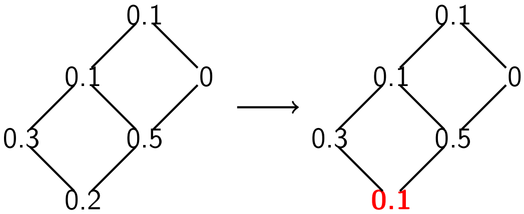

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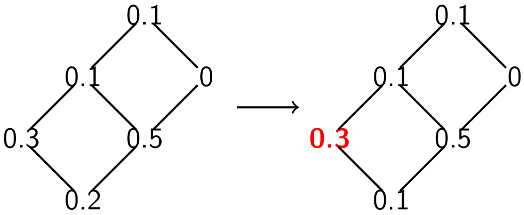

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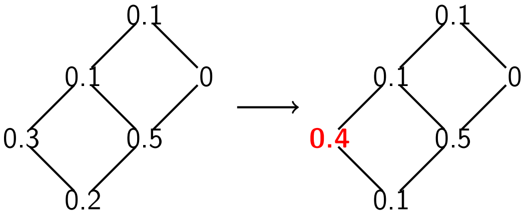

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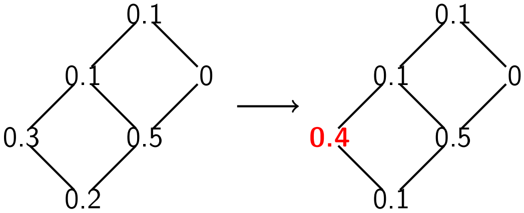

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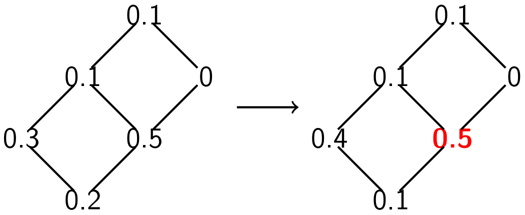

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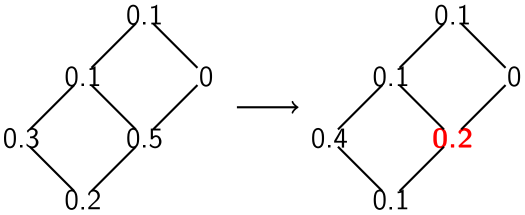

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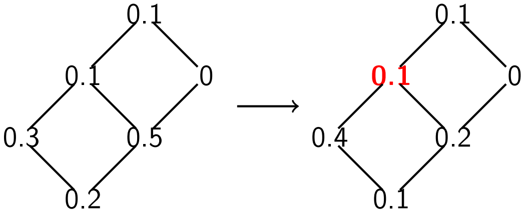

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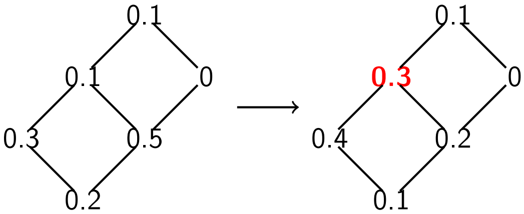

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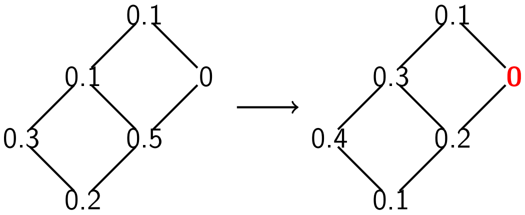

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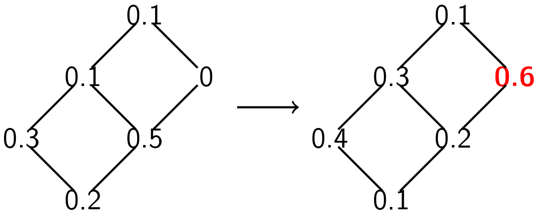

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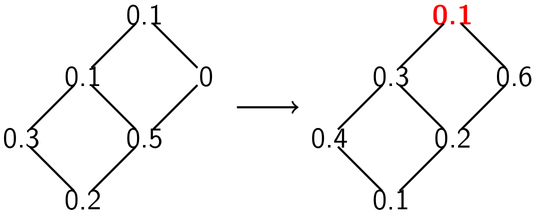

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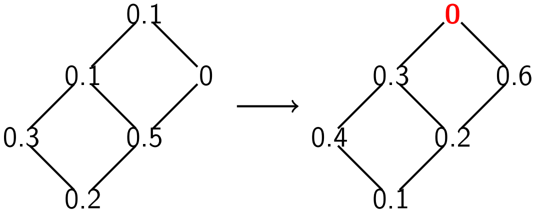

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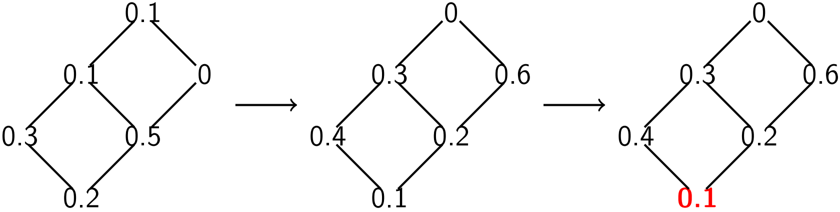

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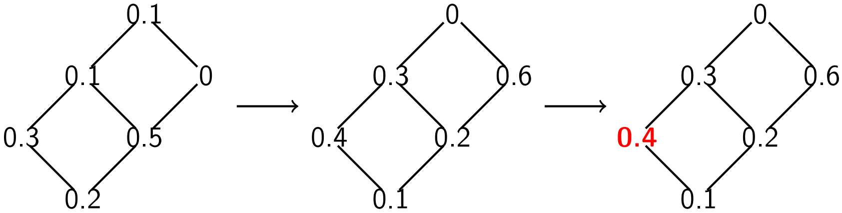

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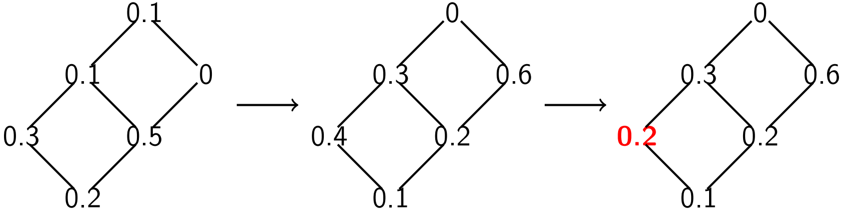

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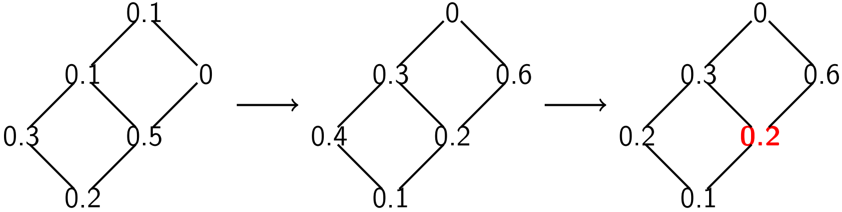

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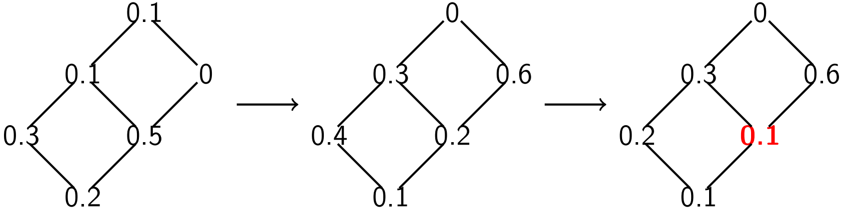

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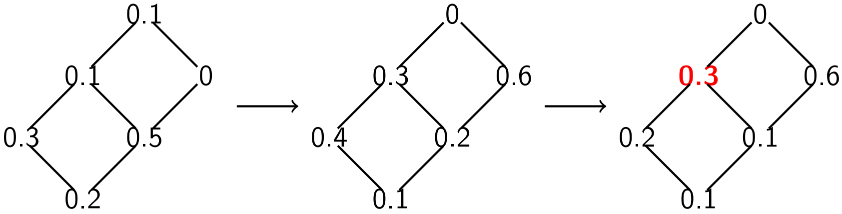

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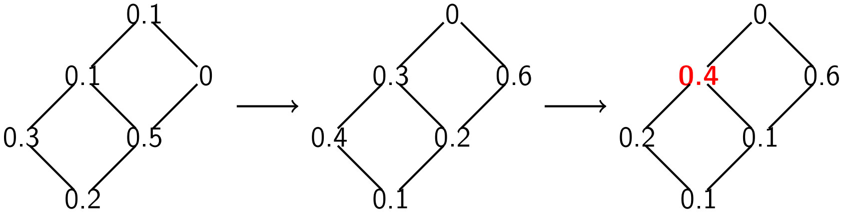

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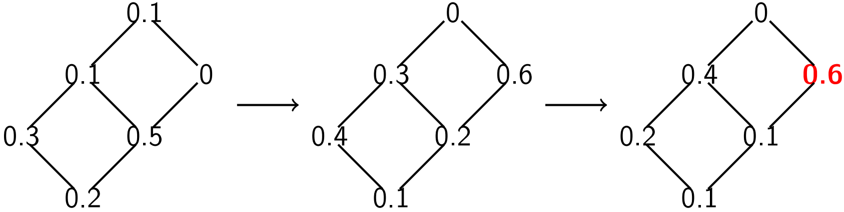

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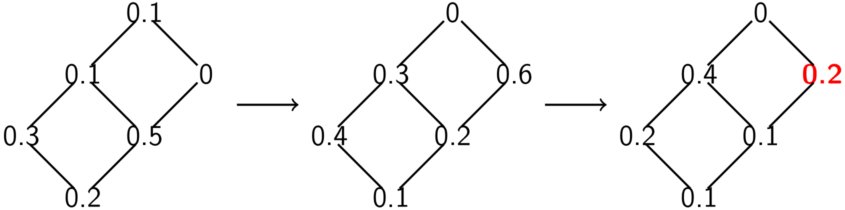

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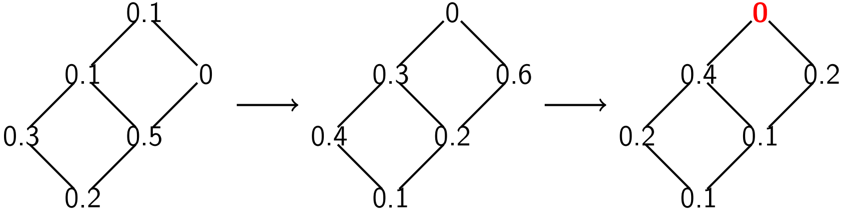
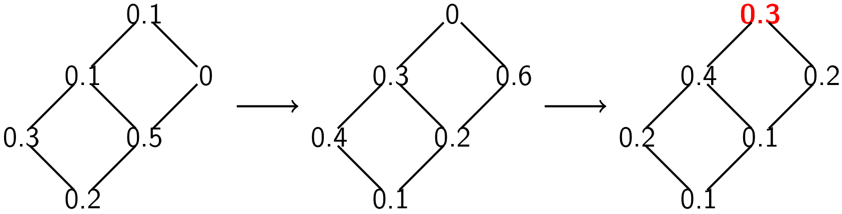
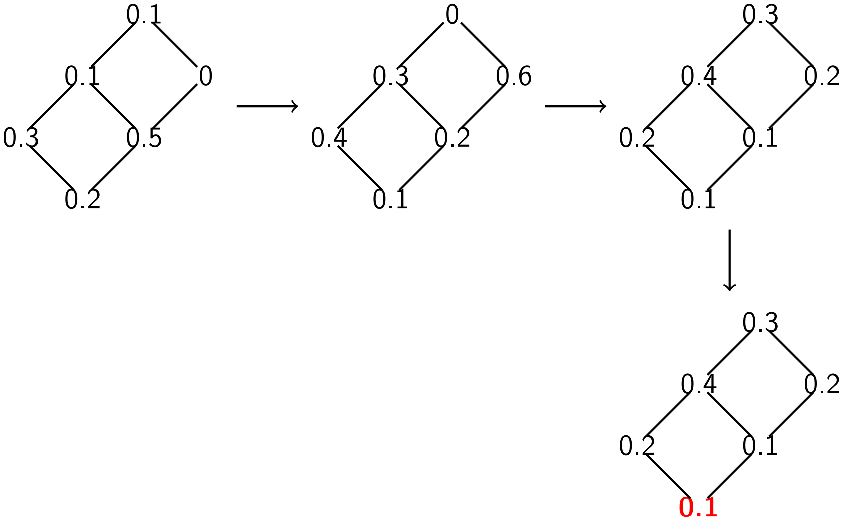


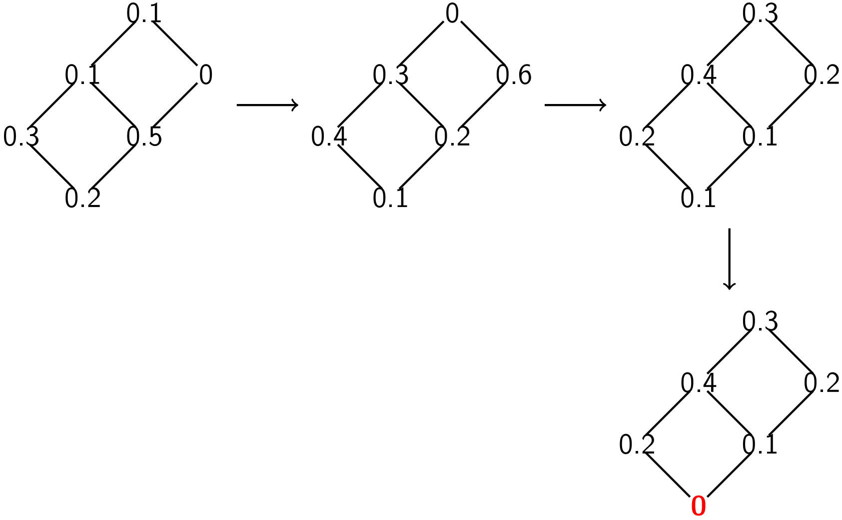
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



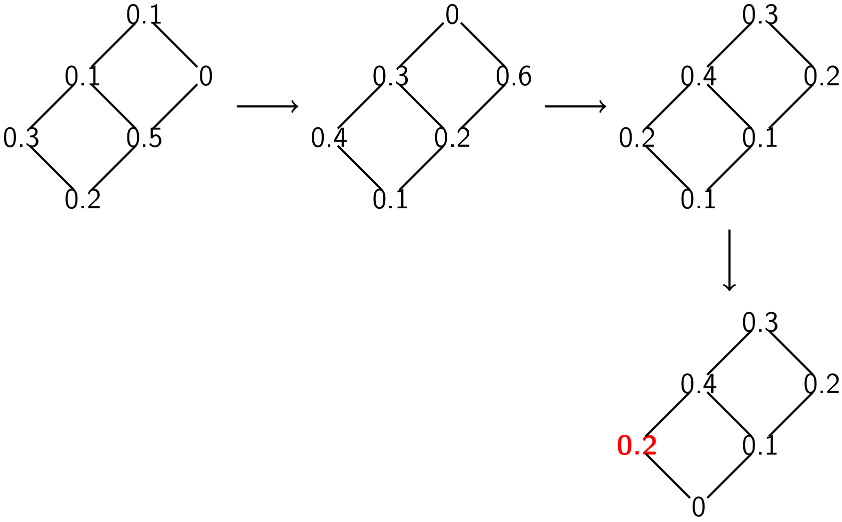
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



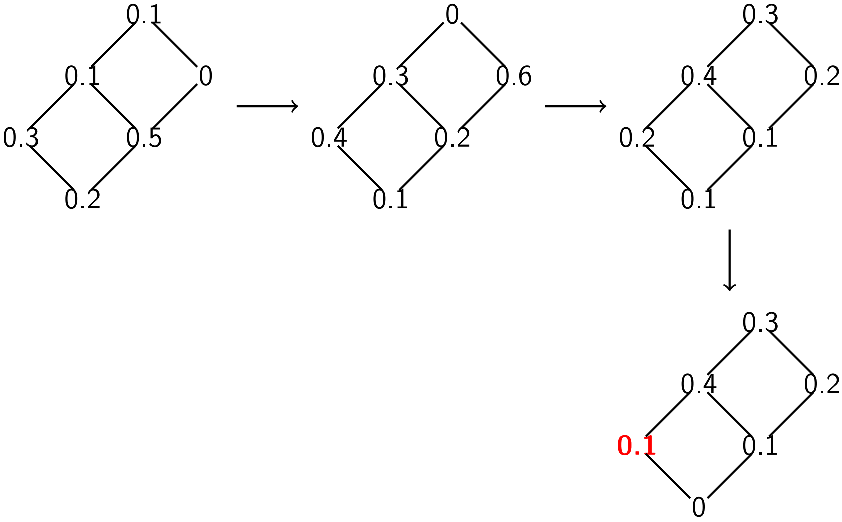
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



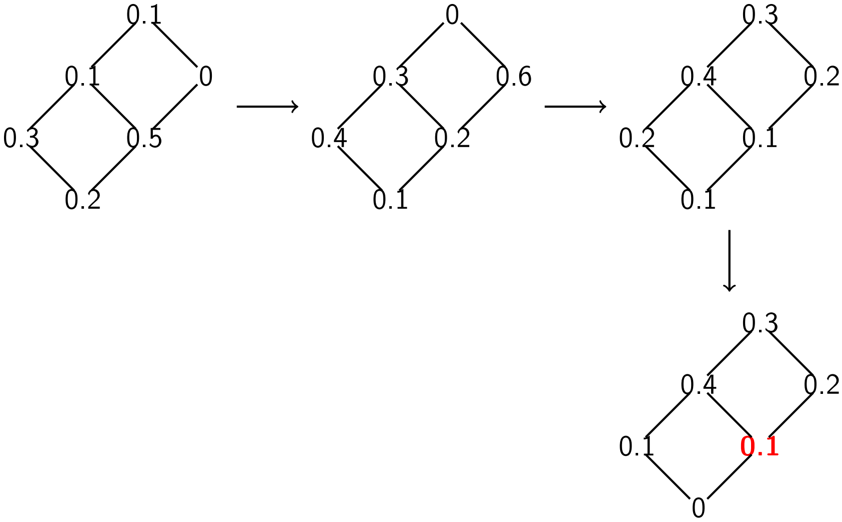
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



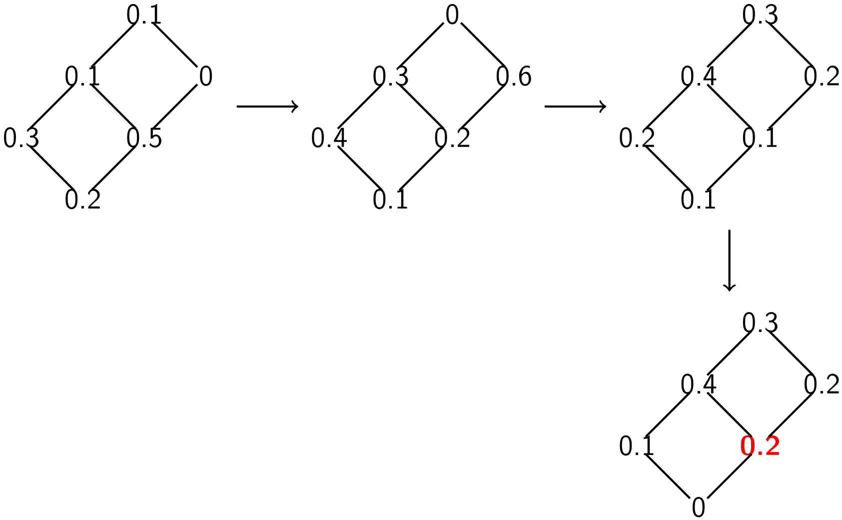
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



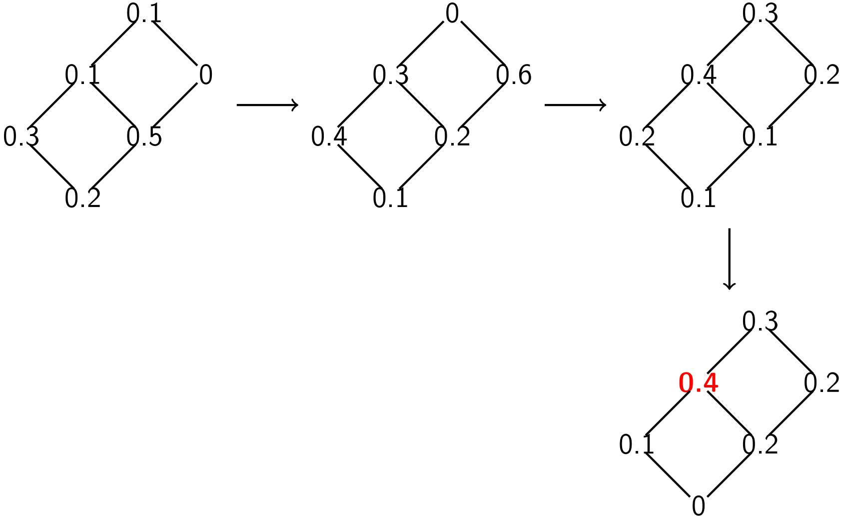
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



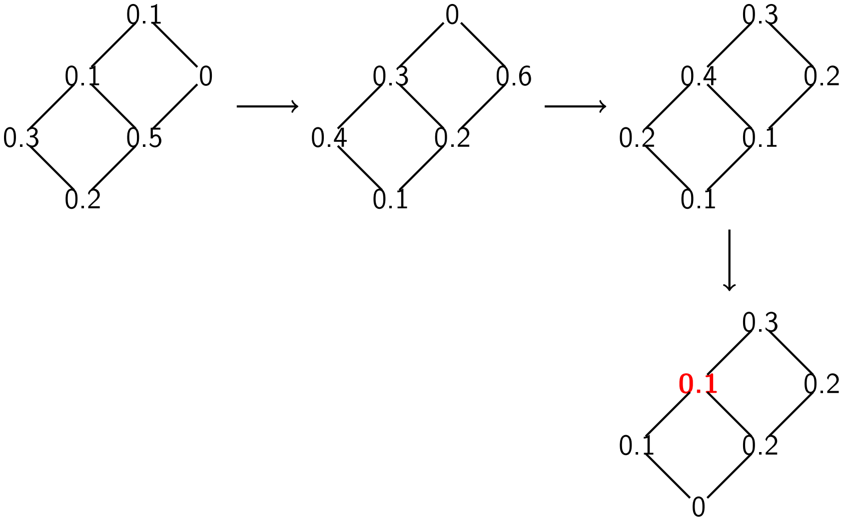
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



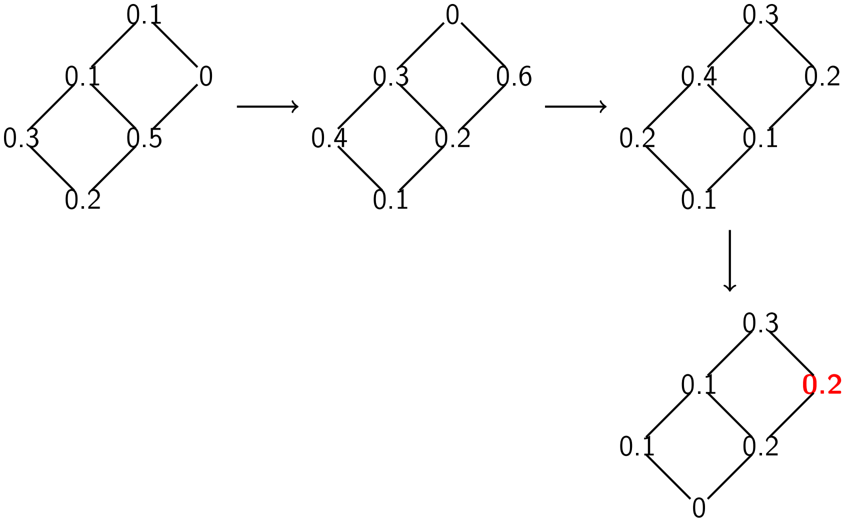
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



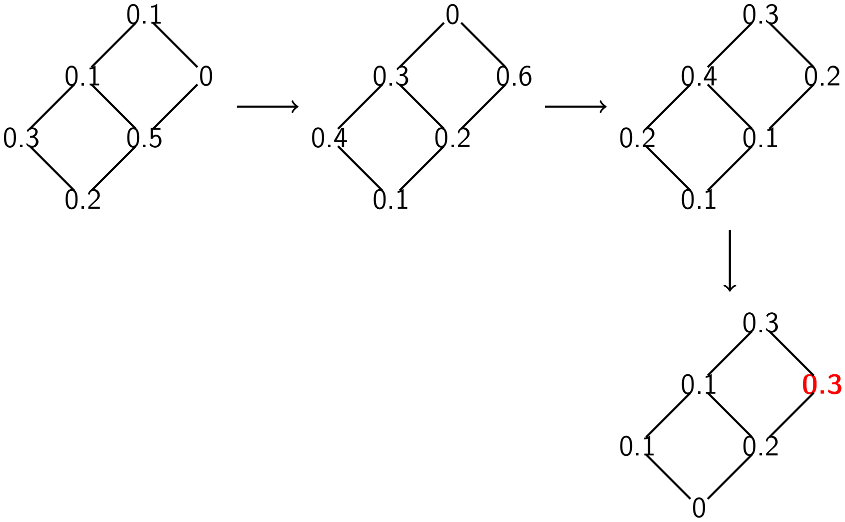
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



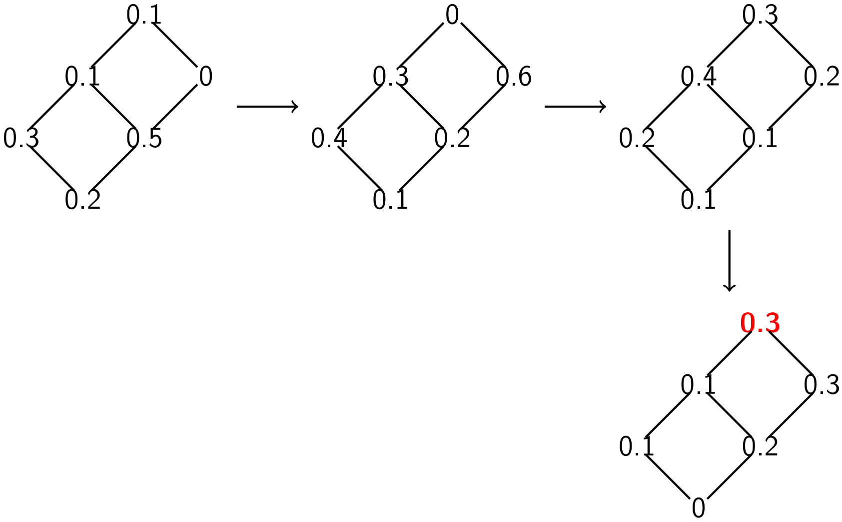
Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



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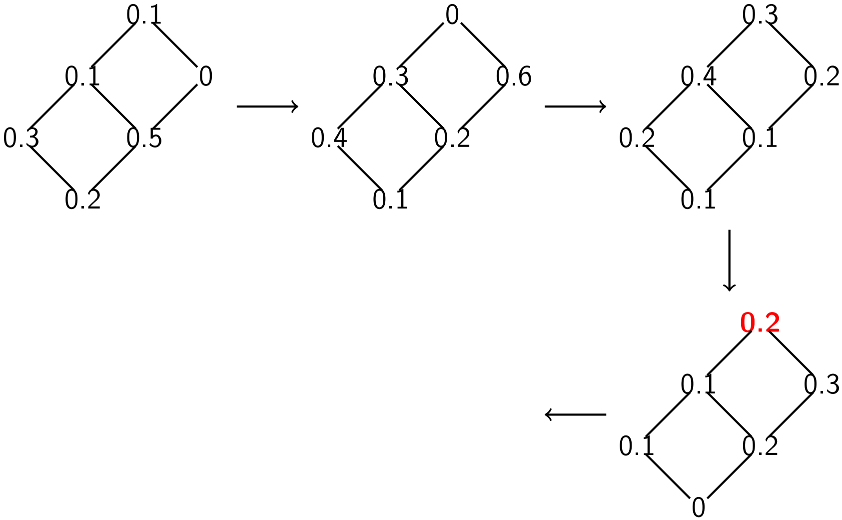

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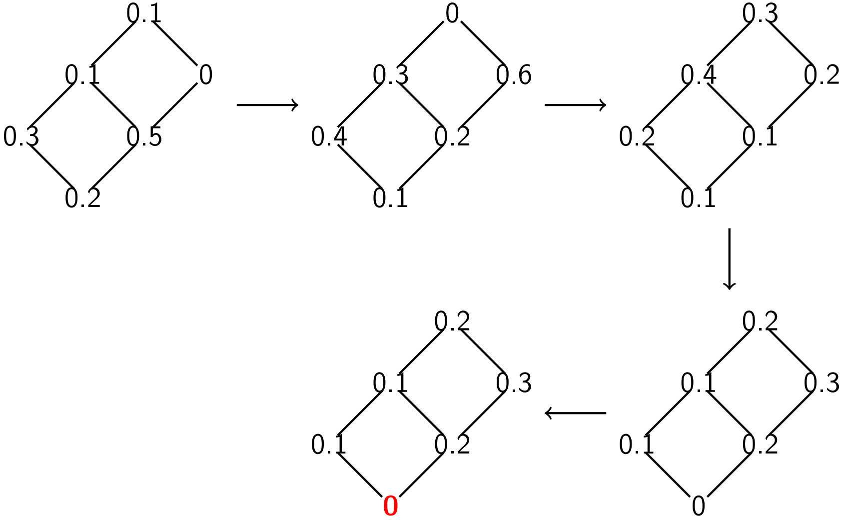

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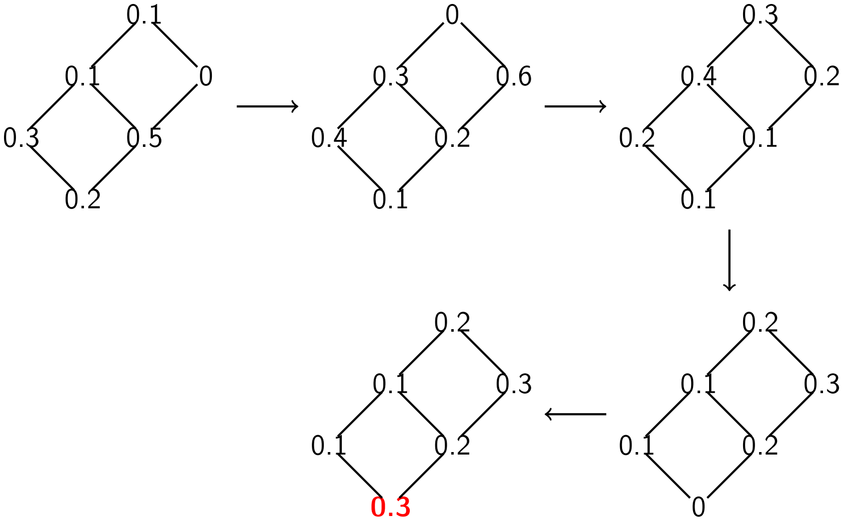

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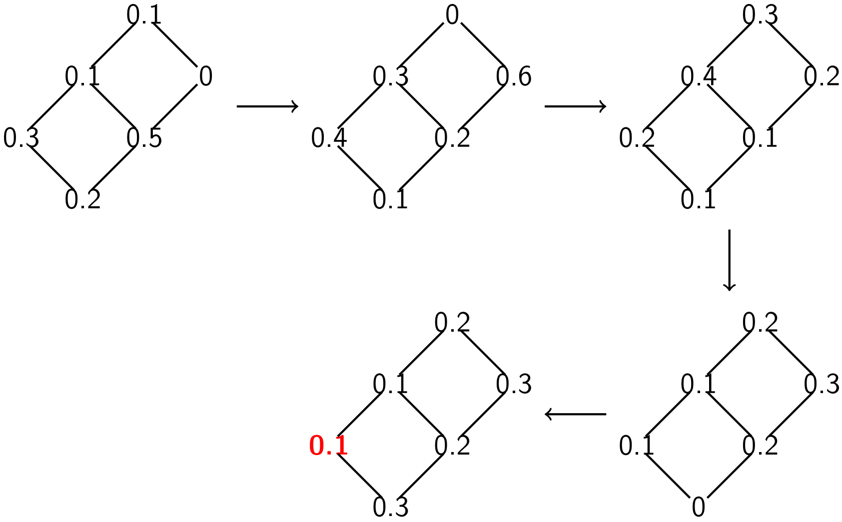

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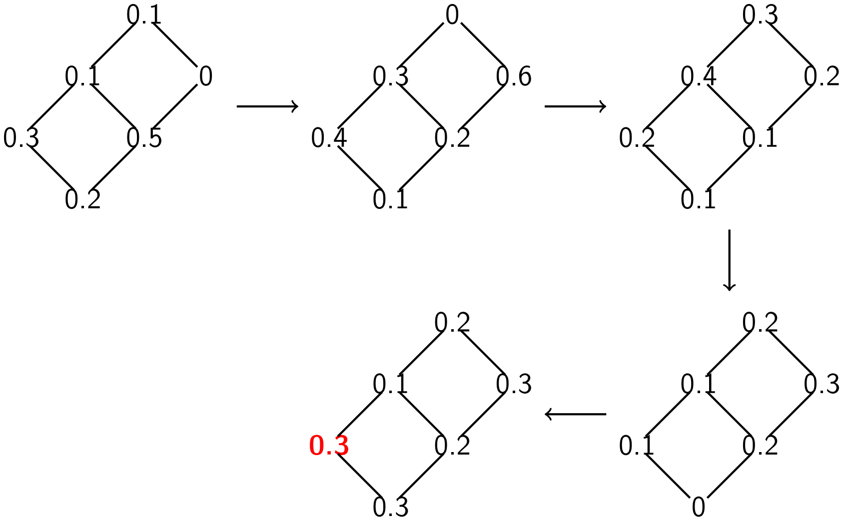

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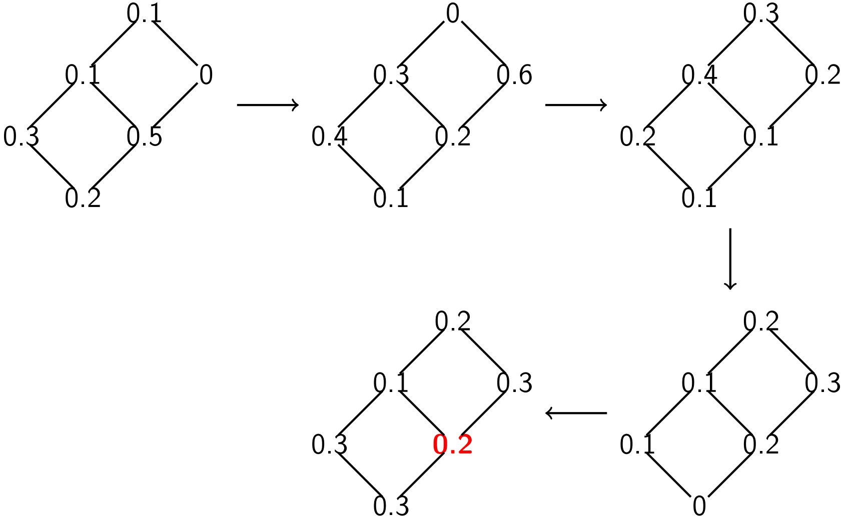

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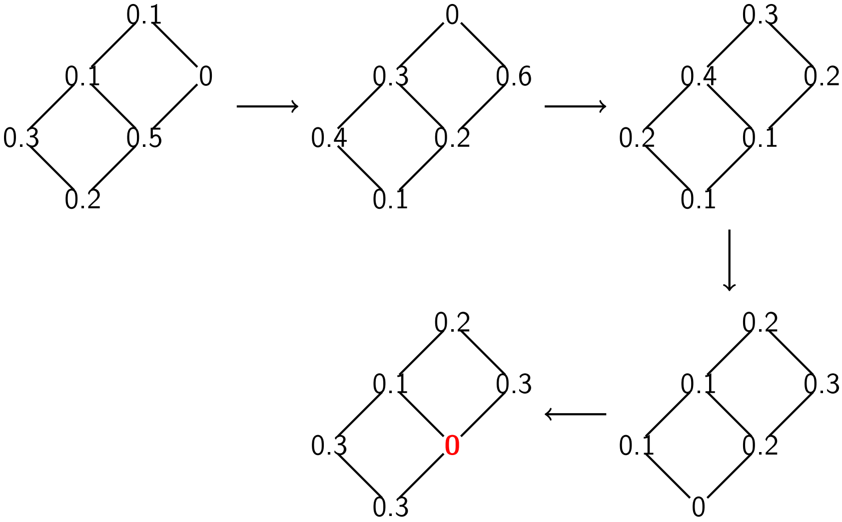

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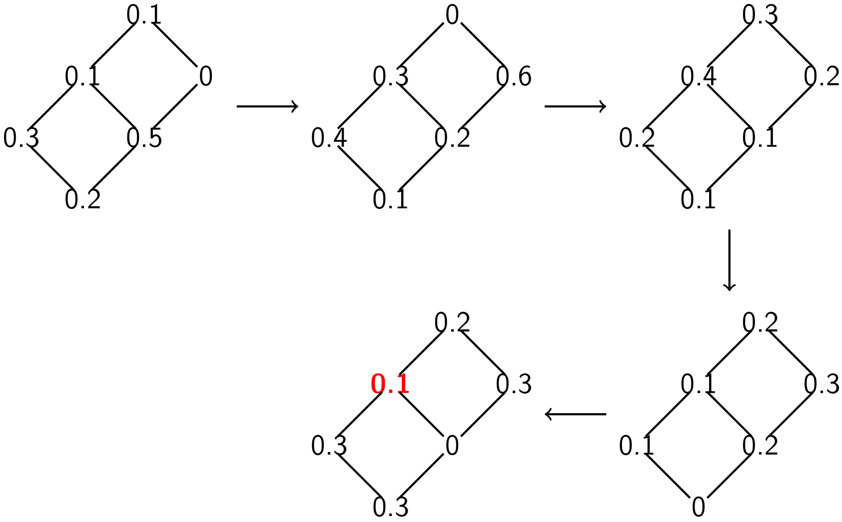
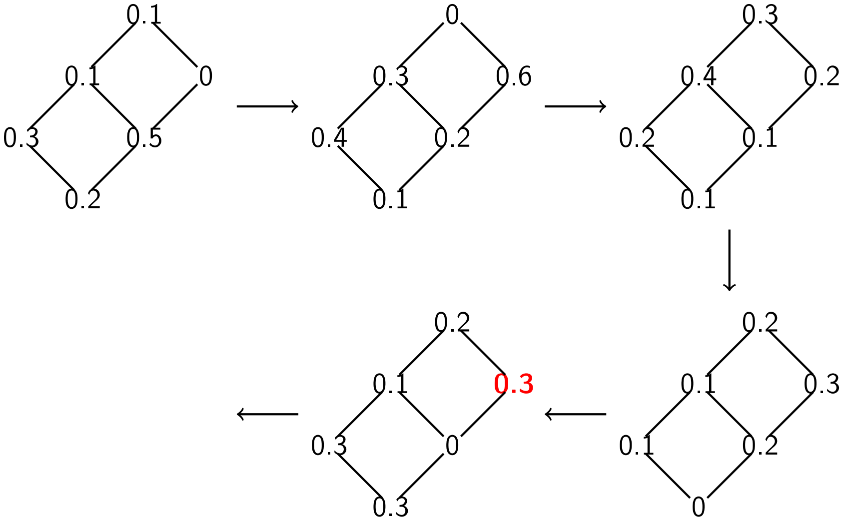
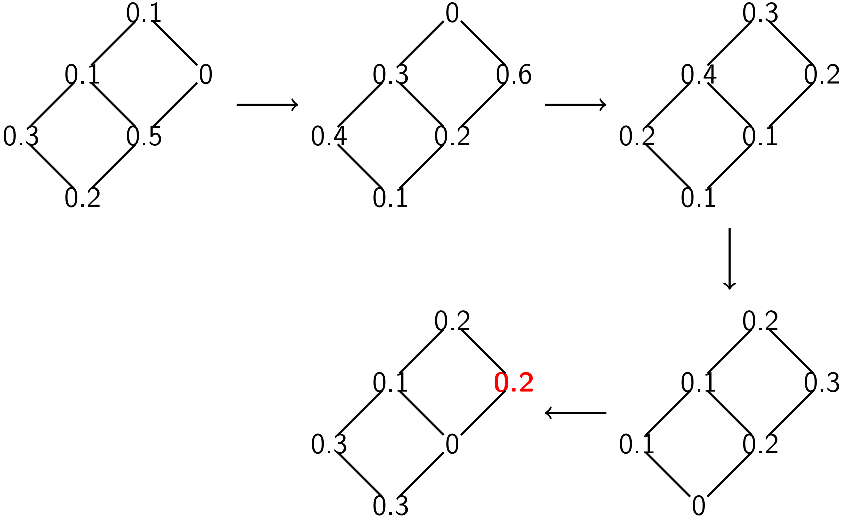


Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



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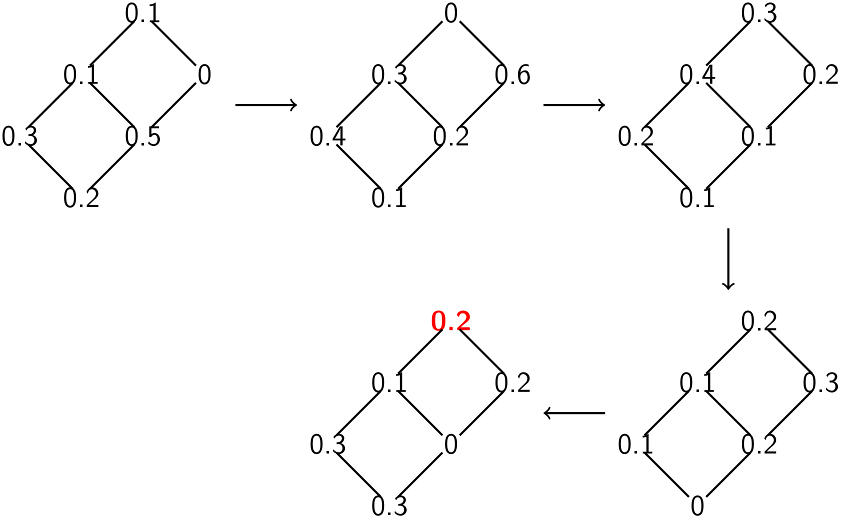

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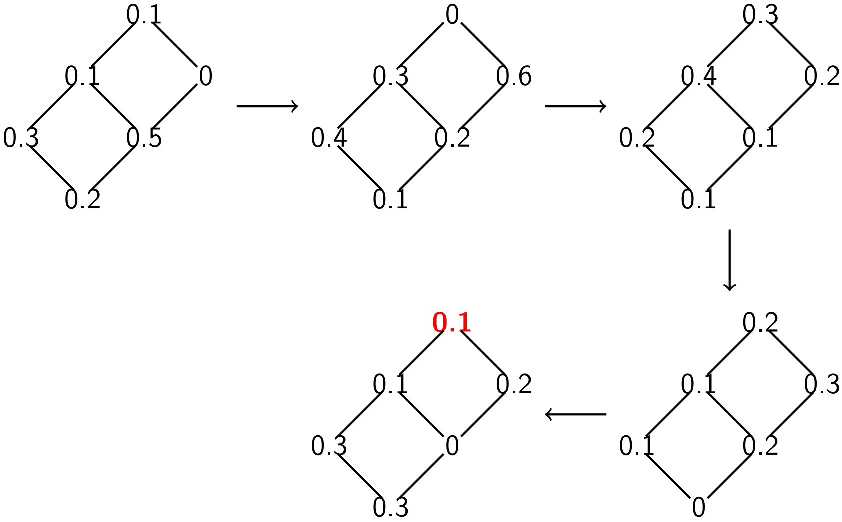
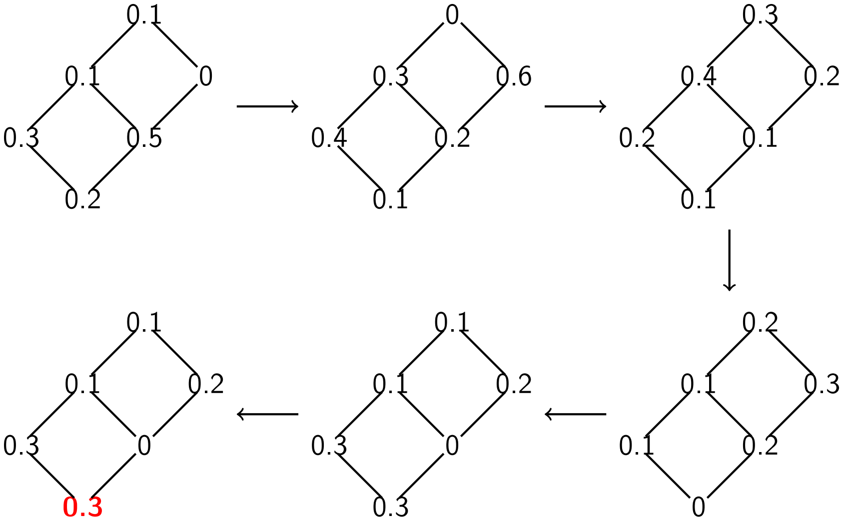


Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



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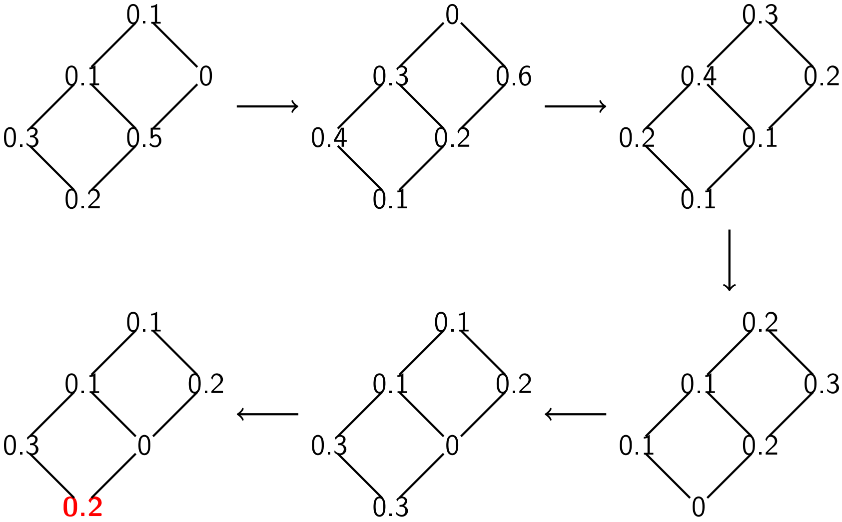

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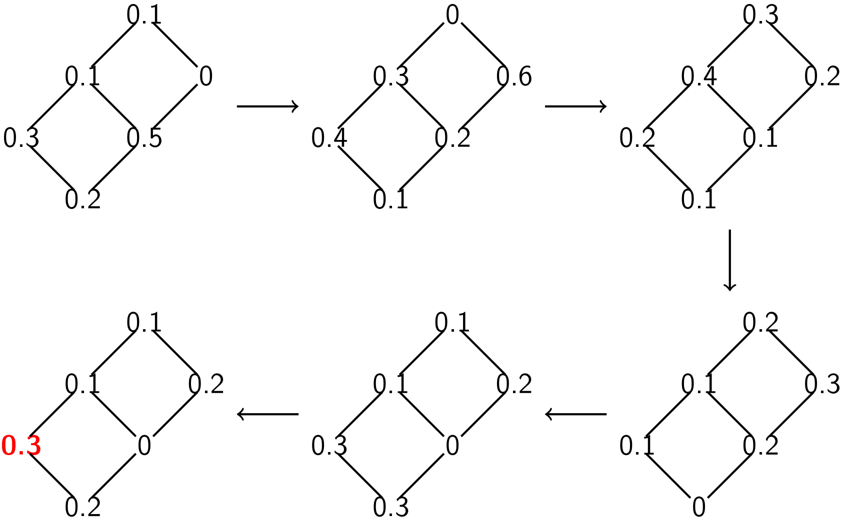

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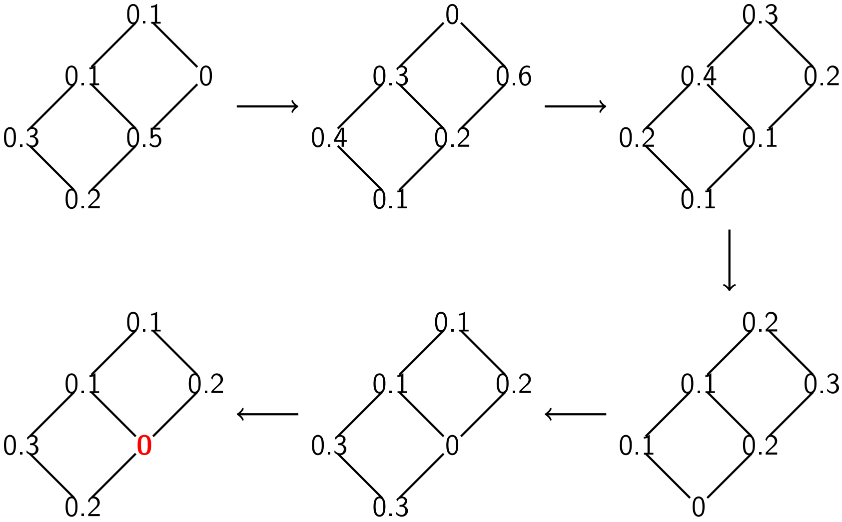

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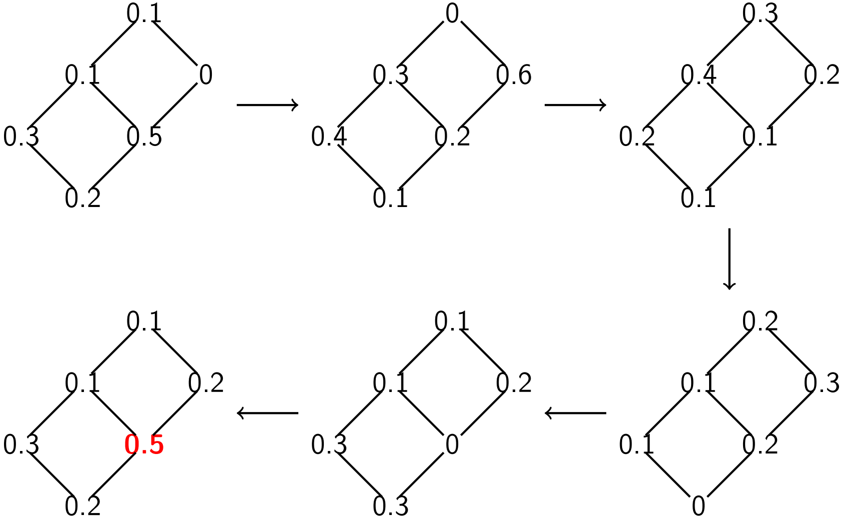

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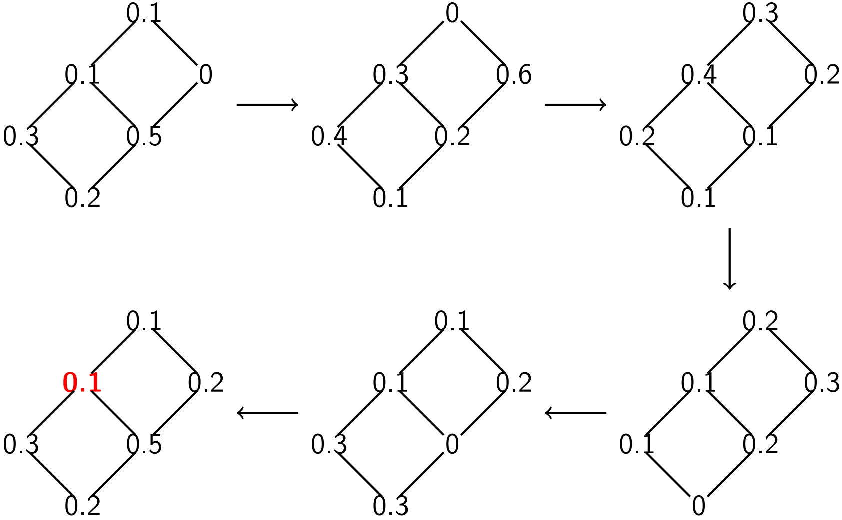

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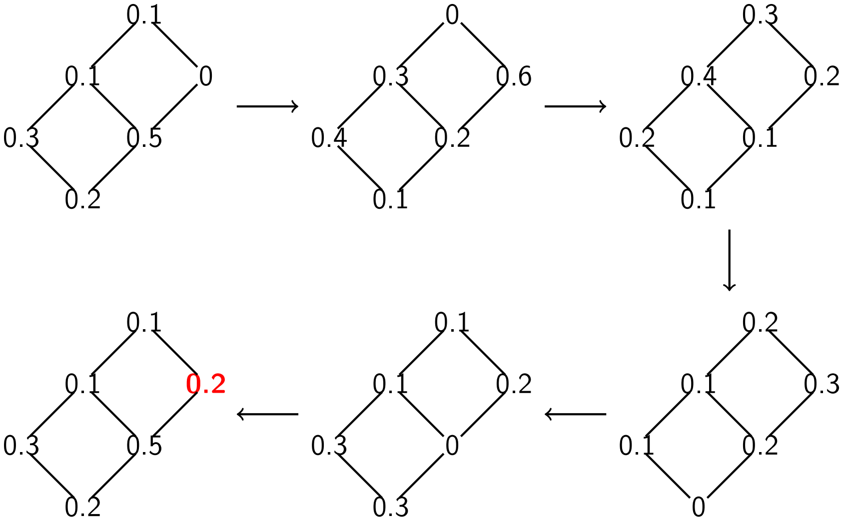
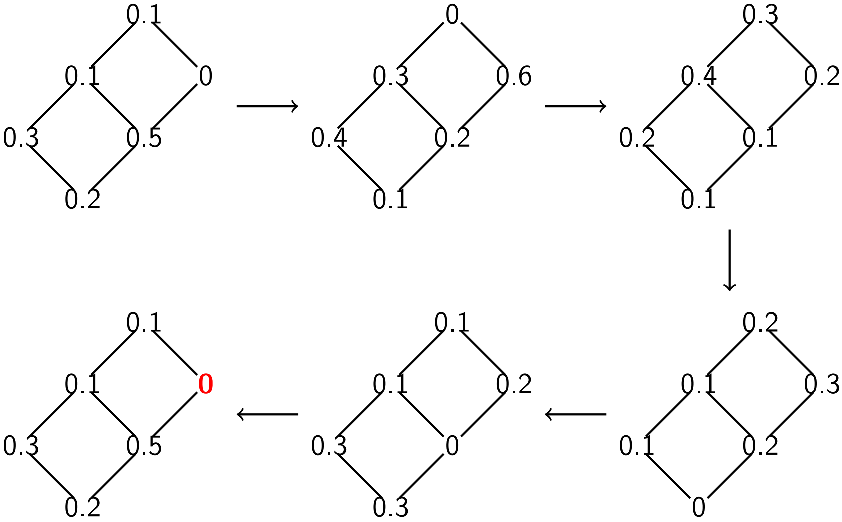
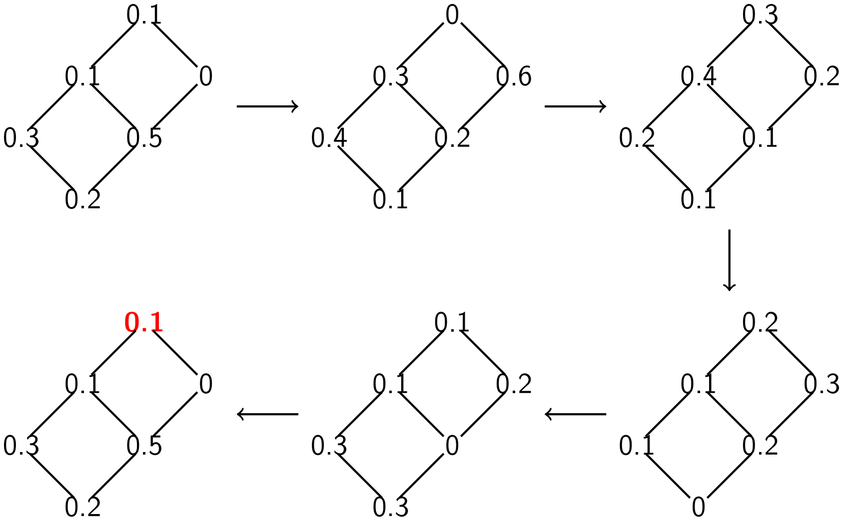


Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



Example of PL (Antichain) Rowmotion on the chain polytope $\mathcal{C}([2] \times [3])$



The birational realm

Detropicalizing from the piecewise-linear realm to the birational realm

Einstein and Propp showed how to lift of order-ideal toggling and rowmotion on $\mathcal{O}(P)$ to the birational realm [EiPr13+]. To do this, we replace max with $+$ and $+$ with multiplication. Under this dictionary

$$\big(\tau_v(g)\big)(v) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } v \end{array} \right\}$$

becomes

$$\big(\tau_v(g)\big)(v) = \frac{C}{\sum \left\{ \prod_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } v \end{array} \right\}}$$

whereas

$$\big(T_v(g)\big)(v) = \max_{y \lessdot v} f(y) + \min_{y \gtrdot v} f(y) - f(v)$$

becomes

$$\frac{\displaystyle\sum_{y \in \widehat{P}, y \lessdot v} f(y)}{f(v) \displaystyle\sum_{y \in \widehat{P}, y \gtrdot v} \frac{1}{f(y)}}$$

Birational Antichain Rowmotion (BAR-motion)

Now we'll define the **birational antichain toggle** corresponding to $e \in P$.

Definition

For $e \in P$, and field $\mathbb{K}$, let $\tau_e : \mathbb{K}^P \to \mathbb{K}^P$ be defined as the birational map that only changes the value at e in the following way.

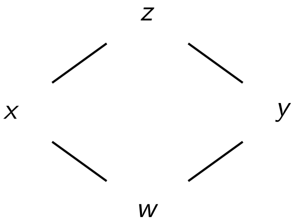
$$\big(\tau_e(g)\big)(e) = \frac{C}{\sum \left\{ \prod_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}}$$

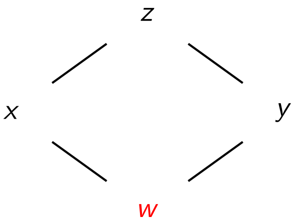
Definition

BAR-motion (birational antichain rowmotion) is the birational map obtained by applying the birational antichain toggles from the bottom to the top.

BAR-motion on $[2] \times [2]$

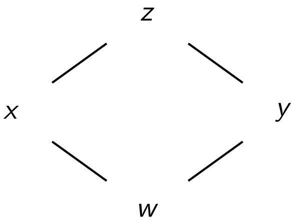
$g =$

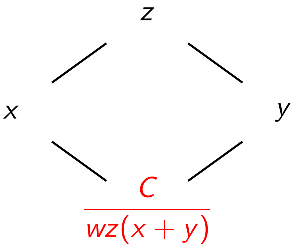




BAR-motion on $[2] \times [2]$

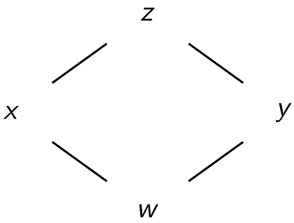
$g =$

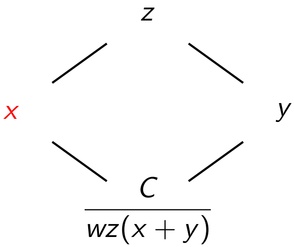




BAR-motion on $[2] \times [2]$

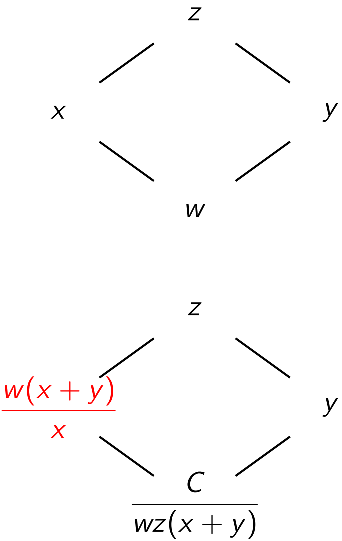
$g =$





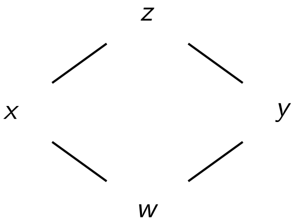
BAR-motion on $[2] \times [2]$

$g =$



BAR-motion on $[2] \times [2]$

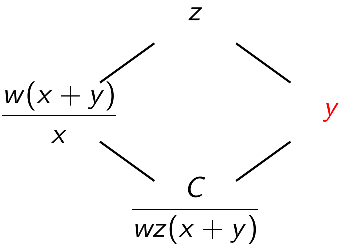
$g =$



z

x y

w



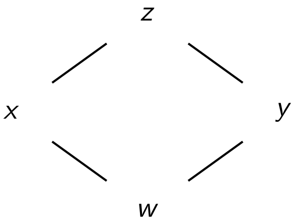
z

$\frac{w(x+y)}{x}$ y

$\frac{C}{wz(x+y)}$

BAR-motion on $[2] \times [2]$

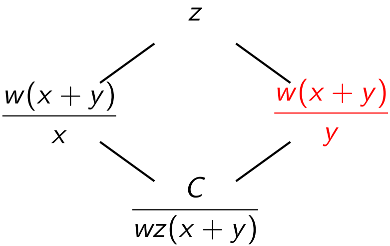
$g =$



z

x $\quad$ y

w



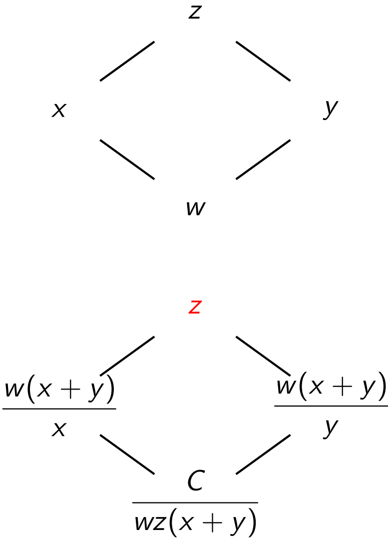
z

$\dfrac{w(x+y)}{x}$ $\quad$ $\dfrac{w(x+y)}{y}$

$\dfrac{C}{wz(x+y)}$

BAR-motion on $[2] \times [2]$

$g =$



z

x y

w

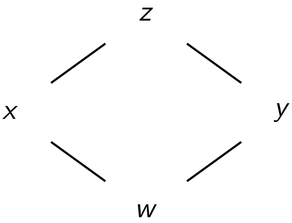
$\color{red}{z}$

$\frac{w(x+y)}{x}$ $\frac{w(x+y)}{y}$

$\frac{C}{wz(x+y)}$

BAR-motion on $[2] \times [2]$

$g =$

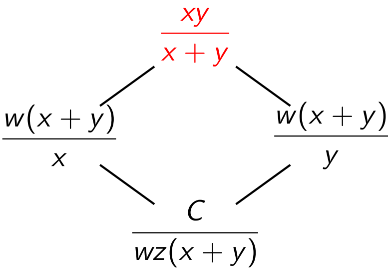


z

x $\quad$ y

w

$\mathrm{BAR}(g) =$



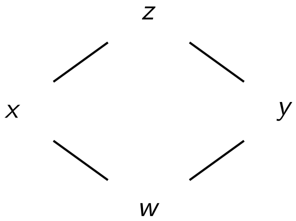
$\frac{xy}{x+y}$

$\frac{w(x+y)}{x}$ $\quad$ $\frac{w(x+y)}{y}$

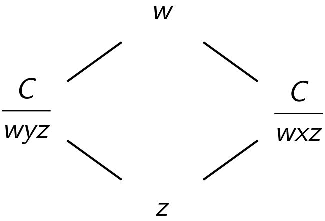
$\frac{C}{wz(x+y)}$

BAR-motion on $[2] \times [2]$

$g =$

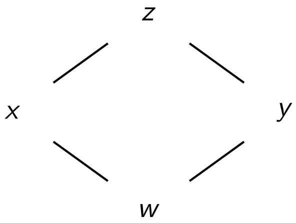


$\mathrm{BAR}^2(g) =$

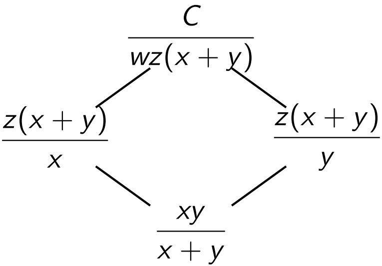


BAR-motion on $[2] \times [2]$

$g =$

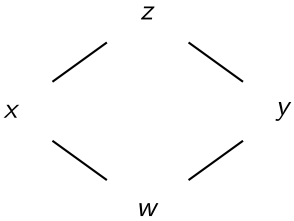


$\mathrm{BAR}^3(g) =$

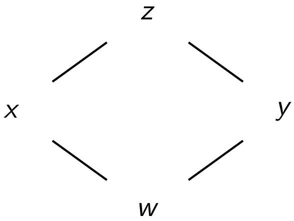


BAR-motion on $[2] \times [2]$

$g =$



$\mathrm{BAR}^4(g) =$



Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the partial map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

  $$\left(T_v f\right)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

  for all $w \in \widehat{P}$.
  Here (and in the following), $\overline{m}$ means m^{-1} whenever $m \in \mathbb{K}$.

Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the partial map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

  $$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

  for all $w \in \widehat{P}$.
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- This is a **partial** map. If any of the inverses does not exist in $\mathbb{K}$, then $T_v f$ is undefined!

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- This is a **partial** map. If any of the inverses does not exist in $\mathbb{K}$, then $T_v f$ is undefined!
- Notice that this is a **local change** to the label at v; all other labels stay the same.
- If $\mathbb{K}$ is commutative, then $T_v^2 = \mathrm{id}$ (on the range of T_v).

Birational Order Rowmotion: definition

- We define **(even noncommutative) birational rowmotion** as the partial map

  $$R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

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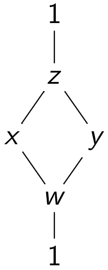
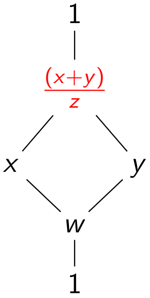
- This is indeed independent on the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (or just don't cover each other);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

BOR-motion on $[2] \times [2]$

Example when $\mathbb{K}$ is commutative:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,1)} f$
	

We are using $\text{BOR} = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

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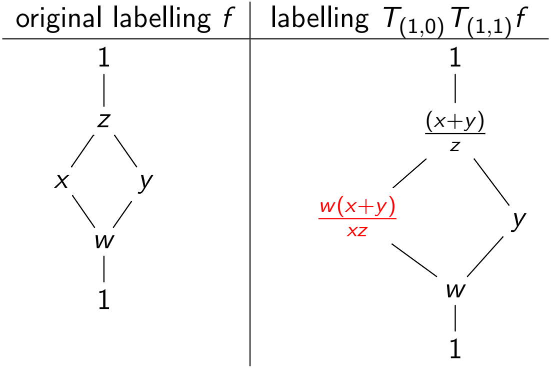

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BOR-motion on $[2] \times [2]$

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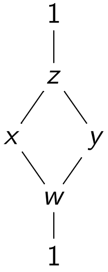
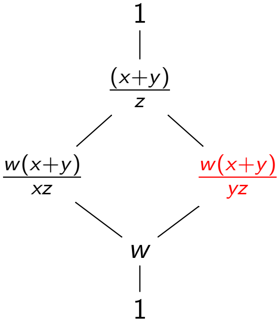
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,1)} T_{(1,0)} T_{(1,1)} f$
	

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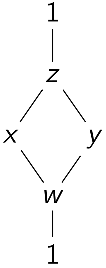
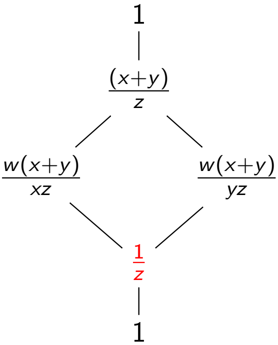
BOR-motion on $[2] \times [2]$

Example when $\mathbb{K}$ is commutative:

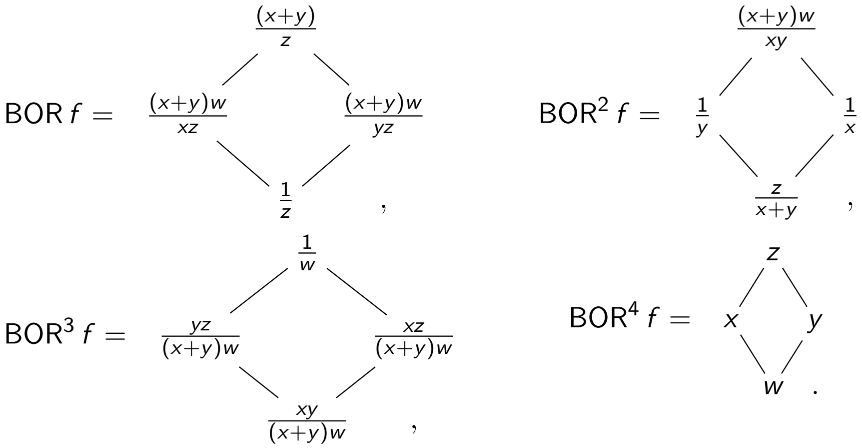
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)}\, T_{(0,1)}\, T_{(1,0)}\, T_{(1,1)} f = \mathrm{BAR}\, f$
	

We are using $\mathrm{BOR} = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

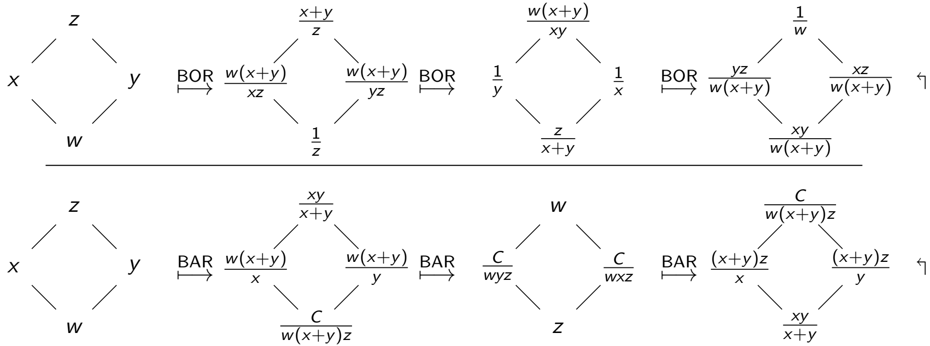
BOR-motion orbit on a product of chains

Example: Iterating this procedure we get



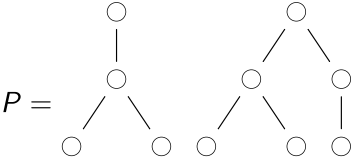
Orbits for BOR-motion and BAR-motion on $[2] \times [2]$

Here are the full orbits of BOR and BAR on a generic labeling for $P = [2] \times [2]$:

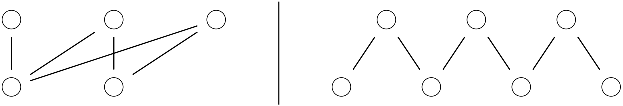


Properties of BOR-motion

- The order of BOR on $[a] \times [b]$ is $a + b$ [GrRo15b, Thm. 30]
- The order of BOR on "graded rooted forests" with all leaves on level n (indexed from 1) is finite and satisfies
  $\mathrm{ord}(\mathrm{BOR}) = \mathrm{ord}(\rho_{\mathcal{J}}) \mid \mathrm{LCM}(1, 2, \ldots, n + 1)$ [GrRo16].
  Example: For P as shown,
  $\mathrm{ord}(\mathrm{BOR}) = \mathrm{ord}(\rho_{\mathcal{J}}) \mid \mathrm{LCM}(1, 2, 3, 4) = 12.$



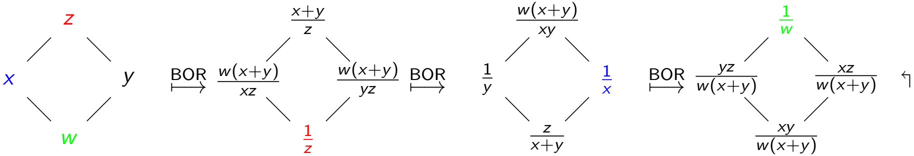
- **NB:** Most posets have $\mathrm{ord}(\mathrm{BOR}) = \infty$, e.g., the Boolean lattices B_3 OR the two below:



Antipodal Homomesy for BOR-motion on rectangular posets

- **Antipodal reciprocity:** [GrRo15b, Thm. 32] Antipodal points in $P = [a] \times [b]$ satisfy:

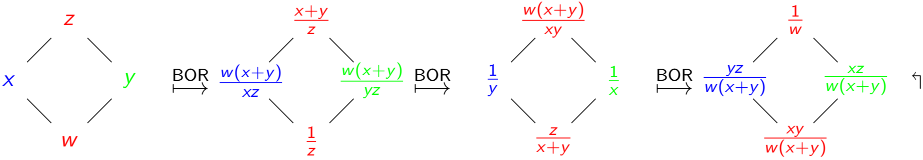
$$f\left(a+1-i, b+1-k\right) = \frac{1}{\left(\mathrm{BOR}^{i+k-1} f\right)(i,k)}.$$



File Homomesy for BOR-motion

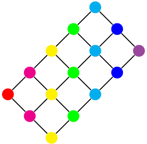
Musiker–R gave a formula for iterates of birational rowmotion in terms of ratios of families of non-intersecting lattice paths (NILPs). This allowed them to reprove the periodicity and antipodal homomesy results, as well as the following refined homomesy, which lifts a known one for $\rho_{\mathcal{J}}$ [MR19].

Given a file F in $[a] \times [b]$, $\displaystyle\prod_{k=1}^{a+b} \prod_{(i,j)\in F} \left(\mathrm{BOR}^k f\right)(i,j) = 1$. i.e., the statistic $\prod_{(i,j)\in F} \widetilde{\mathbb{1}}_{(i,j)}$ is birationally homomesic under BOR.

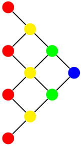


File Homomesy on Minuscule Posets

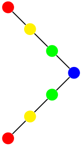
These results generalize to *Minuscule Posets*, where "files" now means "elements of the same color", combinatorially by Rush & Wang [RuWa15+], birationally by Okada [Oka21].



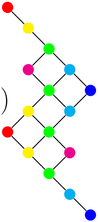
(A_n, ϖ_r)
$(1 \leq r \leq n)$



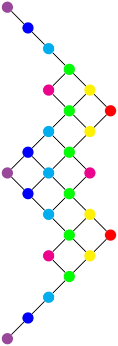
(B_n, ϖ_n)



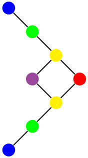
(C_n, ϖ_1)



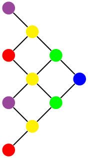
(E_6, ϖ_1)
(E_6, ϖ_6)



(E_7, ϖ_7)



(D_n, ϖ_1)



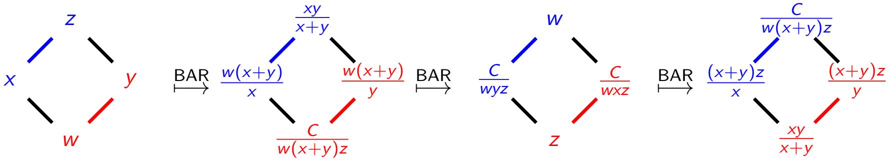
(D_n, ϖ_{n-1})
(D_n, ϖ_n)

(Pictures courtesy of S. Okada)

Properties of BAR-motion

- The order of BAR on $[a] \times [b]$ is $a+b$. This follows from [GrRo15b] via our equivariant toggle-group isomorphisms.
- The homomesy results for antichain cardinality in the combinatorial $\rho_{\mathcal{A}}$ setting lift to this setting. Because...
- We can lift the *Stanley–Thomas* word to this setting as an equivariant *surjection*, cyclically rotating with *BAR*. It proves homomesy, but not periodicity [JR21].

Here is the full orbit of BAR on a generic labeling for $P = [2] \times [2]$, with ST-words.



$$\left(wy, xz, \frac{C}{wx}, \frac{C}{yz}\right) \qquad \left(\frac{C}{yz}, wy, xz, \frac{C}{wx}\right) \qquad \left(\frac{C}{wx}, \frac{C}{yz}, wy, xz\right) \qquad \left(xz, \frac{C}{wx}, \frac{C}{yz}, wy\right)$$

The Noncommutative realm

Lifting to NC toggles and NC Order rowmotion

Our earlier definition of birational toggling was already phrased to work over any semiring $\mathbb{K}$; write $\overline{m}$ for m^{-1}. Set

$$\left(T_v(f)\right)(v) = \left(\sum_{u \in \widehat{P},\, u \lessdot v} f(u)\right) \overline{f(v)} \left(\sum_{u \in \widehat{P},\, u \gtrdot v}^{\#} f(u)\right), \text{ where}$$

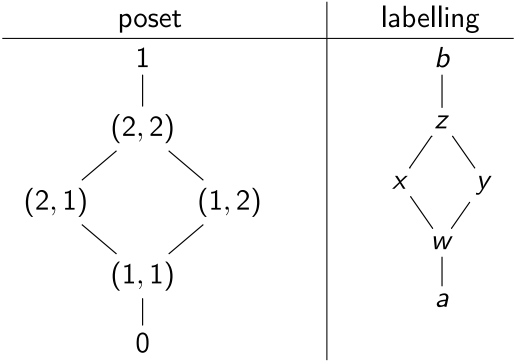
$$\sum_{u \in \widehat{P},\, u \gtrdot v}^{\#} f(u) = \overline{\sum_{u \in \widehat{P},\, u \gtrdot v} \overline{f(u)}}.$$

- These "toggles" are no longer involutions (in general), but we can define their inverses, called "elggots" E_v. Toggles and Elggots for elements which do not cover each other commute (among themselves and with each other).
- As usual, we define Noncommutative Order Rowmotion by $\mathrm{NOR} := T_{x_1} T_{x_2} \dots T_{x_n}$, where $(x_1, \dots, x_n)$ is a linear extension of P. *Henceforth, $R := \mathrm{NOR}$ for simplicity.*
- To spice things up, we can also fix $f(\widehat{0}) = a$ and $f(\widehat{1}) = b$ to see what happens.

NOR-motion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



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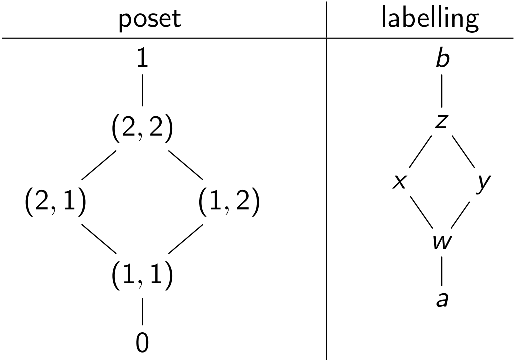


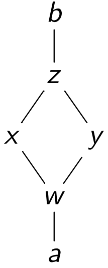
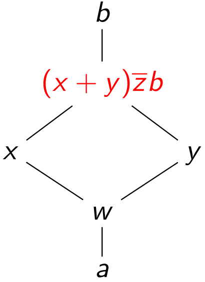
We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1),(1,2),(2,1),(2,2))$).

That is, toggle in the order “top, left, right, bottom”.

NOR-motion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(2,2)} f$
	

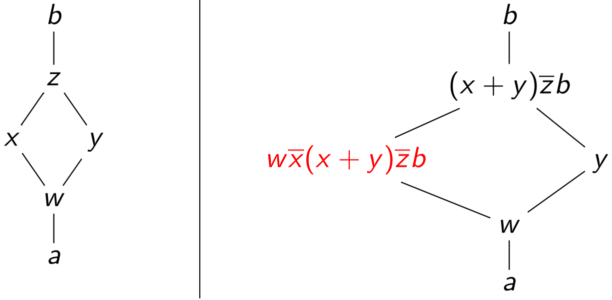
We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

NOR-motion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(2,1)}T_{(2,2)}f$

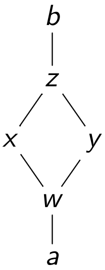
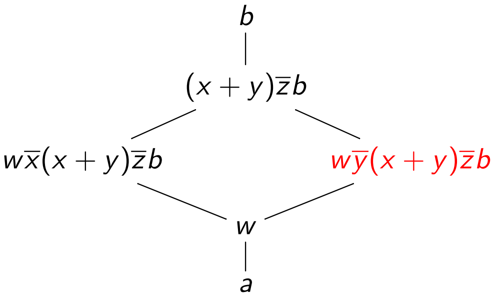


We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

NOR-motion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(1,2)} T_{(2,1)} T_{(2,2)} f$
	

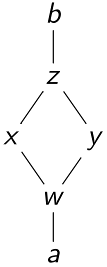
We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

NOR-motion: example

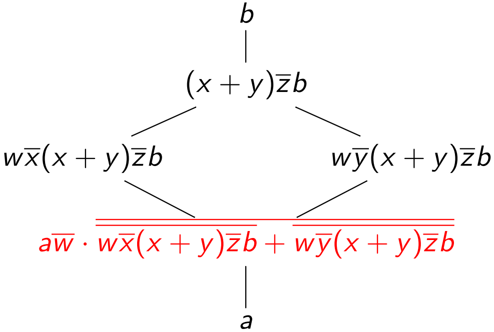
Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f



labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$



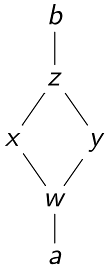
We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

NOR-motion: example

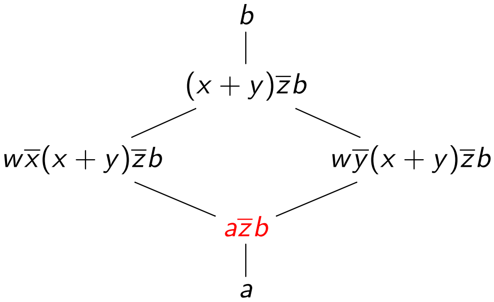
Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f



labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$

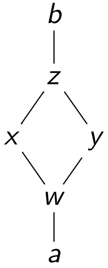


We have used $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ and simplified the result.

NOR-motion: the rectangle case, example

- **Example:** Iteratively apply R to a labelling of the 2×2-rectangle.

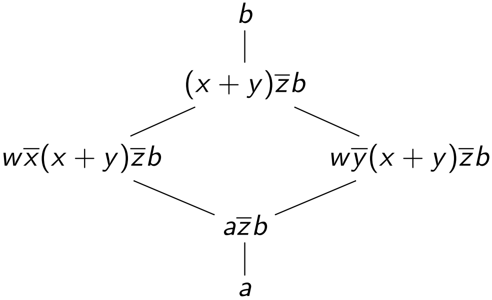
  Here is $R^0 f$:



NOR-motion: the rectangle case, example

- **Example:** Iteratively apply R to a labelling of the 2×2-rectangle.

  Here is $R^1 f$:



b

$(x+y)\overline{z}b$

$w\overline{x}(x+y)\overline{z}b$ $w\overline{y}(x+y)\overline{z}b$

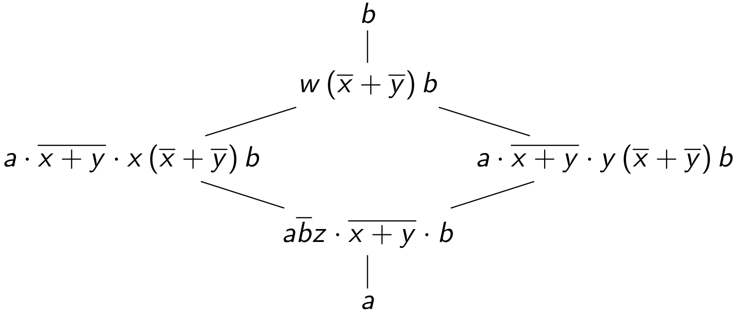
$a\overline{z}b$

a

NOR-motion: the rectangle case, example

- **Example:** Iteratively apply R to a labelling of the 2×2-rectangle.

  Here is $R^2 f$:



b

$w\,(\overline{x}+\overline{y})\,b$

$a \cdot \overline{x+y} \cdot x\,(\overline{x}+\overline{y})\,b$

$a \cdot \overline{x+y} \cdot y\,(\overline{x}+\overline{y})\,b$

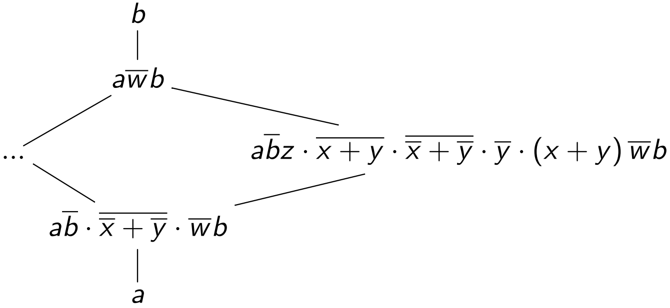
$a\overline{b}z \cdot \overline{x+y} \cdot b$

a

NOR-motion: the rectangle case, example

- **Example:** Iteratively apply R to a labelling of the 2×2-rectangle.

  Here is $R^3 f$:



b

$a\overline{w}b$

$\dots$

$a\overline{b}z \cdot \overline{x+y} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{y} \cdot (x+y)\,\overline{w}b$

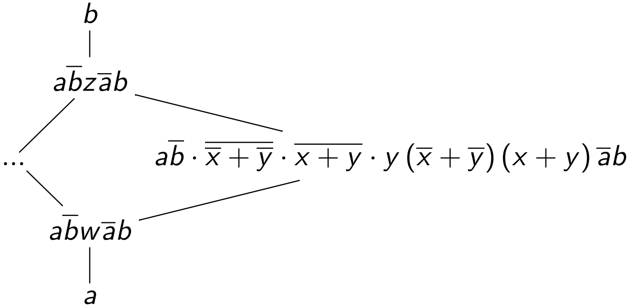
$a\overline{b} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{w}b$

a

NOR-motion: the rectangle case, example

- **Example:** Iteratively apply R to a labelling of the 2×2-rectangle.

  Here is $R^4 f$:



b

$a\overline{b}z\overline{a}b$

$\ldots$

$a\overline{b}\cdot\overline{\overline{x}+\overline{y}}\cdot\overline{x+y}\cdot y\,(\overline{x}+\overline{y})\,(x+y)\,\overline{a}b$

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a

NOR-motion: the rectangle case, example

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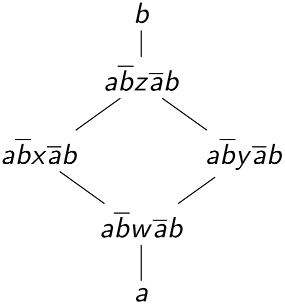

  (after nontrivial simplifications).

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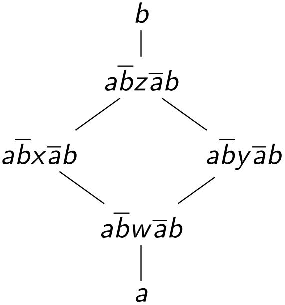
  

  b

  $a\overline{b}z\overline{a}b$

  $a\overline{b}x\overline{a}b$ $a\overline{b}y\overline{a}b$

  $a\overline{b}w\overline{a}b$

  a

  This displays the periodicity theorem for $p = q = 2$.

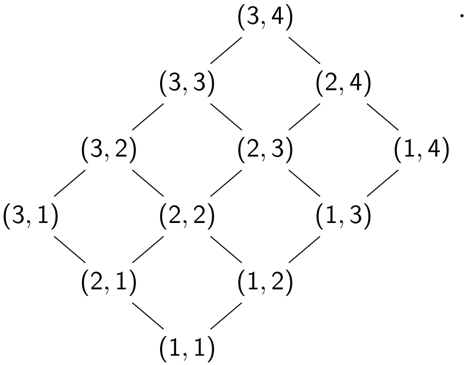
- Note that this is similar to Kontsevich's periodicity conjecture, proved by Iyudu/Shkarin (`arXiv:1305.1965`).

NOR-motion: the rectangle case

- Let p and q be two positive integers. Let $\mathbb{K}$ be a ring. Let P be the $p \times q$-rectangle poset: i.e.,

  $$P := [p] \times [q], \qquad \text{where } [m] := \{1, 2, \ldots, m\}.$$

  (The order on P is entrywise.)
  Example: For $p = 3$ and $q = 4$, this is



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- Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Let $a = f(0)$ and $b = f(1)$.

Periodicity theorem (Grinberg–R [GR22+])

If a and b are invertible and $R^{p+q} f$ is well-defined, then

$$\left(R^{p+q} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a} b \qquad \text{for each } x \in \widehat{P}.$$

Note that $a\overline{b} \cdot f(x) \cdot \overline{a} b$ is **not** generally conjugate to $f(x)$.

NOR-motion: the rectangle case

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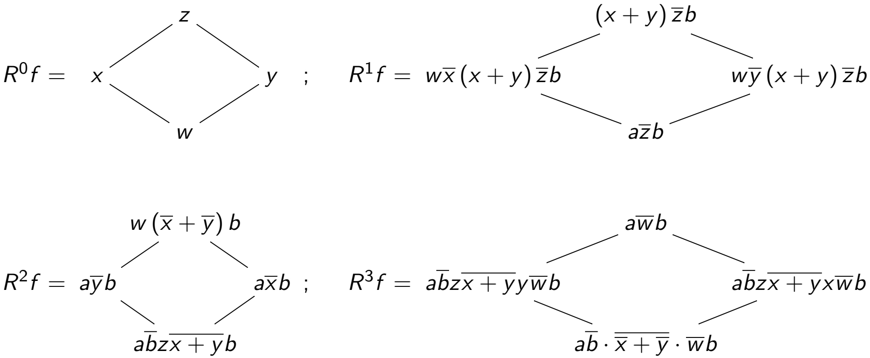
Reciprocity theorem (Grinberg–R [GR22+])

Let $\ell \in \mathbb{N}$. Let $(i, j) \in P$. If $R^{\ell} f$ is well-defined and $\ell \geq i + j - 1$, then
$$\left(R^{\ell} f\right)(i, j) = a \cdot \overline{\left(R^{\ell-i-j+1} f\right) \underbrace{(p + 1 - i, q + 1 - j)}_{=\text{antipode of } (i,j) \text{ in } P}} \cdot b.$$

NOR-motion: the rectangle case, example

- Here are $R^0 f, R^1 f, \ldots, R^4 f$ for a generic $f \in \mathbb{K}^{\widehat{[2]\times[2]}}$ again, this time fully simplified and with the elements and labels $f(0) = a$ and $f(1) = b$ suppressed:



$R^0 f =$

z

x $\quad$ y ;

w

$R^1 f =$

$(x+y)\,\overline{z}b$

$w\overline{x}\,(x+y)\,\overline{z}b$ $\quad$ $w\overline{y}\,(x+y)\,\overline{z}b$

$a\overline{z}b$

$R^2 f =$

$w\,(\overline{x}+\overline{y})\,b$

$a\overline{y}b$ $\quad$ $a\overline{x}b$;

$a\overline{b}z\overline{x+y}b$

$R^3 f =$

$a\overline{w}b$

$a\overline{b}z\overline{x+y}y\overline{w}b$ $\quad$ $a\overline{b}z\overline{x+y}x\overline{w}b$

$a\overline{b}\cdot\overline{\overline{x}+\overline{y}}\cdot\overline{w}b$

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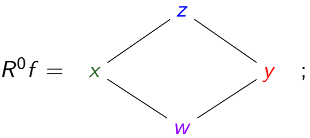
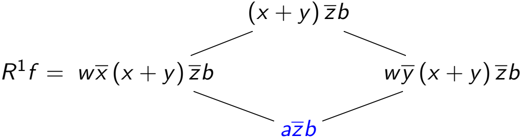
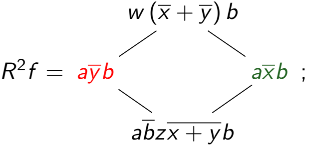


$R^0 f =$

;

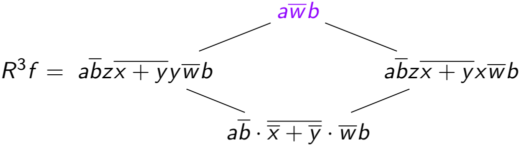


$R^1 f =$



$R^2 f =$

;



$R^3 f =$

Same-colored labels are related by reciprocity. Can you spot some more?

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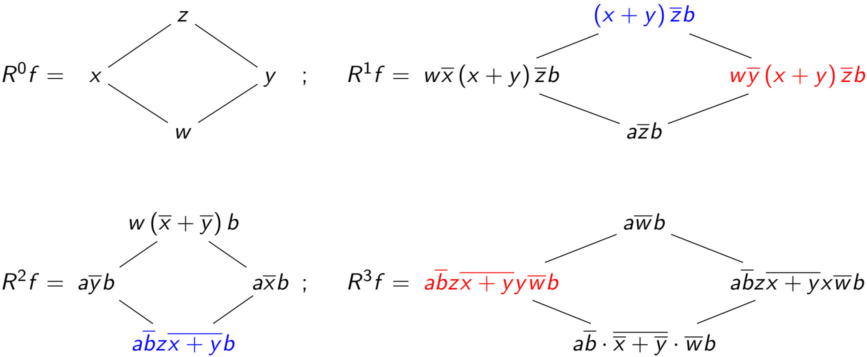


$R^0 f =$ z; x, y; w ;

$R^1 f =$ $(x+y)\,\overline{z}b$; $w\overline{x}\,(x+y)\,\overline{z}b$, $w\overline{y}\,(x+y)\,\overline{z}b$; $a\overline{z}b$

$R^2 f =$ $w\,(\overline{x}+\overline{y})\,b$; $a\overline{y}b$, $a\overline{x}b$; $a\overline{b}z\overline{x+y}b$;

$R^3 f =$ $a\overline{w}b$; $a\overline{b}z\overline{x+y}y\overline{w}b$, $a\overline{b}z\overline{x+y}x\overline{w}b$; $a\overline{b}\cdot\overline{\overline{x}+\overline{y}}\cdot\overline{w}b$

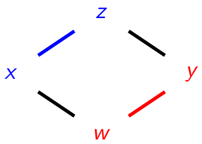
Here are some more instances of reciprocity. (There are more.)

Noncommutative Antichain Rowmotion (NAR-motion)

- Joseph-R. [JR21] lifted birational antichain toggles to the noncommutative setting, and proved that the bijection between the NC order toggle group and the NC antichain toggle lifts as well (again with toggles and elggots).
- We define NAR as usual (toggling from bottom to top), and show that NAR and NOR have the same order.
- However, the Stanley-Thomas word lifts even to this setting, as a tuple that cyclically rotates with the action of NAR.

NAR-motion and NC-Stanley–Thomas Word

The NAR-orbit for a generic labeling on $P = [2] \times [2]$ and Stanley–Thomas words



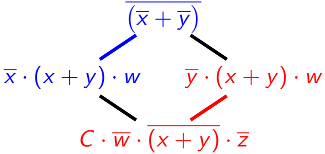
z

x $\qquad$ y

w

$g = \mathrm{NAR}^4(g)$

$\mathrm{ST}_g = (yw, zx, C \cdot \overline{w} \cdot \overline{x}, C \cdot \overline{y} \cdot \overline{z})$



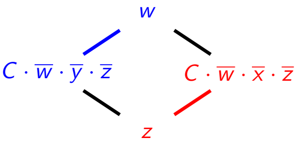
$\overline{(\overline{x} + \overline{y})}$

$\overline{x} \cdot (x + y) \cdot w$ $\qquad$ $\overline{y} \cdot (x + y) \cdot w$

$C \cdot \overline{w} \cdot \overline{(x + y)} \cdot \overline{z}$

$\mathrm{NAR}(g)$

$\mathrm{ST}_{\mathrm{NAR}(g)} = (C \cdot \overline{y} \cdot \overline{z}, yw, zx, C \cdot \overline{w} \cdot \overline{x})$



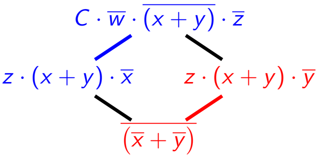
w

$C \cdot \overline{w} \cdot \overline{y} \cdot \overline{z}$ $\qquad$ $C \cdot \overline{w} \cdot \overline{x} \cdot \overline{z}$

z

$\mathrm{NAR}^2(g)$

$\mathrm{ST}_{\mathrm{NAR}^2(g)} = (C \cdot \overline{w} \cdot \overline{x}, C \cdot \overline{y} \cdot \overline{z}, yw, zx)$



$C \cdot \overline{w} \cdot \overline{(x + y)} \cdot \overline{z}$

$z \cdot (x + y) \cdot \overline{x}$ $\qquad$ $z \cdot (x + y) \cdot \overline{y}$

$\overline{(\overline{x} + \overline{y})}$

$\mathrm{NAR}^3(g)$

$\mathrm{ST}_{\mathrm{NAR}^3(g)} = (zx, C \cdot \overline{w} \cdot \overline{x}, C \cdot \overline{y} \cdot \overline{z}, yw)$

Proof of Periodicity for NOR-motion

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- The definition of R yields

  $$(Rf)(v) = \left(\sum_{u \lessdot v} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum_{u \gtrdot v} \overline{(Rf)(u)}} \qquad \text{for each } v \in P.$$

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- In other words,

$$v_1 = \left(\sum_{u \lessdot v} u_0\right) \cdot \overline{v_0} \cdot \overline{\sum_{u \gtrdot v} \overline{u_1}} \qquad \text{for each } v \in P.$$

Transition equation

- We have just shown that

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- So far, we have just rewritten our setup using the (more convenient) $x_\ell := \left(R^\ell f\right)(x)$ notation.

Simplifying the goal

- We must prove:

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  if $x = (i, j)$ and $y = (p+1-i, q+1-j)$.

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Paths, As and $\forall$s

- A **path** shall mean a sequence $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ of elements of $\widehat{P}$. We call it a path from v_0 to v_k.

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- If u and v are elements of $\widehat{P}$, set

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- *Proof idea:* The ℓ is constant. Hence, we omit it, writing $\forall^v$ for $\forall_\ell^v$.

  (a) Rewrite the claim as $\forall^{1 \to u} = b\overline{u_\ell}$.
  Prove this by downwards induction on u.
  Induction step: Given $v \in P$ such that $\forall^{1 \to u} = b\overline{u_\ell}$ for all $u \gtrdot v$.
  Since any path $1 \to v$ passes through a unique $u \gtrdot v$, we have

$$\begin{aligned} \forall^{1 \to v} &= \sum_{u \gtrdot v} \forall^{1 \to u} \forall^v = \sum_{u \gtrdot v} b\overline{u_\ell} \forall^v \qquad \text{(by induction hypothesis)} \\ &= b\overline{v_\ell} \qquad \text{(by definition of } \forall^v\text{)}, \qquad \text{qed.} \end{aligned}$$

Path formulas

- **Path formulas:**
  (a) We have
  $$u_\ell = \overline{\forall_\ell^{1\to u}} \cdot b \qquad \text{for each } u \in P.$$
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  (b) Analogous, but use upwards induction instead.

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  Similarly, **(d)** follows from **(b)**.

Transition equation in A-$\forall$-form

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$$\forall^{v}_{\ell+1} = A^{v}_{\ell} \qquad \text{for each } v \in \widehat{P} \text{ and } \ell \in \mathbb{N}.$$

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Take reciprocals on both sides, multiply by $\overline{\sum_{u \gtrdot v} \overline{u_{\ell+1}}}$ and rewrite using $\forall_{\ell+1}^{v}$ and A_{ℓ}^{v}.

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  Hence, $\forall_{\ell+1}^{u \to v} = A_{\ell}^{u \to v}$ for any $u, v \in \widehat{P}$.

Reciprocity at $(1, 1)$

- Now, for the bottommost element $(1, 1)$ of P, we have

$$\begin{aligned}
(1,1)_1 &= \overline{\forall_1^{(p,q)\to(1,1)}} \cdot b && \text{(by path formula } \textbf{(c)}\text{)} \\
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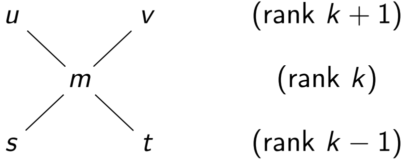
- What now?

The case $j = 1$ suffices: part 1

- We can simplify our goal one bit further. Consider the "neighborhood" of an element of our rectangle P:



u v (rank $k+1$)

m (rank k)

s t (rank $k-1$)

(where the **rank** of an $(i,j) \in P$ is defined to be $i+j-1$). Say we have shown (our "induction hypotheses") that reciprocity holds for each of s, t, m, u; that is, we have

$$s_\ell = a \cdot \overline{s'_{\ell-(k-1)}} \cdot b, \qquad t_\ell = a \cdot \overline{t'_{\ell-(k-1)}} \cdot b,$$
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for all sufficiently high ℓ, where x' denotes the antipode of x (that is, if $x = (i,j)$, then $x' = (p+1-i, q+1-j)$).

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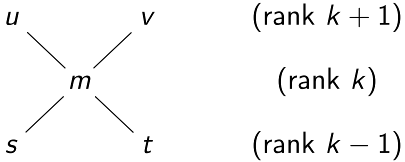

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  for all sufficiently high ℓ, where x' denotes the antipode of x (that is, if $x = (i, j)$, then $x' = (p + 1 - i, q + 1 - j)$).
  Claim: Then, reciprocity also holds for v; that is, we have $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$ for all $\ell \geq k + 1$.

The case $j = 1$ suffices: part 2

- *Proof idea.* Fix $\ell \geq k+1$, and compare the transition equations

$$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}} \qquad \text{and}$$

$$m'_{\ell-k} = \left(u'_{\ell-k-1} + v'_{\ell-k-1}\right) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{\overline{s'_{\ell-k}} + \overline{t'_{\ell-k}}}$$

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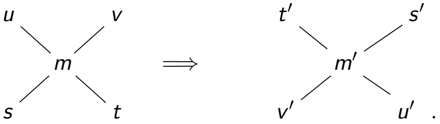

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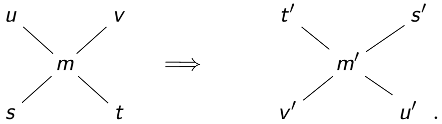

After subtracting $u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b$, out comes $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$.

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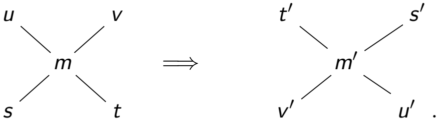

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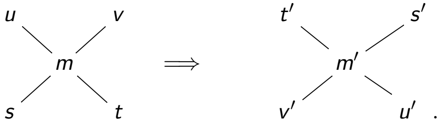

- This argument still works if s, t or u does not exist.
- Thus, in order to prove reciprocity for all (i, j), it suffices (by induction) to prove it in the case when $j = 1$.

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Note the lack of rowmotion in this formula! The ℓ here is constantly 1, so it is a property of a single labeling. Thus, we drop the subscripts.

- **Our new goal:** Prove that

$$A^{(p,q)\to(2,1)} = \forall^{(p-1,q)\to(1,1)}.$$

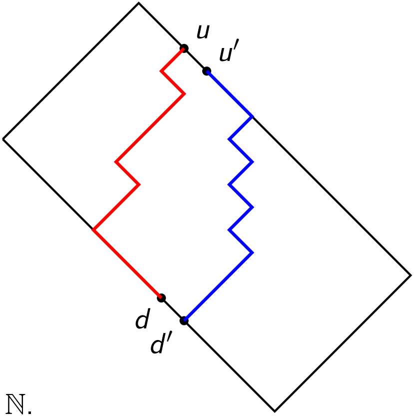
The conversion lemma

- More generally:
- **Conversion lemma:**
  Let u and u' be two adjacent elements on the top-right edge of P (that is, $u = (k, q)$ and $u' = (k-1, q)$). Let d and d' be two adjacent elements on the bottom-left edge of P (that is, $d = (i, 1)$ and $d' = (i-1, 1)$). Then,

  $$A_\ell^{u \to d} = \forall_\ell^{u' \to d'} \qquad \text{for each } \ell \in \mathbb{N}.$$

  In short:

  $$A^{u \to d} = \forall^{u' \to d'}.$$



Rowmotion begone, part 1

- If we can prove the conversion lemma, we will obtain reciprocity not only for $(i, j) = (2, 1)$, but also for all (i, j) on the bottom-left edge of P (that is, for the entire case $j = 1$), because we can argue as follows:

Rowmotion begone, part 2

$$
\begin{aligned}
(i,1)_i &= \overline{\forall_i^{(p,q)\to(i,1)}} \cdot b && \text{(by path formula } \textbf{(c)}\text{)} \\
&= \overline{A_{i-1}^{(p,q)\to(i,1)}} \cdot b && \left(\text{since } \forall_{\ell+1}^{u\to v} = A_\ell^{u\to v}\right) \\
&= \overline{\forall_{i-1}^{(p-1,q)\to(i-1,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \overline{A_{i-2}^{(p-1,q)\to(i-1,1)}} \cdot b && \left(\text{since } \forall_{\ell+1}^{u\to v} = A_\ell^{u\to v}\right) \\
&= \overline{\forall_{i-2}^{(p-2,q)\to(i-2,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \cdots \\
&= \overline{\forall_{1}^{(p-i+1,q)\to(1,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \overline{A_{0}^{(p-i+1,q)\to(1,1)}} \cdot b && \left(\text{since } \forall_{\ell+1}^{u\to v} = A_\ell^{u\to v}\right) \\
&= a \cdot \overline{(p-i+1,q)_0} \cdot b && \text{(by path formula } \textbf{(d)}\text{)}\,.
\end{aligned}
$$

Rowmotion begone, part 3

- This proves reciprocity

$$(i, 1)_\ell = a \cdot \overline{(p - i + 1, q)_{\ell - i}} \cdot b$$

  for $\ell = i$.

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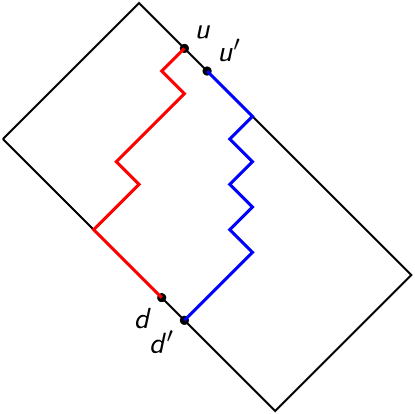
  The case $\ell > i$ follows by applying this to $R^{\ell-i}f$ instead of f.
- Thus, we only need to prove the conversion lemma. We can now drop all subscripts forever!

Proving the conversion lemma: the intuition

- Let us again look at the picture:



We must prove $A^{u \to d} = \forall^{u' \to d'}$.

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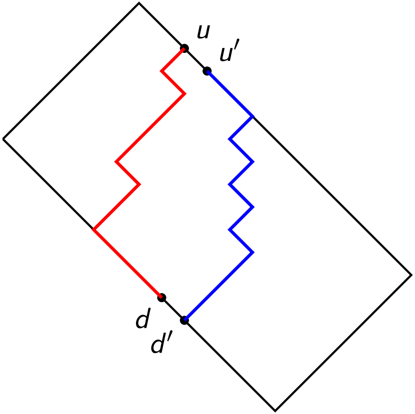

We must prove $A^{u\to d} = \forall^{u'\to d'}$.

- How do we interpolate between paths $u \to d$ and paths $u' \to d'$?

Proving the conversion lemma: path-jump-paths

- We define a **path-jump-path** to be a sequence

  $$\mathrm{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$$

  of elements of P, where the relation $x \blacktriangleright y$ means "y is one step down and some steps to the right of x" (that is, if $x = (r, s)$, then $y = (r - k, s + k - 1)$ for some $k > 0$). We say that this path-jump-path p has **jump at** i.

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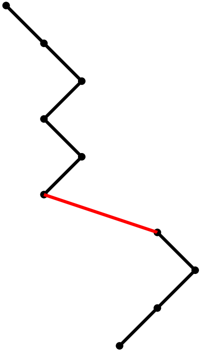
  Example of a path-jump-path:



(The red edge is the jump.)

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  For any such path-jump-path p, we set

$$E_{\mathrm{p}} := A^{v_0} A^{v_1} \cdots A^{v_{i-1}} v_i \overline{v_{i+1}} \forall^{v_{i+2}} \forall^{v_{i+3}} \cdots \forall^{v_k}.$$

  (Here, we are omitting the ℓ subscripts – so v_i means $(v_i)_\ell$ and v_{i+1} means $(v_{i+1})_\ell$.)

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- Now, if $k = \operatorname{rank} u - \operatorname{rank}(d')$, then

  $$A^{u \to d} = \sum_{\substack{\text{p is a path-jump-path } u \to d' \\ \text{with jump at } k-1}} E_{\mathrm{p}},$$

  since $A^d = d\overline{d'}$, and similarly

  $$\forall^{u' \to d'} = \sum_{\substack{\text{p is a path-jump-path } u \to d' \\ \text{with jump at } 0}} E_{\mathrm{p}}.$$

Proving the conversion lemma: moving the jump

- So we need to show that

$$\sum_{\substack{\mathsf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } k-1}} E_{\mathsf{p}} = \sum_{\substack{\mathsf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } 0}} E_{\mathsf{p}}.$$

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- Reasonable to expect that

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for each $0 \leq i < k - 1$.

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- This is indeed true and can be proved by a "local" argument (rewriting two consecutive steps of the path).
- This is similar to the "zipper argument" in lattice models. (Is there a Yang–Baxter equation lurking?)

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- However, the path-jump-path argument is somewhat messy. We can make it slicker by rewriting it in matrix notation:
- Define three $P \times P$-matrices A, $\forall$ and U by

  $$\begin{aligned} \mathsf{A}_{x,y} &:= A^x \left[x \gtrdot y\right], & \qquad \forall_{x,y} &:= \forall^y \left[x \gtrdot y\right], \\ \mathsf{U}_{x,y} &:= x\overline{y} \left[x \blacktriangleright y\right] & \qquad & \text{for all } x, y \in P. \end{aligned}$$

  Here, $[\mathcal{A}]$ is the Iverson bracket (i.e., truth value) of a statement $\mathcal{A}$; the relation $x \blacktriangleright y$ means "y is one step down and some steps to the right of x" as before. And again, we are omitting the ℓ subscripts, so $x\overline{y}$ actually means $x_\ell \overline{y_\ell}$.
- Now, we claim that

  $$\mathsf{AU} = \mathsf{U}\forall.$$

Proving the conversion lemma: the civilized version, part 2

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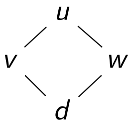
  Indeed, this follows easily from the following neat lemma: If

  are four adjacent elements of P, then

  $\overline{w} \cdot \forall^d \cdot d = \overline{u} \cdot A^u \cdot v$ and $\overline{v} \cdot \forall^d \cdot d = \overline{u} \cdot A^u \cdot w$.

  (The u and d here are unrelated to the u and d from the conversion lemma!)

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  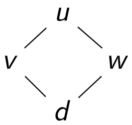

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- From $\mathrm{A}\mathrm{U} = \mathrm{U}\forall$, we easily obtain

  $$\mathrm{A}^{\circ k}\mathrm{U} = \mathrm{U}\forall^{\circ k} \qquad \text{for any } k \in \mathbb{N},$$

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  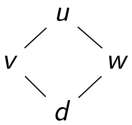

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  where $A^{\circ k}$ means the k-th power of a matrix A.
- Setting $k = \operatorname{rank} u - \operatorname{rank} d$ and comparing the (u, d')-entries of both sides, we quickly obtain $A^{u \to d} = \forall^{u' \to d'}$ (since $x \blacktriangleright d'$ holds only for $x = d$, and since $u \blacktriangleright x$ holds only for $x = u'$).
  This proves the conversion lemma again.

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  In the **commutative** case, the theorems hold for semifields (and, more generally, commutative semirings) because they hold for fields and because they are "essentially" polynomial identities (once you clear denominators).
  This **fails** for noncommutative $\mathbb{K}$!
- **Scary example** (David Speyer, MathOverflow #401273): If x and y are two elements of a ring such that $x + y$ is invertible, then

  $$x \cdot \overline{x+y} \cdot y = y \cdot \overline{x+y} \cdot x.$$

  But this is not true if "ring" is replaced by "semiring"!

Is that all? Part 2: The semiring question

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Question:

Are the periodicity and reciprocity theorems still true if “ring” is replaced by “semiring”? (I.e., we no longer require $\mathbb{K}$ to have a subtraction.)

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- We have partial results, e.g., for $p = q = 3$ and for $p = 2$.

Is that all? Part 3: Other posets

- Other posets remain to be studied.

Conjecture:

Let P be the triangle-shaped poset $\Delta(p)$ or its reflection $\nabla(p)$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a labelling such that $R^p f$ exists. Let $a = f(0)$ and $b = f(1)$. Then, for each $x \in \widehat{P}$, we have

$$(R^p f)(x) = a\overline{b} \cdot f\left(x'\right) \cdot \overline{a}b,$$

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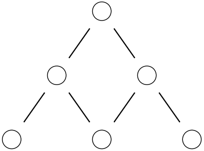

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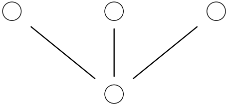
- As already mentioned, other simple posets such as



do not have periodic behavior for noncommutative $\mathbb{K}$.

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Question:

What other results like ours are known in the noncommutative case?

Johnson and Liu 2022

- A recent preprint by Joseph Johnson and Ricky Ini Liu (*Birational rowmotion and the octahedron recurrence*, `arXiv:2204.04255`) reproves the "order $p+q$" theorem for commutative $\mathbb{K}$ in a simpler way (besides doing a number of other interesting things).

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- We don't know if the octahedron recurrence is well-behaved for noncommutative $\mathbb{K}$ (too many options to check), but LGV certainly is not available.
- Lemma 4.1 in the Johnson-Liu preprint generalizes our conversion lemma in the commutative case from single paths to k-tuples of nonintersecting paths. We don't know how this could be done in the noncommutative case; it is unclear in what order to multiply labels from different paths.

The Y-system connection

- **Zamolodchikov periodicity conjecture in type AA** (proved by A. Yu. Volkov, `arXiv:hep-th/0606094v1`): Let r and s be positive integers. Let $Y_{i,\ j,\ k}$ be elements of a commutative ring for $i \in [r]$ and $j \in [s]$ and $k \in \mathbb{Z}$. Assume that

  $$Y_{i,\ j,\ k+1} Y_{i,\ j,\ k-1} = \frac{(1 + Y_{i+1,\ j,\ k})(1 + Y_{i-1,\ j,\ k})}{(1 + 1/Y_{i,\ j+1,\ k})(1 + 1/Y_{i,\ j-1,\ k})}$$

  for all $i,\ j,\ k$, where sums involving "off-grid" points (e.g., $1 + Y_{0,\ j,\ k}$) are understood as 1.
  Then, $Y_{i,\ j,\ k+2(r+s+2)} = Y_{i,\ j,\ k}$ for all $i,\ j,\ k$.

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  Then, $Y_{i,\ j,\ k+2(r+s+2)} = Y_{i,\ j,\ k}$ for all $i,\ j,\ k$.
- **Observation (Max Glick and others, ca. 2015?):** This is equivalent to periodicity of birational rowmotion ($R^{p+q} = 1$) for $[p] \times [q]$, where $p = r + 1$ and $q = s + 1$, when the ring is commutative. Explicitly,

  $$Y_{i,\ j,\ i+j-2k} = (R^k f)(i, j+1) \ \big/ \ (R^k f)(i+1, j).$$

  (Fine points omitted.)

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- **Disappointment:** Zamolodchikov periodicity does not generalize to noncommutative rings (no matter how we order the five factors).

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting at a variety of levels: combinatorial, piecewise-linear, birational, and noncommutative.
- All of our themes apply at all levels:
1) *Periodicity/order, orbit structure*; 2) *Homomesy*; and 3) *Equivariant bijections*.
- Maps which can be built out of toggling involutions seem particularly fruitful.
- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at higher level.
- The noncommutative level is challenging!

Slides for this talk are available online at: Google "Tom Roby"

Thanks very much for coming to this talk!

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Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to questions of periodicity/order, orbit structure, homomesy, and equivariant bijections.
- Actions that can be built out of smaller, simpler actions (toggles and whirls) often have interesting and unexpected properties.
- Much more remains to be explored, perhaps for combinatorial objects or actions that you work with for other reasons.

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