Some recent orbits of homomesy

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AND virtually over Zoom

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Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

Abstract: Natural maps on sets of discrete combinatorial objects often display interesting dynamics. Questions about periodicity and orbit structure natural arise, requiring a variety of approaches for their solutions. One particular phenomenon of interest is "homomesy", where a statistic on the set of objects has the same average for each orbit of an action. Along with its intrinsic interest as a kind of hidden "invariant", homomesy can be used to help understand certain properties of the action.

Proofs of homomesy often lead one to develop tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection. Maps that can be decomposed as products of "toggling" involutions or "whirling" maps are particularly amenable to this line of analysis. This talk will discuss actions on order ideals, independent sets, and other sets related to posets and graphs. The underlying shapes will mostly be products of a chain with itself or with a 3-element V-shaped poset, highlighting some recent work with Matthew Plante.

This talks discusses joint work, mostly with (chronologically) Jim Propp, Mike Joseph, and Matthew Plante.

I'm grateful to Matthew, Mike, and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

Thanks also to Drew Armstrong, Arkady Berenstein, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, David Einstein, Darij Grinberg, Shahrzad Haddadan, Sam Hopkins, Mike La Croix, Svante Linusson, Gregg Musiker, Nathan Williams, Vic Reiner, Bruce Sagan, Richard Stanley, Jessica Striker, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to put questions and comments in the chat, and the moderator will convey them with appropriate timing and finese. Or someone else may answer them!

Some themes in dynamical algebraic combinatorics

1. Periodicity/order;
2. Orbit structure;
3. Homomesy;
4. Equivariant bijections; and
5. Lifting from combinatorial to piecewise-linear and birational settings.

Cyclic rotation of binary strings

"Immer mit den einfachsten Beispielen anfangen." — David Hilbert

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \underset{C_R}{\longmapsto} 010100$$

- *Periodicity* is clear here. The map has order $n = 6$.
- *Orbit structure* is very nice; every orbit size must divide n.
- *Homomesy?* Need a statistic, first.
- *Equivariant bijection?* No need.

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6$, $k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Cyclic rotation of binary strings

Example

$n = 6,\ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Example

$n = 6,\ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2	100100	6
100010	5	000011	0	010010	4
010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well:
Let $\chi_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic?

Coxeter Toggling

Independent Sets

of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the factor 11.

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex i** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

Toggles

Proposition

Each toggle τ_i is an involution, i.e., τ_i^2 is the identity. Also, τ_i and τ_j commute if and only if $|i-j| \neq 1$.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Order & Orbits

- The order of this action grows quite fast as n increases and is difficult to describe in general. It is the LCM of the orbit sizes, which are not all divisors of some small number (relative to n): $2, 3, 6, 15, 24, 231, 210, 1989, 240, 72105, 18018, 3354725, \ldots$
- For $n = 6$ the three orbits have sizes 3, 7, 11, giving order LCM(3,7,11)$=$ 231.
- The number of orbits appeared to be OEIS A000358 ("Number of binary necklaces of length n with no subsequence 00") , but we didn't understand why at first.
- This means that this action is unlikely to exhibit interesting Cyclic Sieving.
- But we can still find homomesy.

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph–R. [JR18])

Define $\chi_i : \mathcal{I}_n \to \{0,1\}$ to be the indicator function of vertex i.

For $1 \leq i \leq n$, $\chi_i - \chi_{n+1-i}$ is 0-mesic on φ-orbits of $\mathcal{I}_n$.

Also $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ are 1-mesic on φ-orbits of $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\chi_i - \chi_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1)$ mod 4.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using elementary Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Hanaoka & Sadahiro have generalized the "palindromic" homomesy to the case of "*m*-independent sets", leading them to an interesting variation of bitstring rotation [HS19+]. Video lecture from BIRS-DAC (Kelowna) is available at `https://www.birs.ca/events/2021/5-day-workshops/21w5514/videos`

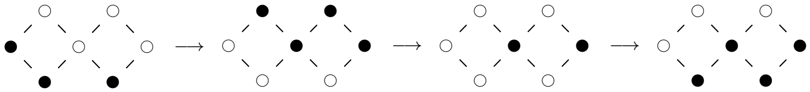
Rowmotion on Order Ideals of a Poset

Rowmotion: an invertible operation on order ideals

We define the (cyclic) group action of **rowmotion** on the set of order ideals $\mathcal{J}(P)$ via the map $\mathrm{Row} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the following three-step process.

Start with an order ideal, and

1. Θ: Take the complement (giving an order filter)
2. ∇: Take the minimal elements (giving an antichain)
3. Δ^{-1}: Saturate downward (giving a order ideal)

ρ_J :


This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Dynamical properties of rowmotion: cardinality is homomesic

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

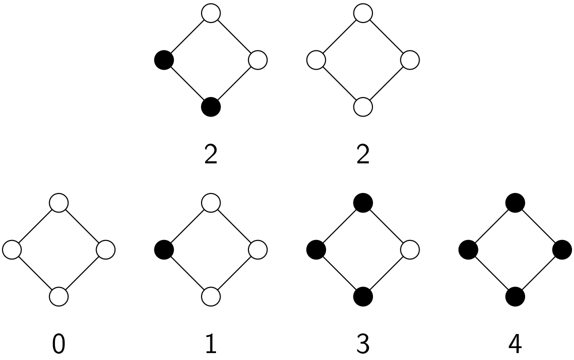
Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary rowmotion orbit in $\mathcal{J}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Ideals in $[a] \times [b]$: the case $a = b = 2$

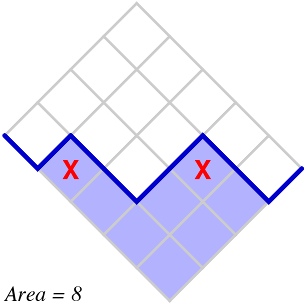
We have an orbit of size 2 and an orbit of size 4:



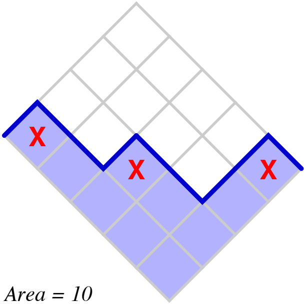
Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Example in lattice cell form

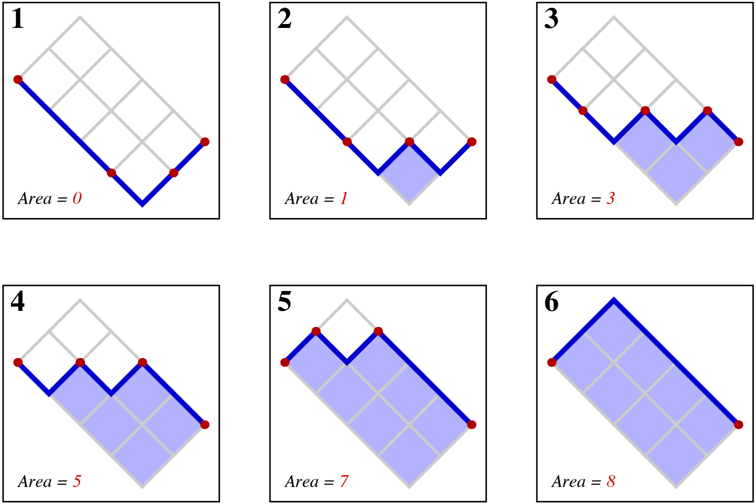
Viewing the elements of the poset as **squares** below, we would map:



$\longrightarrow$

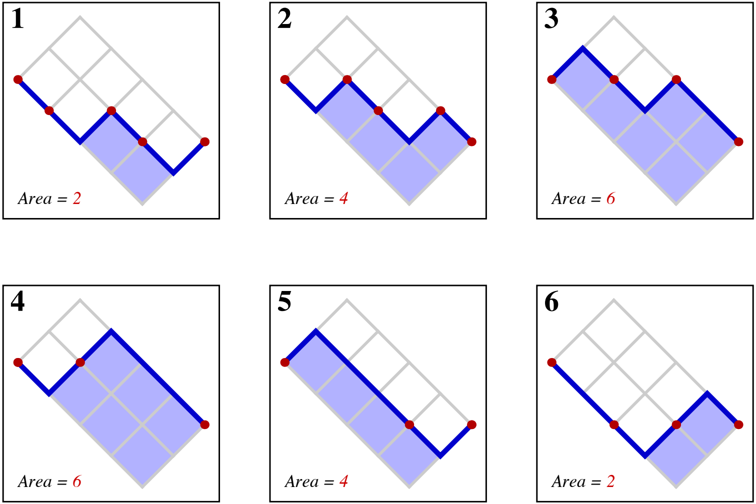


Rowmotion on $[4] \times [2]$: Orbit 1



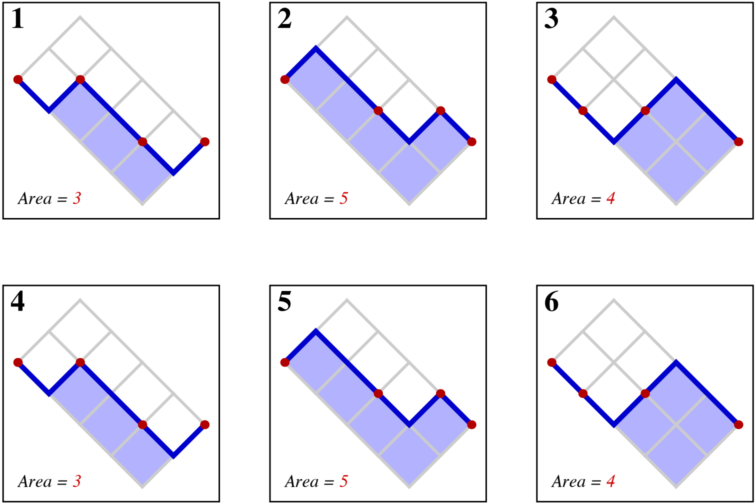
$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 2



$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 3



$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Toggling order ideals

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order ideals* (equivalently *order filters*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{J}(P)$ be the set of order ideals of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{J}(P) \to \mathcal{J}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{J}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{J}(P), \\ U & \text{otherwise.} \end{cases}$$

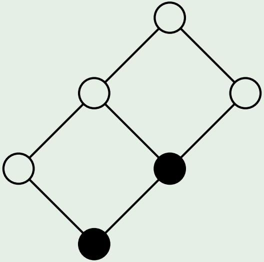
Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order ideals of P.

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example



Theorem (Cameron and Fon-Der-Flaass 1995)

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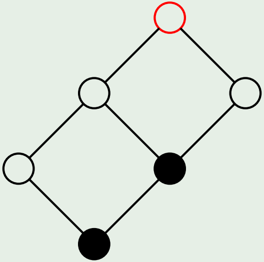
Example



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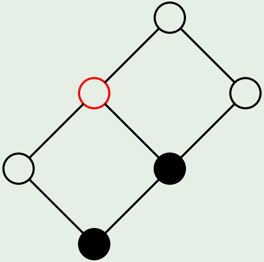
Example

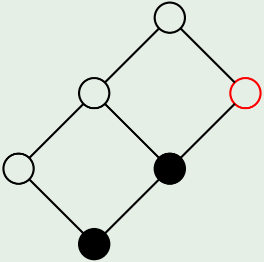


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

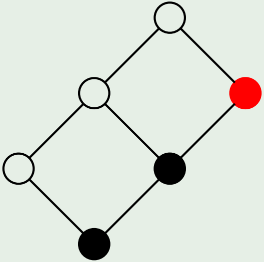
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

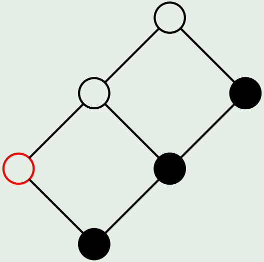
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

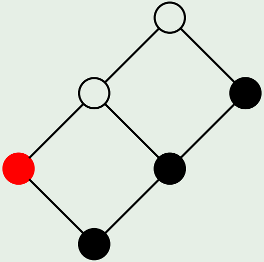
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example



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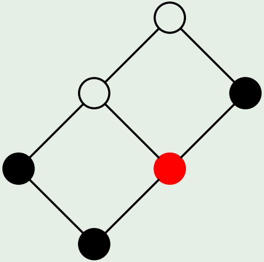
Example



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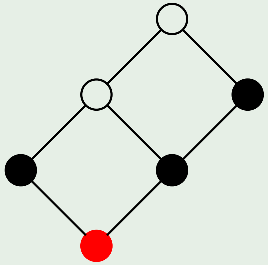
Example

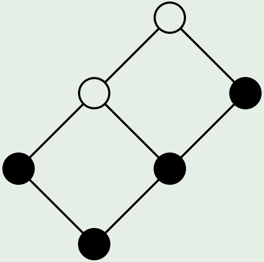


Toggling order ideals

Theorem (Cameron and Fon-Der-Flaass 1995)

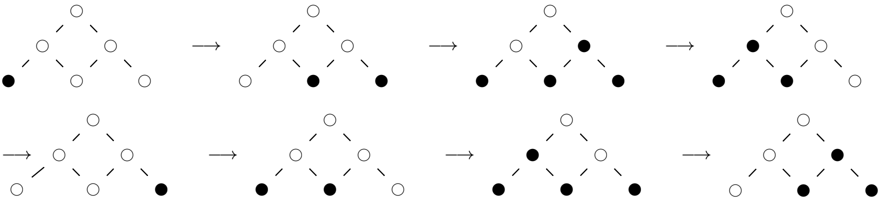
Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

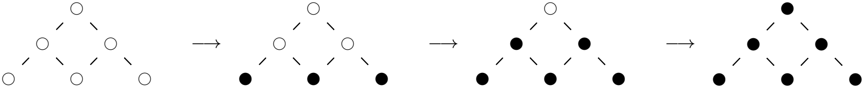
Example

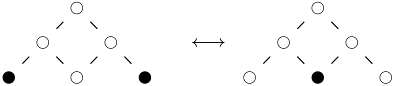


Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:





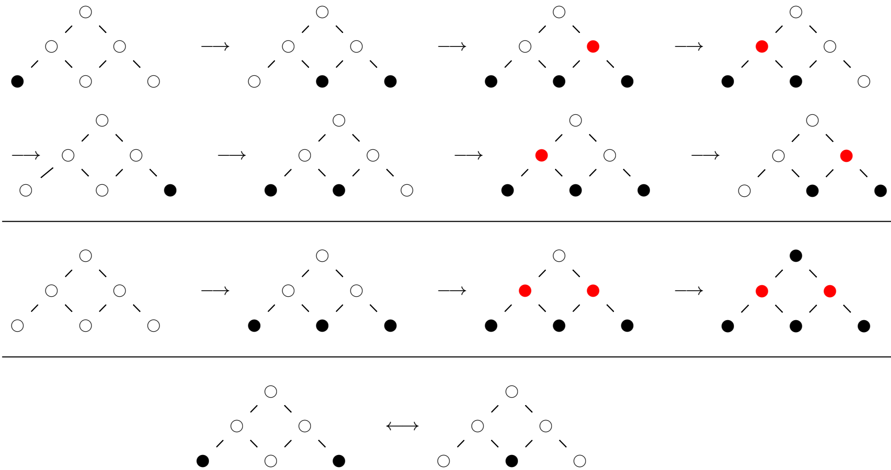


Checking the average cardinality for each orbit we find that

$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}; \quad \frac{0+3+5+6}{4} = \frac{7}{2}; \quad \frac{2+1}{2} = \frac{3}{2}. \text{ Darn!}$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ-orbits, of sizes 8, 4, 2:



Checking the average rank-alternating cardinality for each orbit we find:

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{1+2+2+1}{4} = \frac{2+1}{2} = \frac{3}{2} \text{ Yay!}$$

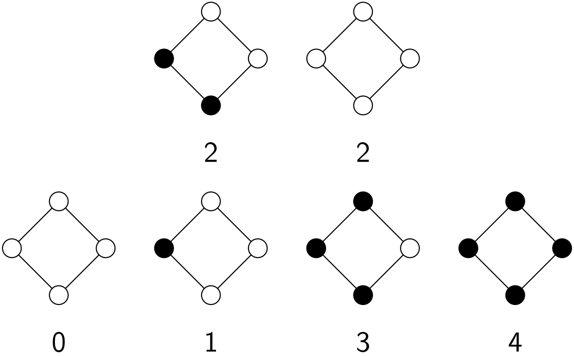
Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan)

Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign an order ideal $I \in \mathcal{J}(P)$ weight $f(I) = \sum_{x \in I} \mathrm{wt}(x)$, then f is homomesic under rowmotion and promotion, with average $n/2$.

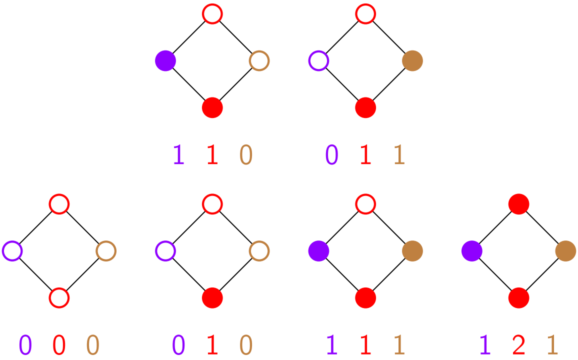
Ideals in $[a] \times [b]$: the case $a = b = 2$

We have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ of a violet element, 1 red element, and $1/2$ of a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

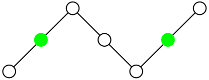
Theorem (Propp, R.)

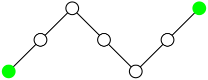
For every ρ-orbit $\mathcal{O}$ in $J([a] \times [b])$:

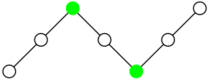
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

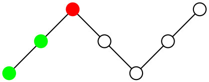
Some homomesies for (order-ideal) rowmotion on fence posets

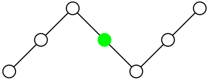
Periodicity and homomesy for rowmotion on *fence posets* was explored in recent work of Elizalde–Plante–Roby–Sagan [EPRS].

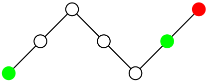

Average:1


Average:1

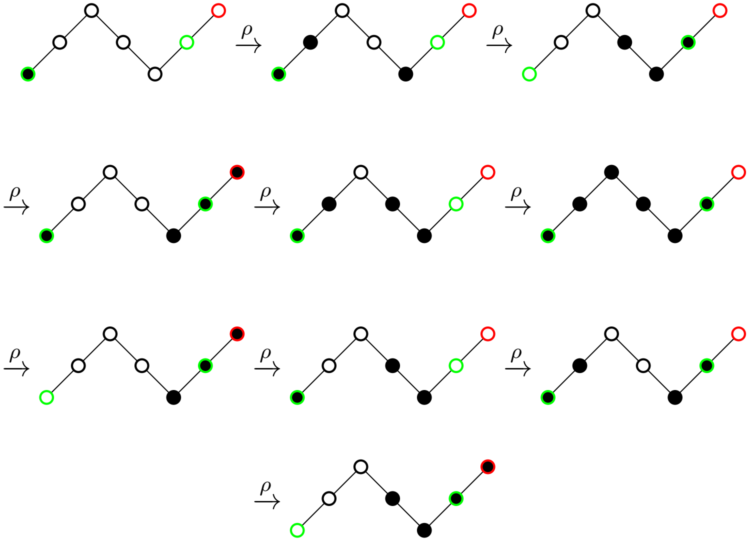

Average:1


Average:1


Average:$\frac{1}{2}$


Average:1

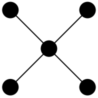
One orbit of rowmotion on a fence poset, highlighting a homomesy

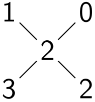


Checking the statistic we see $\frac{13-3}{10} = 1$

Whirling on posets

Let $\mathcal{F}_k$ be the set of order-reversing functions from P to $\{0, 1, 2, \ldots, k\}$.

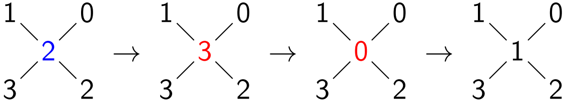
$P =$ 



$\in \mathcal{F}_3(P)$

Definition ([JPR18])

Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$ define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at* x, as follows: repeatedly add 1 $(\mathrm{mod}\ k+1)$ to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.



$\in \mathcal{F}_3(P)$

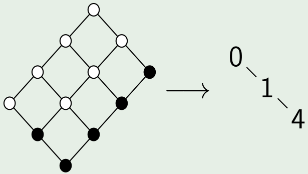
Equivariant bijection between whirling and rowmotion

Now let $\{x_1, x_2, \ldots, x_n\}$ be any linear extension of P (with $\#P = n$.) It is easy to show that w_x and w_y commute when $x, y \in P$ are *incómparable*. Thus the *whirling operator* $w := w_{x_1} w_{x_2} \cdots w_{x_n}$ is well-defined (whirling each poset element from top to bottom).

Theorem (Plante)

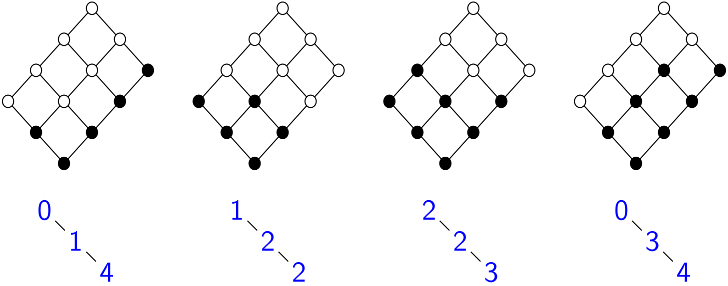
There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w to $\rho_{\mathcal{J}}$.

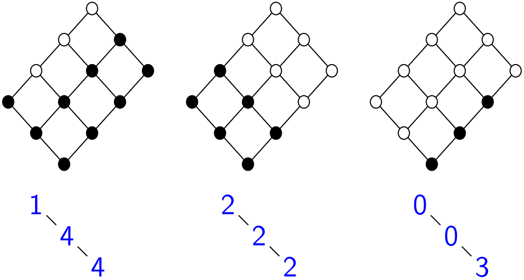
Example ($\mathcal{J}([3] \times [4])$ to $\mathcal{F}_4([3])$)



The number of order ideal elements in each fiber is recorded as an order-reversing function on [3].

Product of two chains orbit bijection example





0	1	2	0	1	2	0
1	2	2	3	4	2	0
4	2	3	4	4	2	3

Theorem (Plante)

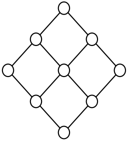
Let w denote the whirling operator on order-reversing functions $\mathcal{F}_k([m])$. Consider a superorbit board of w with length $k+m$.

1. *The board can be partitioned into m snakes of length $k+m$ under the following rules:*
    1. *Start at zero in the top row.*
    2. *Stay in a row until the value does not increase then move down.*
    3. *End once the snake contains k in the bottom row.*
2. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the segments of a snake α, that is, α_i is the number of blocks of the snake in row i. Each snake in the board has segments which are a cyclic rotation of $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.*
3. *The average sum of values along a snake is $k(m+k)/2$.*

An orbit board of $(0,1,4) \in \mathcal{F}_4([3])$:

0	1	2	0	1	2	0
1	2	2	3	4	2	0
4	2	3	4	4	2	3

Orbits of a product of two chains

$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

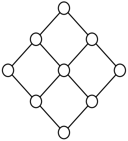
0	0	0	1	2	3
0	0	1	2	3	3
0	1	2	3	3	3

0	0	1	0	1	2
0	1	1	2	3	2
3	1	2	3	3	2

0	1	0	0	1	2
3	1	0	1	2	2
3	1	2	3	2	3

0	1
2	1
2	3

Orbits of a product of two chains

$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

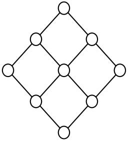
0	0	0	1	2	3
0	0	1	2	3	3
0	1	2	3	3	3

0	0	1	0	1	2
0	1	1	2	3	2
3	1	2	3	3	2

0	1	0	0	1	2
3	1	0	1	2	2
3	1	2	3	2	3

0	1
2	1
2	3

Orbits of a product of two chains

$[3] \times [3] =$ 

The 4 orbits of $\mathcal{F}_3([3])$ under the action of w.

0	0	0	1	2	3
0	0	1	2	3	3
0	1	2	3	3	3

0	0	1	0	1	2
0	1	1	2	3	2
3	1	2	3	3	2

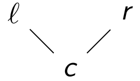
0	1	0	0	1	2
3	1	0	1	2	2
3	1	2	3	2	3

0	1	0	1	0	1
2	1	2	1	2	1
2	3	2	3	2	3

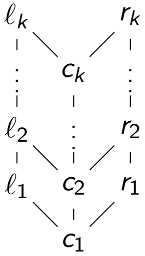
The $\vee \times [k]$ poset

The poset $\mathsf{V} \times [k]$

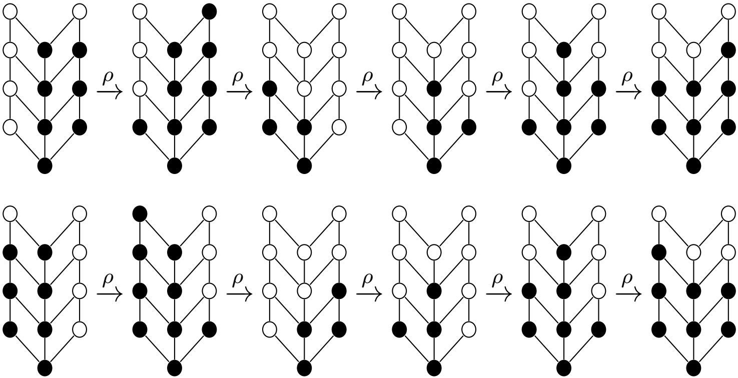
- Let V be the three-element partially ordered set with Hasse diagram



- The poset of interest is $V(k) := \mathsf{V} \times [k]$



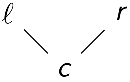
Order-ideal rowmotion on $\mathrm{V} \times [k]$



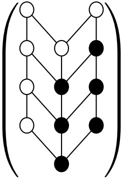
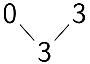
Theorem (Plante)

Order ideals of $V(k)$ are reflected about the center chain after $k + 2$ iterations of ρ, and furthermore, the order of ρ on order ideals of $V(k)$ is $2(k + 2)$.

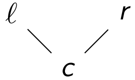
Map to order-reversing functions on V



1. Define $\mathcal{F}_k(\mathsf{V}) = \{(\ell, c, r) \in \{0, \ldots, k\}^3 : \ell, r \leq c\}$.
2. Define $\phi : \mathcal{J}(\mathsf{V}(k)) \to \mathcal{F}_k(\mathsf{V})$ by
   $\phi(I) = \left(\sum \chi_{\ell_i}, \sum \chi_{c_i}, \sum \chi_{r_i}\right)$.

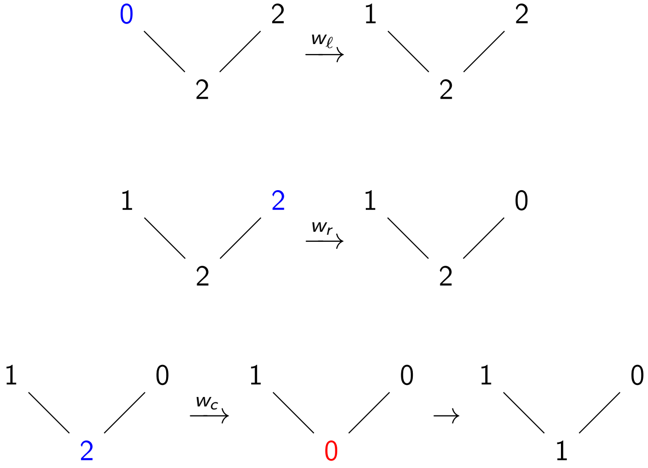
ϕ  $= (0, 3, 3) \leftrightarrow$ 

Example of whirling V

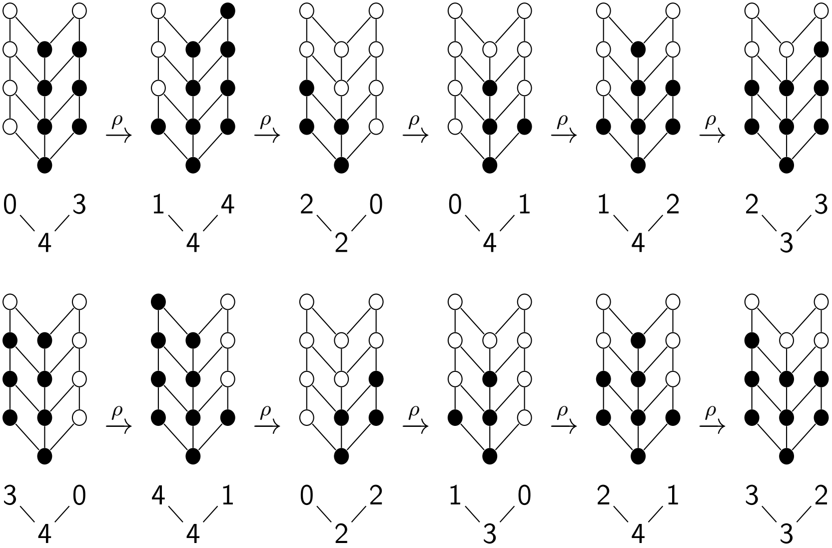
We whirl the example 

first at ℓ, r, then c.

Start with $(0, 2, 2) \in \mathcal{F}_2(\mathrm{V})$.



Example of rowmotion orbit with triples



Equivariant bijection example

Alternatively we may define w on $(\ell, c, r) \in \mathcal{F}_k(\mathsf{V})$ as the process:

1. $\ell \to \ell + 1$ unless $\ell = c$, then $\ell \to 0$.
2. Repeat step 1 with r instead of ℓ.
3. $c \to c + 1$ unless $c = k$, then $c \to \max(\ell, r)$.

Corollary

The map ϕ is an equivariant bijection that sends ρ to w.

$$\begin{tikzcd}
\mathcal{J}(\mathsf{V}(k)) \arrow[r, "\rho"] \arrow[d, leftrightarrow, "\phi"'] & \mathcal{J}(\mathsf{V}(k)) \arrow[d, leftrightarrow, "\phi"'] \\
\mathcal{F}_k(\mathsf{V}) \arrow[r, "w"] & \mathcal{F}_k(\mathsf{V})
\end{tikzcd}$$

Theorem (Plante)

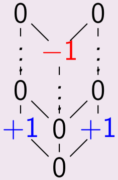
Order ideals of $V(k)$ are reflected about the center chain after $k + 2$ iterations of ρ, and furthermore, the order of ρ on order ideals of $V(k)$ is $2(k + 2)$.

Direct inspection of order-reversing functions on V as tuples gives a straightforward proof of periodicity.

Theorem (Plante)

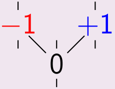
For the action of rowmotion on order ideals of $\mathsf{V}(k)$:

1. *The statistic $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ is $\frac{2(k-1)}{k+2}$-mesic.*



.

2. *The statistic $\chi_{r_i} - \chi_{\ell_i}$ is 0-mesic*



for each $i = 1, \ldots, k$, where χ_x is the indicator function.

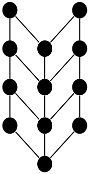
Center Seeking Snakes

We decompose the orbit board into 6 snakes of length $k + 2$. Or 2 two-tailed snakes if the order-reversing functions are symmetric. Recall that snakes start at the top of a poset and move down. Since the least element of V is in the center, we call these snakes, *center-seeking snakes*.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Sketch of Proof of Homomesy



$$\sum \chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$$

$$= (2(k+2)-3)+(2(k+2)-3)-6$$

Thus we see

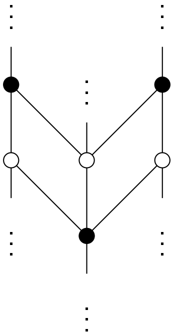
$$\frac{4(k+2)-12}{2(k+2)} = \frac{2k-2}{k+2}.$$

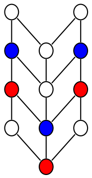
1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

$2(k+2)$

Another Potential Homomesy

There is another nice looking homomesy not yet proved. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the following flux-capacitor shape in $\mathsf{V}(k)$.



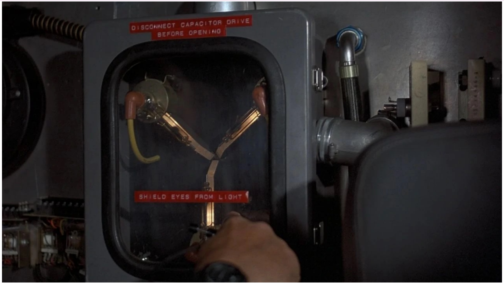


$F_3 - F_2$

Conjecture

The difference $F_i - F_{k-i+1}$ is homomesic.

Flux Capacitor??



https://www.youtube.com/watch?v=VcZe8_RZO8c

Whirling Proper k-Colorings of the Path and Cycle Graph

Let $G = (V, E)$ be a graph with $K_k(G)$ being the set of proper k-colorings $\kappa : V \to [k]$.

All Proper 3-Colorings of $\bullet - \bullet - \bullet$

$1 - 2 - 1$	$1 - 2 - 3$	$1 - 3 - 1$	$1 - 3 - 2$
$2 - 1 - 2$	$2 - 1 - 3$	$2 - 3 - 1$	$2 - 3 - 2$
$3 - 1 - 2$	$3 - 1 - 3$	$3 - 2 - 1$	$3 - 2 - 3$

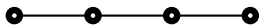
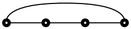
Definition (JPR18, Def 2.1)

Define $w_v : K(G) \to K(G)$ (whirl at v) by incrementing the color of vertex v by 1 modulo k repeatedly until arriving at a proper k-coloring.

$$w_b\left(\underset{a}{\textcircled{3}} - \underset{b}{\textcircled{2}} - \underset{c}{\textcircled{3}}\right) = \underset{a}{\textcircled{3}} - \underset{b}{\textcircled{1}} - \underset{c}{\textcircled{3}}$$

Whilring k-Colorings

Let $\mathcal{P}_n$ be the path graph with n vertices, and let $\mathcal{C}_n$ be the cycle graph with n vertices.

$\mathcal{P}_4 =$  $\mathcal{C}_4 =$ 

We set $V = [n]$ labeled from left to right and consider the action $w = w_n \dots w_1$. Thus the proper k-colorings of $\mathcal{P}_n$ are maps $\kappa : [n] \to [k]$ such that $\kappa(i-1) \neq \kappa(i) \neq \kappa(i+1)$ (modulo n if we are on a cyclic graph.) We also represent colorings with $[k]$-words of length n.

$$2 - 1 - 2 \to 212$$

$$w(212) = w_3 w_2 w_1(212) = w_3 w_2(312) = w_3(312) = 313$$

Homomesies for w on Paths

Fix any color $j \in [3]$. Set χ_i to be the indicator function for when vertex i is colored with j.

Theorem (Plante)

Under the action of w on $K_3(\mathcal{P}_n)$,

1. $\chi_i - \chi_{n+1-i}$ *is 0-mesic, and*
2. $2\chi_1 + \chi_2$ *is 1-mesic and* $\chi_{n-1} + 2\chi_n$ *is 1-mesic.*

Orbit from $K_3(\mathcal{P}_n)$:

1	3	2	3	1	2	1
2	1	2	3	1	3	2
3	1	2	3	2	1	3
2	3	1	3	2	1	2
1	2	1	3	2	3	1
3	2	1	3	1	2	3

Difference Vector

Definition

The *difference vector*, d of a proper 3-coloring of $\mathcal{P}_n$ is the string of $n-1$ **+**'s and **−**'s depending on whether the coloring increases by 1 or decreases by 1 respectively from left to right.

1	2	3	2	+	+	−
3	1	3	1	+	−	+
2	1	2	3	−	+	+
3	1	2	1	+	+	−
2	3	2	3	+	−	+
1	3	1	2	−	+	+
2	3	1	3	+	+	−
1	2	1	2	+	−	+
3	2	3	1	−	+	+

Affect of w on the Difference Vector

Lemma

If $\kappa \in K_3(\mathcal{P}_n)$ or $\kappa \in K_3(\mathcal{C}_n)$ with difference vector d,

1. *If i is an interior vertex (degree two), then the difference vector of $w_i(\kappa)$ is d but with d_{i-1} and d_i swapped.*
2. *If i is an exterior vertex (degree one) and $i = 1$ (resp. $i = n$), then the difference vector of $w_i(\kappa)$ is d but with d_1 (resp. d_{n-1}) changed from $\boldsymbol{+}$ to $\boldsymbol{-}$ or vise versa.*

Here is an example where w acts on κ one whirl at a time with the difference vector updated at each step.

$$\begin{array}{lcccccclccccc}
 & 1 & 2 & 1 & 2 & 3 & 1 & \quad & + & - & + & + & + \\
w_1 & 3 & 2 & 1 & 2 & 3 & 1 & & - & - & + & + & + \\
w_2 & 3 & 2 & 1 & 2 & 3 & 1 & & - & - & + & + & + \\
w_3 & 3 & 2 & 3 & 2 & 3 & 1 & & - & + & - & + & + \\
w_4 & 3 & 2 & 3 & 1 & 3 & 1 & & - & + & + & - & + \\
w_5 & 3 & 2 & 3 & 1 & 2 & 1 & & - & + & + & + & - \\
w_6 & 3 & 2 & 3 & 1 & 2 & 3 & & - & + & + & + & +
\end{array}$$

Implication of Rotation of Difference Vector

Proposition

If $\kappa \in K_3(\mathcal{P}_n)$, d is the difference vector of κ, and τ is leftward cyclic-shift on strings of $\mathbf{+}$'s and $\mathbf{-}$'s, then the difference vector $w(\kappa)$ is $\tau(d)$.

$$\begin{array}{ccc} 1\ 2\ 1\ 3\ 1\ 2 & \xrightarrow{w} & 3\ 2\ 1\ 2\ 3\ 1 \\ \mathbf{+}\ \mathbf{-}\ \mathbf{-}\ \mathbf{+}\ \mathbf{+} & \xrightarrow{\tau} & \mathbf{-}\ \mathbf{-}\ \mathbf{+}\ \mathbf{+}\ \mathbf{+} \end{array}$$

Theorem (Plante)

Let $\kappa \in K_3(\mathcal{P}_n)$ have difference vector d. Let ℓ be the smallest natural number such that $w^\ell(\kappa) = \kappa$, t be the smallest natural number such that $\tau^t(d) = d$.

1. *If the sum of d is 0, then $\ell = t$.*
2. *Otherwise $\ell = 3t$.*

Theorem (Plante)

Fix any color $j \in [3]$. Set $\chi_i := \chi_{i,j}$. Under the action of w on $K_3(\mathcal{C}_n)$,

1. *If $3 \not| n$, then χ_i is $1/3$-mesic, and*
2. *If $3|n$, then $\chi_{3a+i} - \chi_{3b+i}$ is 0-mesic for $i \in [3]$ and $0 \leq a, b \leq \frac{n}{3} - 1$.*

Similarly the difference vector, d, of a proper 3-coloring of $\mathcal{C}_n$ is the same as the difference vector of $\mathcal{P}_n$ but with an extra **+** or **−** for the difference between the last color and the first color

$$3\ 2\ 1\ 2\ 1 \qquad \mathbf{-\ -\ +\ -\ -}$$

We prove the theorem using a similar argument to the one for Path Graphs.

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Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to questions of periodicity/order, orbit structure, homomesy, and equivariant bijections.
- Actions that can be built out of smaller, simpler actions (toggles and whirls) often have interesting and unexpected properties.
- Much more remains to be explored, perhaps for combinatorial objects or actions that you work with for other reasons.

Slides for this talk will be available online at

Google "Tom Roby".

Thanks very much for coming to this talk!