Equivalences relations of permutations generated by constrained transpositions

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Outline

- The intermediary game
- Knuth relations and tableaux
- Right superior game
- Typical results
- The general framework
- Table of results
- Involutions & the Chinese monoid
- Further work & Open problems

The Intermediary Game on Permutations

We view permutations in S_n as words: $a_1 a_2 \cdots a_n$, e.g., 314652 $\in S_6$, and allow moves of the following type:

If $a_i < a_j < a_k$ or $a_i > a_j > a_k$ for some $i < j < k$, then we may interchange (transpose, swap) a_i and a_k.

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Try this now with one of the following permutations (working with a partner encouraged):

314652 124356 213456

Intermediary Game Examples

213456	124356	314652
243156	624351	514632
243651	654321	512634
643251	634521	542631
143256	631524	245631
123456	136524	265431
	436521	263451
	436125	213456
	136425	...
	156423	
	153426	
	123456	

Equivalence Relation

The transitive closure of the relations defined by these moves defines an equivalence relation on the set of permutations. We are interested in the sizes of and possible characterizations of the resulting equivalence classes.

More concretely, we can think of this game as creating a *graph*, whose vertex set is S_n, and with edges between any two permutations connected by a legal move. We are interested in questions about the connected components of this graph.

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For $n \geq 6$ we get a single equivalence class (the "fully mixed" case), which of course contains the identity.

A Bar Game

This leads to the following bar game:

- Demonstrate your skill at obtaining the identity from random permutations in S_6 using only the legal moves.
- Bet someone that they won't be able to do it.
- When they get stuck (quite likely), take pity on them and give them a "easier" permutation in S_5.
- There's an 80% chance that a randomly chosen $\sigma \in S_5$ is NOT legally obtainable by this set of moves.
- Compare your winnings with the expected value you computed before going to the bar...

Basic questions

This example illustrates the basic questions we will consider, not just for this game, but for ones with other sets of rules P:

A. Compute the number of equivalence classes $\#\mathrm{Classes}^\star(n, P)$ into which S_n is partitioned.
B. Compute the size of $\#\mathrm{Eq}^\star(\iota_n, P)$ of the equivalence class containing the identity, ι_n.
C. (More generally) characterise the set $\mathrm{Eq}^\star(\iota_n, P)$ of permutations equivalent to the identity.

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   5,10,3,1,1,1,. . .
B. Compute the size of $\#\mathrm{Eq}^\star(\iota_n, P)$ of the equivalence class containing the identity, ι_n.
   2,4,24,720,5040,40320,. . .
C. (More generally) characterise the set $\mathrm{Eq}^\star(\iota_n, P)$ of permutations equivalent to the identity.
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- **C** (More generally) characterise the set $\mathrm{Eq}^\star(\iota_n, P)$ of permutations equivalent to the identity.
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So the relation given by $P = \{123 \leftrightarrow 321\}$ (with no adjacency constraints) is not a particularly interesting example from this standpoint.

A more formal description

Let $P = \{B_1, B_2, \ldots, B_t\}$ be a (set) partition of S_k (e.g., $k = 3$). Each block B_l of P represents a list of k-length patterns which can replace one another within some $\pi \in S_n$.

Call π, σ $\mathbf{P}^{::}$**-equivalent** if one can be obtained from the other by a sequence of such replacements; $\mathrm{Eq}^{::}(\pi, P)$ is the eq. class of π.

Similarly we discuss $\mathbf{P}^{||}$**-equivalence** and $\mathrm{Eq}^{||}(\pi, P)$ when all moves involve only (positionally) adjacent entries:

We use $\mathbf{P}^{\square}$ and $\mathrm{Eq}^{\square}(\pi, P)$ when both positions and values are constrained:

The set of distinct equivalence classes into which S_n splits under an equivalence P^* is denoted by $\mathrm{Classes}^*(n, P)$.

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Knuth's relations

An interesting and important example of a relation on S_n was given by Knuth in his study of the Robinson-Schensted-Knuth (RSK) correspondence: $P_K^{|\,|} = \big\{\{213, 231\}, \{132, 312\}\big\}$.

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Here we are only allowed to swap **(positionally) adjacent** entries, and only if there is **a (positionally) adjacent** intermediary. The values, however, simply need to be in the same relative order as 231. So another way of expressing this is: Whenever $a < b < c$ allow any adjacent swap:

$$bac \leftrightarrow bca \qquad \text{or} \qquad acb \leftrightarrow cab\,.$$

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$\{54123, \quad 51423, 51243, 15423, 15243, \quad 12543\}$

Facts about Knuth relations

Here are some facts about the Knuth relations and RSK.

- Two permutations are Knuth-eq. iff they map to the same standard Young tableaux P under RSK-insertion. $[\ \pi \leftrightarrow (P, Q)]$.
- So one way to think of SYT is as equivalence classes of permutations under this relation. The size of this class is the number of SYT of the same shape.
- Because of RSK's symmetry $[\ \pi \leftrightarrow (P, Q) \Leftrightarrow \pi^{-1} \leftrightarrow (Q, P).]$, SYT with n-boxes correspond bijectively to involutions in S_n. Hence the number of equivalence classes is $\mathrm{Inv}(n) = [t^n]e^{t+\frac{1}{2}t^2}$.
- The identity is isolated by this relation, so the answers to B) and C) are trivial.

Dual Equivalence Graphs

- Dual equivalence was introduced by Mark Haiman c. 1990 to prove conjectures of Bob Proctor & Richard Stanley.
- He gave a bijection between standard tableaux of shifted staircase shape and reduced expressions for the longest element in the Coxeter group B_l.
- In her dissertation S. Assaf constructed graphs (with some extra structure) whose vertices are tableaux of a fixed shape (which may be viewed as permutations via their "reading words"), and whose edges represent (elementary) dual equivalences between vertices. She characterised the local structure of these graphs, which she later used to give a combinatorial formula for the Schur expansion of LLT polynomials and Macdonald Polynomials. She also used these, along with crystal graphs, to give a combinatorial realization of Schur-Weyl duality.

Dual Equivalence Graphs

Two permutations are Knuth equivalent iff their inverses are dual-Knuth equivalent.

So from the enumerative standpoint of our work, there's no difference between between these relations.

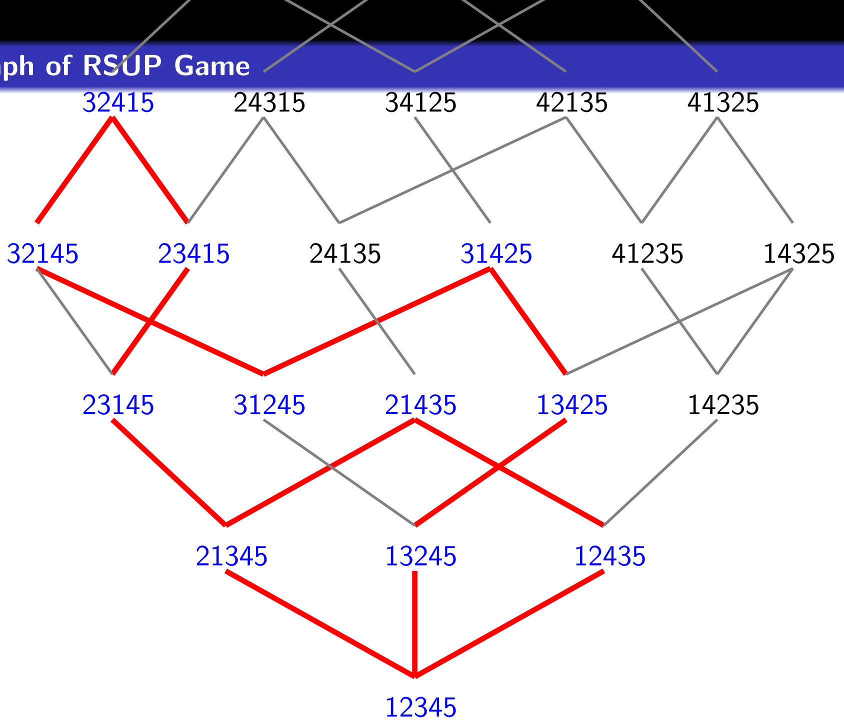
OPEN: Understand the structure of the graphs one gets with differently defined games.

The Right Superior Game

Here are the rules for a new game on S_n. Say that two n-permutations are equivalent if they differ by an **adjacent** transposition:
$a_i a_{i+1} \leftrightarrow a_{i+1} a_i$, where both inequalities
$a_i < a_{i+2}$ and $a_{i+1} < a_{i+2}$ hold.

$P_2^{|\,|} = \{123 \leftrightarrow 213\}$.

Graph of RSUP Game



Size of the Class of ι_n

How many permutations are equivalent to the identity?

n	3	4	5	6	7	8	9	10
$\#\mathrm{Eq}^{\circ}(\iota_n, P)$	2	4	12	36	144	576	2880	14400

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Theorem 1. For the Right Superior Game, the number of n-permutations in the equivalence class of the identity is

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i.e., for $n = 2r$, $\#\{\pi : \pi \leftrightarrow 123\ldots n\} = r!r!$.

and for $n = 2r + 1$, $\#\{\pi : \pi \leftrightarrow 123\ldots n\} = r!(r+1)!$.

Proof that this is an upper bound.

Proof that this is an upper bound:
The largest element must be in the rightmost position.

This implies that the second-largest element must be in one of the three rightmost positions.

This implies that the third-largest element ...

Now, placing the elements from largest to smallest, we have the following number of choices for each placement:

$1 \cdot 2 \cdot 3 \cdot \cdots \cdot \lceil n/2 \rceil \cdot \lfloor n/2 \rfloor \cdot \cdots \cdot 3 \cdot 2 \cdot 1$

Proof that the upper bound is attained

Proof of equality.

It remains to show that all permutations meeting these constraints are in fact reachable.

Imagine a target permutation meeting the constraints. That is, the first element (even case) or first two elements (odd case) are less than $\lceil n/2 \rceil + 1$, the next two elements are less than $\lceil n/2 \rceil + 2$, etc.

`Target:    kgfOdiPahNQcbTRjUmSVWXelYZ`

Step one. Working from the identity, move all the "large" elements leftwards as far as they will go:

Proof of Attainability

```
.............NOPQRSTUVWXYZ
............N.OPQRSTUVWXYZ
............NO.PQRSTUVWXYZ
............NOP.QRSTUVWXYZ
............NOPQ.RSTUVWXYZ
            :
............NOPQRSTUVWXY.Z
            :
            :
...........NOPQRSTUVWX.Y.Z
            :
            :
..........NOPQRSTUVW.X.Y.Z
            :
            :
            :
.N.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

Proof of Attainability 2

`Target:   kgfOdiPahNQcbTRjUmSVWXelYZ`

```
.N.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

Now observe that the "small" elements can be permuted freely while leaving the "large" elements in place.

```
fRjS  ->  fjRS  ->  jfRS  ->  jRfS
```

Step two. Using this observation, move the correct element into the first position. (In the odd case, move the two correct elements into the first two positions.) Because the target permutation obeys the constraints, this element (or pair of elements) will be small compared with the fixed skeleton of large elements which is facilitating their movement.

```
kN.O.P.Q.R.S.T.U.V.W.X.Y.Z
```

Proof of Attainability 3

Target: kgfOdiPahNQcbTRjUmSVWXelYZ

Continue to place elements two at a time:

```
kN.O.P.Q.R.S.T.U.V.W.X.Y.Z
kgfO.P.Q.R.S.T.U.V.W.X.Y.Z
kgfOdP.Q.R.S.T.U.V.W.X.Y.Z
kgfOdiPQ.R.S.T.U.V.W.X.Y.Z
kgfOdiPahR.S.T.U.V.W.X.Y.Z
            :
kgfOdiPahNQcbTRjUmSVWXelYZ
```

□

$n = 5$ Example 1

As one one might expect, this algorithm constructs some permutations efficiently, but not others.
Construction of the "furthest" permutation 32145:

```
1 2 3 4 5
1 3 2 4 5
1 3 4 2 5
3 1 4 2 5  <-- build the skeleton

3 1 2 4 5  <-- 3 is already at front, so advance 2
3 2 1 4 5  <-- we're there, stop!
```

$n = 5$ Example 2

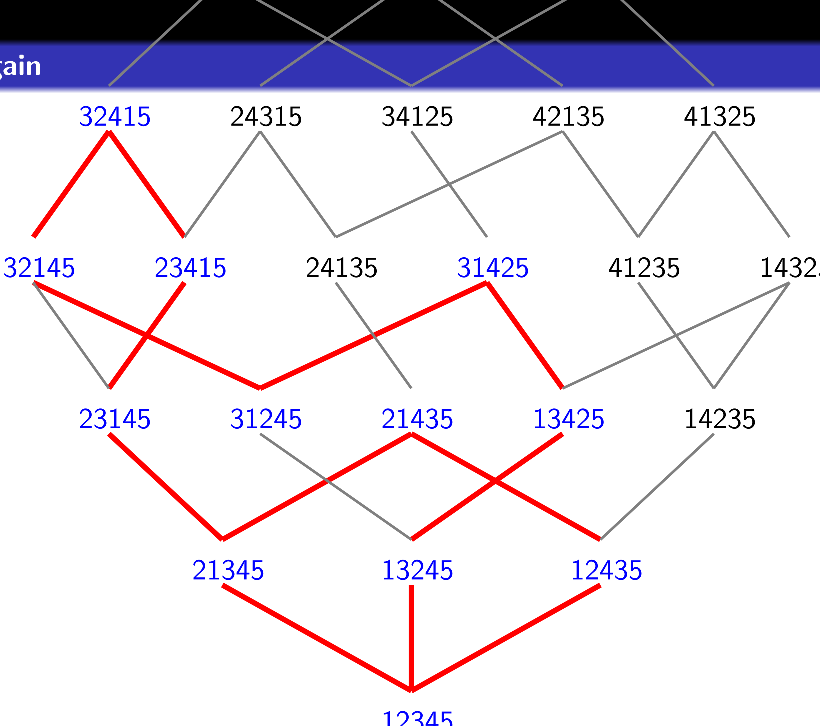
An example where the algorithm is inefficient, 12435:

```
1 2 3 4 5
1 3 2 4 5
1 3 4 2 5
3 1 4 2 5  <-- build the skeleton

1 3 4 2 5  <-- advance 1 (it just came from there!)

1 3 2 4 5  )
1 2 3 4 5  ) <-- 3-step procedure for advancing 2
1 2 4 3 5  )
```

Graph Again



32415
24315
34125
42135
41325
32145
23415
24135
31425
41235
23145
31245
21435
13425
14235
21345
13245
12435

Propp's Proposition

```
From: James Propp <jpropp@cs.uml.edu>
Date: Wed, 8 Jul 2009 17:07:07 -0400
Subject: two hundred and ten questions

:
:

I'd like to know the partition of n! determined by the transitive
closure of each of the following seven relations on S_n:

:

The two most interesting
numbers are probably the number of components and the size of
the component containing the permutation 1,2,3,...,n.
:

I should say that I want this information for _three_ distinct
interpretations of what "123 <--> 213" means:
(a) In the narrowest sense, it could mean that if pi(i+1) = pi(i)+1
    and pi(i+2) = pi(i)+2, then you can swap the values of pi(i) and
    pi(i+1).
(b) More broadly, it could mean that if pi(i) < pi(i+1) < pi(i+2),
    then you can swap the values of pi(i) and pi(i+1).
(c) More broadly still, it could mean that if pi(i) < pi(j) < pi(k)
    for i < j < k, then you can swap the values of pi(i) and pi(j).

Jim
```

General Framework

- Consider interchanges of subwords of "type" $\sigma_1 \leftrightarrow \sigma_2$, where $\sigma_i \in S_3$.
- As Jim described, this can be taken in three sense: (a) both indices and values must be adjacent; (b) entries must be in adjacent positions; (c) unrestricted in value or position
- Restricting entries to be adjacent values (but not necessarily positions) is equivalent to (b) by the map that sends $\pi \to \pi^{-1}$.
- In theory one could consider any of the $B(6) = 203$ partitions of S_3 as defining a relation (or three) of this type, although some of these will be trivially equivalent.
- To keep the problem within bounds, we currently consider only sets of relations of the form $\iota_3 \leftrightarrow \sigma$, where $\sigma \in S_3$. Equivalently, these are partitions of S_3 with a single nontrivial block (containing ι_3).

Number of Classes

How many equivalences classes for each relation?

$\#\mathrm{Classes}(n, P)$

Transpositions	general	indices adjacent	indices & values adjacent
123 ↔ 132 123 ↔ 213	[5, 14, 42, 132, 429] Catalan	[5, 16, 62, 284, 1507, 9104]	[5, 20, 102, 626, 4458, 36144]
123 ↔ 321	[5, 10, 3, 1, 1, 1] trivial	[5, 16, 60, 260, 1260, 67442]	[5, 20, 102, 626, 4458, 36144]
123 ↔ 132 ↔ 213	[4, 8, 16, 32, 64, 128] powers of 2	[4, 10, 26, 76, 232, 764] involutions	[4, 17, 89, 556, 4011, 32843]
123 ↔ 132 ↔ 321 123 ↔ 213 ↔ 321	[4, 2, 1, 1, 1, 1] trivial	[4, 8, 14, 27, 68, 159, 496]	[4, 16, 84, 536, 3912, 32256]
123 ↔ 132 ↔ 213 ↔ 321	[3, 2, 1, 1, 1, 1] trivial	[3, 4, 5, 8, 11, 20, 29, 57]	[3, 13, 71, 470, 3497]

Size of class containing identity: $\#\mathrm{Eq}^*(\iota, P)$

Transpositions	general	indices adjacent	indices & values adjacent
123 ↔ 132 123 ↔ 213	[2, 6, 24, 120, 720] (n-1)!	[2, 4, 12, 36, 144, 576, 2880] product of two factorials	[2, 3, 5, 8, 13, 21, 34, 55] Fibonacci numbers
123 ↔ 321	[2, 4, 24, 720] trivial	[2, 3, 6, 10, 20, 35, 70, 126] central binomial coefficients	[2, 3, 4, 6, 9, 13, 19, 28] A000930
123 ↔ 132 ↔ 213	[3, 13, 71, 461] connected A003319	[3, 7, 35, 135, 945, 5193] terms are always odd	[3, 4, 8, 12, 21, 33, 55, 88] A052952
123 ↔ 132 ↔ 321 123 ↔ 213 ↔ 321	[3, 23, 120, 720] trivial	[3, 9, 54, 285, 2160, 15825] proven for odd terms	[3, 5, 9, 17, 31, 57, 105, 193] tribonacci numbers A000213
123 ↔ 132 ↔ 213 ↔ 321	[3, 23, 120, 720] trivial	[4, 21, 116, 713, 5030]	[4, 6, 13, 23, 44, 80, 149, 273] tribonacci A000073 $-[n \text{ even}]$

The Chinese Monoid

Note that $\#\mathrm{Classes}^{||}(n, \{\{123, 132, 213\}\}) = \mathrm{inv}_n$. In other words, there is an equivalence relation on S_n other than the Knuth relations which gives the same number of classes.

This relation was studied in detail at the level of words as an analogue of Lascoux and Schützenberger's plactic monoid.

Duchamp & Krob showed that there is exactly one other regular monoid with the same Hilbert series as the plactic one.

The key observation is that any of these relations make sense when applied to **words with repeated entries**: $w = a_1 \cdots a_n$ where each $a_i \in [n]$. Just consider an entry b occurring to the right of the same entry b to be larger. (Or add subscripts to repeated entries from left to right.)

Chinese Monoid 2

In other words: *Plactic Monoid* is $[n]^*/P_K^{||}$ and *Chinese Monoid* is $[n]^*/P_3^{||}$, where: $P_3 = \{\{123, 132, 213\}\}$ (up to reversal of words).

In [CEHKN], the authors give an analogue of Robinson-Schensted to characterize the equivalence classes and study the conjugacy classes.

[CEHKN] J. Cassaigne, M. Espie, D. Krob, J.-C. Novelli, F. Hivert, *The Chinese Monoid*, Int'l. J. Algebra and Comp., **11** #3 (2001), 301–334.

Why do we get involutions?

Idea of Pf: Write any involution $\tau \in \mathrm{Inv}_n$ as product of 1- & 2-cycles; order cycles decreasing by largest elt., then drop parentheses to get an "involution word".

(59)(8)(17)(46)(3)(2) $\mapsto$ 598174632

We claim the set $\mathcal{D}_n$ of such words is canonical list of representatives of the P_3-equivalence classes.

Given any $\pi \in S_n$, either n is leftmost, or we can move it leftwards using one of $123 \to 132$ or $213 \to 132$, eventually to 2nd position. This leaves us with: $Mna_3 \cdots a_n$, where $M =$ minimal elt. to left of n in π. Now proceed inductively on the remaining elements $a_3 \cdots a_n$.

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Example of transforming a permutation to an involution word

EG:

$$
\begin{aligned}
6214375 &\mapsto 6214375 \\
&\mapsto 6213745 \\
&\mapsto 6217345 \\
&\mapsto 6172345 \\
&\mapsto (17)62345
\end{aligned}
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&\mapsto [17][6]2345
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&\mapsto [17]62345 \\
&\mapsto [17][6]2345 \\
&\mapsto [17][6]2345 \\
&\mapsto [17][6]2354 \\
&\mapsto [17][6][25]34
\end{aligned}
$$

Example of transforming a permutation to an involution word

EG:

$$
\begin{aligned}
6214375 &\mapsto 6214375 \\
&\mapsto 6213745 \\
&\mapsto 6217345 \\
&\mapsto 6172345 \\
&\mapsto (17)62345 \\
&\mapsto [17]62345 \\
&\mapsto [17][6]2345 \\
&\mapsto [17][6]2345 \\
&\mapsto [17][6]2354 \\
&\mapsto [17][6][25]34 \\
&\mapsto [17][6][25][34]
\end{aligned}
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&\mapsto [17][6]2354 \\
&\mapsto [17][6][25]34 \\
&\mapsto [17][6][25][34] \\
&\mapsto 1762534 \in \mathcal{D}_\backslash
\end{aligned}
$$

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Further Work & Open Problems

We've just begun the more general study of these kinds of relations. Plenty of open problems remain, including:

1. Find formulae for the unknown data in the table.
2. Recall our initial "Intermediary in-between" rule, but in the adjacent context. We prove that

$$\#\mathrm{Eq}^{||}(\iota_n, \{\{123, 321\}\}) = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$$

   in a fairly indirect way. Is there a simple combinatorial proof?
3. Understand the structure of the graphs one gets on these relations. Are the (like Bruhat order in the unconstrained case) posets?
4. Is there a useful length (distance from the identity) function?

Further Work & Open Problems 2

5. Answer more generally what the sizes of all the equivalence classes are, or whether there's a simple way to characterize them (as insertion tableaux characterizes all permutations which are Knuth equivalent).
6. Consider more general relations, defined by partitioning S_3 in different ways (more general block structures or connecting non-transpositions. Or even using relations within S_4?
7. Pierrot, Rossin, & West (FPSAC 2011) handle the other case of including non-transpositions within a unique non-singleton block containing ι_3 of a partition of S_3 (e.g., $\{123, 231\}$).

Thanks!

Thanks for your attention!

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Any questions?