Combinatorial Ergodicity

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Describing joint research with Jim Propp

Algebra Seminar
University of Connecticut
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Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

Acknowledgments

This talk discusses ongoing work with Jim Propp.

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Overview

For many actions τ on a finite set S of combinatorial objects, and for many natural real-valued statistics ϕ on S, one finds that the ergodic average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\tau^i(x))$$

is **independent** of the starting point $x \in S$.

We say that ϕ is **homomesic** (from Greek: "same middle") with respect to the combinatorial dynamical system (S, τ).

I'll give numerous examples of **homomesies** (homomesic functions), some proved and others conjectural.

Please interrupt with questions!

Introductory examples

1. Rotation of bit-strings;
2. Bulgarian solitaire;
3. Promotion of Near-Standard Young Tableaux; and
4. Suter's symmetries.

Example 1: Rotation of bit-strings

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\phi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.

Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \phi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \phi(s).$$

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$0110 \mapsto 2$	
$0011 \mapsto 0$	

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$1001 \mapsto 2$	$1010 \mapsto 3$
$1100 \mapsto 4$	$0101 \mapsto 1$
$0110 \mapsto 2$	AVG $= \frac{4}{2} = 2$
$0011 \mapsto 0$	
AVG$= \frac{8}{4} = 2$	

More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001

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100001	100010	100100
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011000	101000	001001
001100	010100	
000110	001010	
000011	000101	

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110000	010001	010010
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More rotation

EG: $n = 6, k = 2$ gives us three orbits:

000011	000101	001001
$100001 \mapsto 4$	$100010 \mapsto 5$	$100100 \mapsto 6$
$110000 \mapsto 8$	$010001 \mapsto 3$	$010010 \mapsto 4$
$011000 \mapsto 6$	$101000 \mapsto 7$	$001001 \mapsto 2$
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	

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$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	
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We know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate b (replacing a 01 somewhere in b by 10 or vice versa), or one can write the number of inversions in b as $\sum_{i<j} b_i(1 - b_j)$ and then perform algebraic manipulations.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define $\tau(\lambda)$ as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \underline{\mathbf{3}}\mathbf{311} \to \underline{\mathbf{4}}\mathbf{22} \to \dots$$

and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \underline{\mathbf{3}}\mathbf{32} \to \underline{\mathbf{3}}\mathbf{221} \to \underline{\mathbf{4}}\mathbf{211} \to \underline{\mathbf{4}}\mathbf{31} \to \dots$$

(the new heaps are underlined).

Let $\phi(\lambda)$ be the number of parts of λ.
In the forward orbit of $\lambda = (5, 3)$, the average value of ϕ is

$$(4 + 3)/2 = 7/2;$$

in the forward orbit of $\lambda = (6, 2)$, the average value of ϕ is

$$(3 + 4 + 4 + 3)/4 = 14/4 = 7/2.$$

Proposition

If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of ϕ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.)

So the number-of-parts statistic on partitions of n is homomesic under the Bulgarian solitaire map.

The same is true for the size of the largest part, the size of the second largest part, etc.

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of ϕ for the forward orbit that starts at x is just the average of ϕ over the periodic orbit that x eventually goes into.

So an equivalent way of stating our main definition in this case is, ϕ is homomesic with respect to (S, τ) iff the average of ϕ over each periodic τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$.

In the rest of this talk, we'll restrict attention to maps τ that are invertible on S, so transience is not an issue.

Example 3: Promotion of Near-Standard Young Tableaux

Given a positive integer N, define a Near-Standard Young Tableau (NSYT) with "ceiling" N as a Young tableau T in which entries are distinct integers between 1 and N.

(When N equals the number of cells of T, this is just the definition of a Standard Young Tableau.)

For each $1 \leq i \leq N-1$, let s_i be the action on NSYT's with ceiling N that replaces i (if it occurs in T) by $i+1$, and vice versa, provided that this does not violate the weak-increase condition in the definition of Young tableaux, and let ∂ be the composition of the maps $s_1, s_2, \ldots, s_{N-1}$. This generalizes promotion of SYT's.

A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942; http://arxiv.org/abs/1108.1172):

J. Striker, N. Williams / European Journal of Combinatorics 33 (2012) 1919–1942 1927

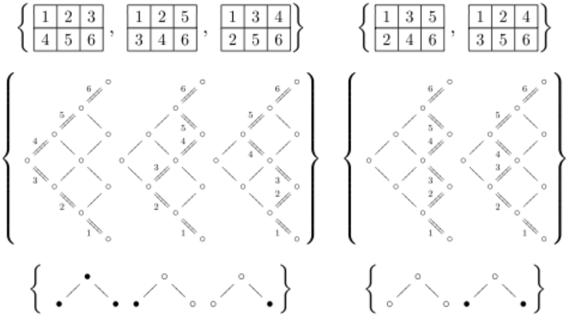

Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

A small example of promotion: centrally symmetric sums

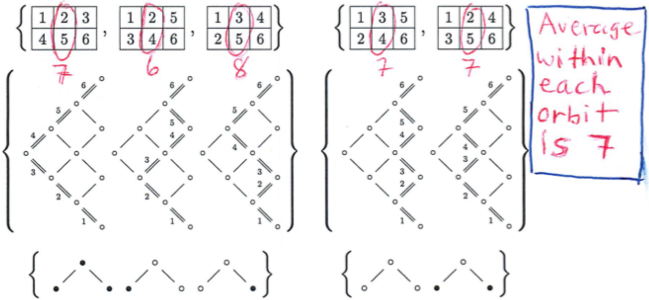

Fig. 5. The two orbits of SYT of shape (3, 3) under promotion, the same two orbits as maximal chains, and the same two orbits as order ideals under Pro.

Promotion of Near-Standard Young Tableaux: homomesies

Conjecture

Let T be a Near-Standard Young Tableau of rectangular shape λ, and ceiling N. If c and c' are opposite cells, i.e., c and c' are related by 180-degree rotation about the center, (note: the case $c = c'$ is permitted when λ is odd-by-odd), and $\phi(T)$ denotes the sum of the numbers in cells c and c', then ϕ is homomesic under ∂ with average value $N + 1$.

A standard method for proving combinatorially that Schur functions are symmetric is to use **Bender-Knuth Involutions**. Given $T \in SSYT(\lambda, \alpha)$ and $i \in \mathbb{P}$, consider all the entries paired within in the same column $\substack{i \\ i+1}$ to be **married**, which the involution ignores. Then in a row with r i's and s $i+1$'s, β_i replaces these with s copies of i and r of $i+1$.

$$\begin{array}{ccccc} & & & & i \\ i & i & \underbrace{i \quad i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1 \quad i+1}_{s=4} & i+1 \\ i+1 & i+1 & & & \end{array}$$

$$\mapsto \begin{array}{ccccc} & & & & i \\ i & i & \underbrace{i \quad i \quad i \quad i}_{r=4} & \underbrace{i+1 \quad i+1}_{s=2} & i+1 \\ i+1 & i+1 & & & \end{array}$$

Bender-Knuth Action on SSYT

Consider the set $SSYT(\lambda, [N])$ [shape λ, entries in $[N]$].
Let $\beta := \beta_{N-1}\beta_{N-2}\cdots\beta_2\beta_1$ be the composition of all possible BK involutions. Set $\phi_i(T) := \#_i(T)$.
Then the triple $(SSYT(\lambda, [N]), \beta, \phi_i)$ is Comb. Erg. for each i.
EG: Let $N = 5$ and $T = \begin{smallmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\ 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 3 & 4 & 4 & 5 \end{smallmatrix}$
Then the content vectors $v = [\phi_1, \phi_2, \ldots, \phi_N]$ that arise as we successively apply β_i's behave as follows, starting from $[3, 4, 6, 5, 2]$:

$[4,3,6,5,2]$	$[6,4,5,2,3]$	$[5,6,2,3,4]$	$[2,5,3,4,6]$	$[3,2,4,6,5]$
$[4,6,3,5,2]$	$[6,5,4,2,3]$	$[5,2,6,3,4]$	$[2,3,5,4,6]$	$[3,4,2,6,5]$
$[4,6,5,3,2]$	$[6,5,2,4,3]$	$[5,2,3,6,4]$	$[2,3,4,5,6]$	$[3,4,6,2,5]$
$[4,6,5,2,3]$	$[6,5,2,3,4]$	$[5,2,3,4,6]$	$[2,3,4,6,5]$	$[3,4,6,5,2]$

Central Symmmetry Homomesy for Semi-Standard Young Tableaux

The Bender-Knuth involutions generalize the maps $s_1, \ldots, s_{N-1}$ for promotion discussed above. Does the central symmetry homomesy conjectured above extend to semi-standard tableaux (with repeated entries)?

Central Symmmetry Homomesy for Semi-Standard Young Tableaux

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Conjecture

If the shape of a skew tableau has central symmetry, and $\phi(T)$ denotes the sum of the numbers in cells c and c' where cells c and c' are opposite one another, then ϕ is homomesic under promotion with average value $N+1$.

This is known when λ has one row or one column.

Example 4: Suter's symmetries

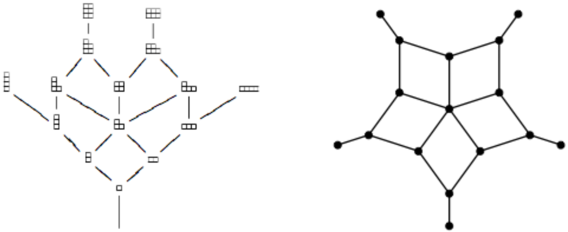
Let $\mathbb{Y}_N$ be the set of number-partitions λ whose maximal hook lengths are strictly less than N (i.e., whose Young diagrams fit inside some rectangle that fits inside the staircase shape $(N-1, N-2, ..., 2, 1)$).

Suter showed that the Hasse diagram of $\mathbb{Y}_N$ has N-fold cyclic symmetry (indeed, N-fold dihedral symmetry) by exhibiting an explicit action of order N.

Suter's action, $N = 5$

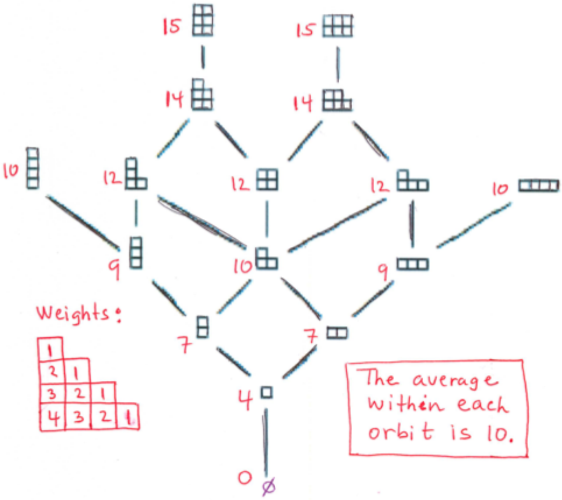
(taken from R. Suter, *Young's lattice and dihedral symmetries revisited: Möbius strips and metric geometry*; http://arxiv.org/abs/1212.4463):

Example (Hasse diagram of $\mathbb{Y}_5$ and its undirected Hasse graph)



This graph has a 5-fold (cyclic) symmetry, and its full symmetry group is a dihedral group of order 10.

Suter's action, $N = 5$: weighted sums



Suter's action: homomesies

Assign weight 1 to the cells at the diagonal boundary of the staircase shape, weight 2 to their neighbors, ..., and weight $N-1$ to the cell at the lower left, and for $\lambda \in \mathbb{Y}_N$ let $\phi(\lambda)$ be the sum of the weights of all the cells in the Young diagram of λ.

Prop. (Einstein, P.): ϕ is homomesic under Suter's map with average value $(n^3-n)/12$.

More refined result: If $i+j=N$ (note: $i=j$ is permitted), and $\phi_{i,j}(\lambda)$ is the sum of the weights of all the cells in λ with weight i plus the sum of the weights of all the cells in λ with weight j, then $\phi_{i,j}$ is homomesic under Suter's map with average ij in all orbits.

An invertible operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

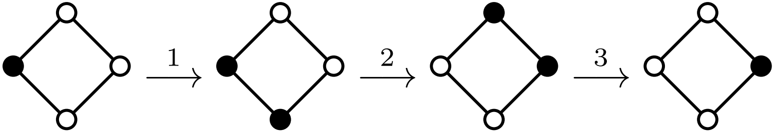
Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the downward-saturation of A.
τ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams.

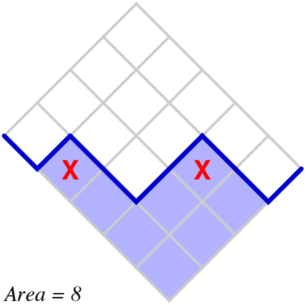
An example

1. Saturate downward
2. Complement
3. Take minimal element(s)

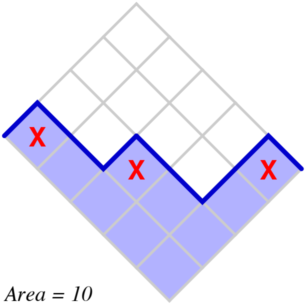


Example in lattice cell form

If we elements of the poset as **squares** below, we would map:



$\longrightarrow$



Panyushev's conjecture

Let Δ be a reduced irreducible root system in $\mathbf{R}^n$.
Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.
Conjecture (Conjecture 2.1(iii) in D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. 30 (2009), 586-594): Let $\mathcal{O}$ be an arbitrary τ-orbit. Then

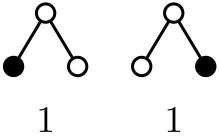
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{n}{2}.$$

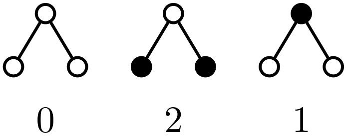
(Two other assertions of this kind, Panyushev's Conjectures 2.3(iii) and 2.4(ii), appear to remain open.)

Panyushev's Conjecture 2.1(iii) (along with much else) was proved by Armstrong, Stump, and Thomas in their article *A uniform bijection between nonnesting and noncrossing partitions*, http://arxiv.org/abs/1101.1277.

Panyushev's conjecture: The A_n case, $n = 2$

Here we have just an orbit of size 2 and an orbit of size 3:

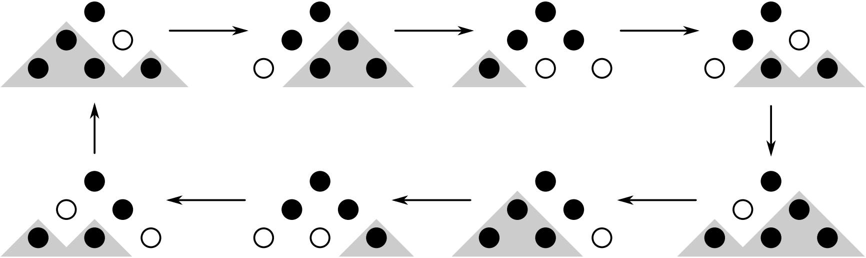




Within each orbit, the average antichain has cardinality $n/2 = 1$.

The case A_3.

Here's an example orbit taken from [AST] for the A_3 root poset:



For A_3 this action has three orbits (sized 2, 4, and 8), and the average cardinality of an antichain is

$$\frac{1}{8}(2+1+1+2+2+1+1+2) = \frac{3}{2}$$

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (where $[k]$ denotes the linear ordering of $\{1, 2, \ldots, k\}$):

Theorem (Propp, R.)

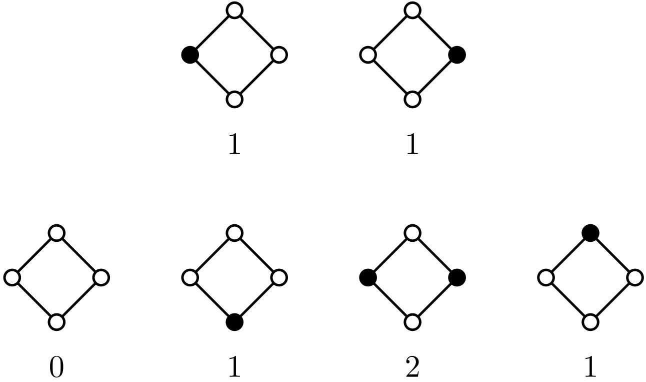
Let $\mathcal{O}$ be an arbitrary τ-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{ab}{a+b}.$$

This is an easy consequence of unpublished work of Hugh Thomas building on earlier work of Richard Stanley: see the last paragraph of section 2 of R. Stanley, *Promotion and evacuation*, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i2r9 .

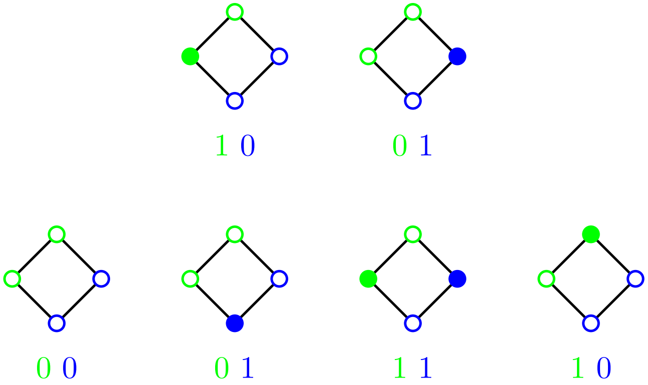
Antichains in $[a] \times [b]$: the case $a = b = 2$

Here we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has $1/2$ a green element and $1/2$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#(A) = \sum_i f_i(A)$.
Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#(A) = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i, j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under τ, even though the indicator functions $1_{i,j}$ aren't.

Antichains in $[a] \times [b]$: centrally symmetric homomesies

Theorem (Propp, R.)

In any orbit, the number of A that contain (i,j) equals the number of A that contain the opposite element $(i', j') = (a+1-i, b+1-j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under τ, with average value 0 in each orbit.

Linearity

Useful triviality: every linear combination of homomesies is itself homomesic.

E.g., consider the adjusted major index statistic defined by $\mathrm{amaj}(A) = \sum_{(i,j) \in A} (i-j)$.

P. and Roby proved that amaj is homomesic under τ
by writing it as a linear combination of the functions $1_{i,j} - 1_{i',j'}$.
Haddadan gave a simpler proof,
writing amaj as a linear combination of the functions f_i and g_j.

Question: Are there other homomesic combinations of the indicator functions $1_{i,j}$ (with $(i,j) \in [a] \times [b]$),
linearly independent of the functions f_i, g_j, and $1_{i,j} - 1_{i',j'}$?

From antichains to order ideals

Given a poset P and an antichain A in P, let $\mathcal{I}(A)$ be the order ideal $I = \{y \in P \ : y \leq x \text{ for some } x \in A\}$ associated with A, so that for any order ideal I in P, $\mathcal{I}^{-1}(I)$ is the antichain of maximal elements of I.

As usual, we let $J(P)$ denote the set of (order) ideals of P.

We define $\overline{\tau} : J(P) \to J(P)$ by $\overline{\tau}(I) = \mathcal{I}(\tau(\mathcal{I}^{-1}(I)))$. That is, $\overline{\tau}(I)$ is the downward saturation of the set of minimal elements of the complement of I.

For $(i,j) \in P$ and $I \in J(P)$, let $\overline{1}_{i,j}(I)$ be 1 or 0 according to whether or not I contains (i,j).

One action, two vector spaces

$\overline{\tau}$ is "the same" τ in the sense that the standard bijection from $\mathcal{A}(P)$ to $J(P)$ (downward saturation) makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\tau} & \mathcal{A}(P) \\ \downarrow & & \downarrow \\ J(P) & \xrightarrow{\overline{\tau}} & J(P) \end{array}$$

However, the bijection from $\mathcal{A}(P)$ to $J(P)$ does **not** carry the vector space generated by the functions $1_{i,j}$ to the vector space generated by the functions $\overline{1}_{i,j}$ in a linear way.

So the homomesy situation for $\overline{\tau} : J(P) \to J(P)$ could be (and, as we'll see, is) different from the homomesy situation for $\tau : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{P})$.

Ideals in $[a] \times [b]$: cardinality is homomesic

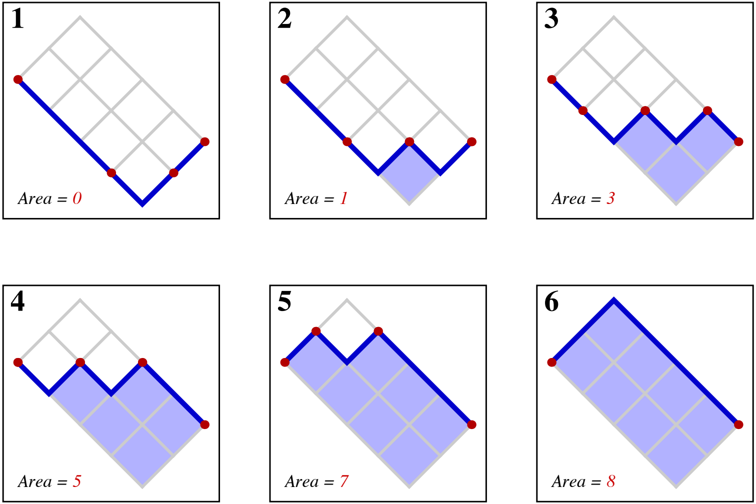
Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\overline{\tau}$-orbit in $J([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

Rowmotion on $[4] \times [2]$ A

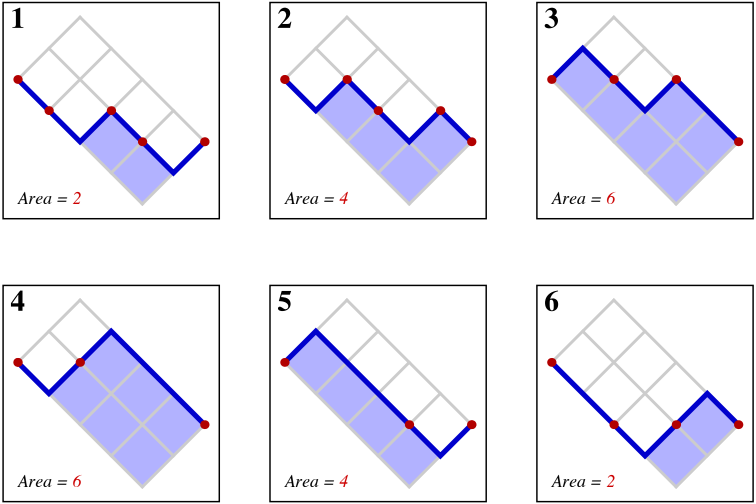
Rowmotion on $[4] \times [2]$ A



$$(0+1+3+5+7+8) / 6 = 4$$

Rowmotion on $[4] \times [2]$ B

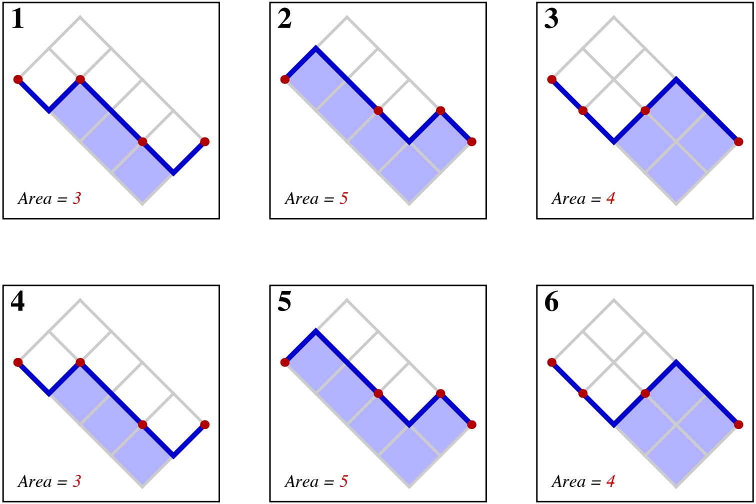
Rowmotion on $[4] \times [2]$ B



$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ C

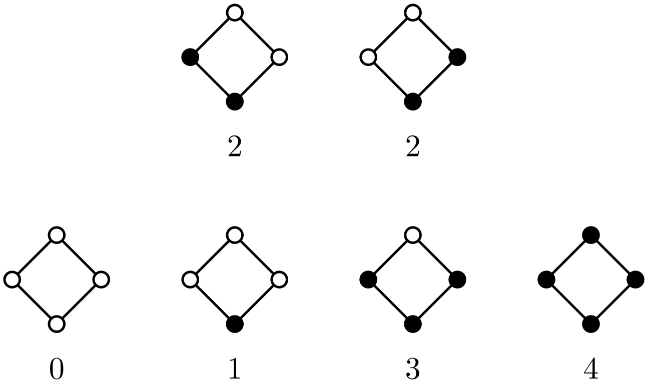
Rowmotion on $[4] \times [2]$ C



$$(3+5+4+3+5+4)\ /\ 6 = 4$$

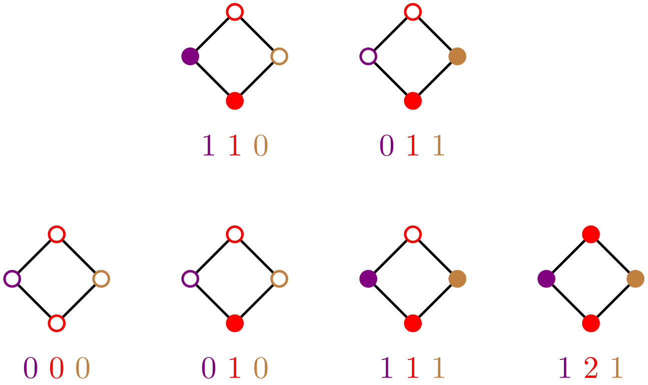
Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has
$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#(I) = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every $\overline{\tau}$-orbit $\mathcal{O}$ in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Ideals in $[a] \times [b]$: centrally symmetric homomesies

Recall that for $(i, j) \in [a] \times [b]$, and I an ideal in $[a] \times [b]$, $\overline{1}_{i,j}(I)$ is 1 or 0 according to whether or not I contains (i, j).

Write $(i', j') = (a + 1 - i, b + 1 - j)$, the point opposite (i, j) in the poset.

Theorem (Propp, R.)

$\overline{1}_{i,j} + \overline{1}_{i',j'}$ is homomesic under $\overline{\tau}$.

Question: In addition to the functions h_k and $\overline{1}_{i,j} + \overline{1}_{i',j'}$, are there other homomesic functions in the span of the functions $\overline{1}_{i,j}$?

The two vector spaces, compared

In the space associated with antichains:

  fiber-cardinalities and

  centrally symmetric **differences**

are homomesic.

In the space associated with order ideals:

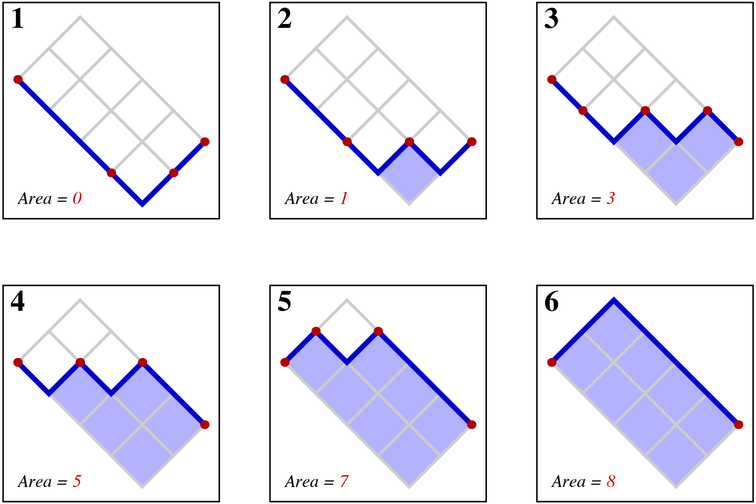
  file-cardinalities and

  centrally symmetric **sums**

are homomesic.

Rowmotion on $[4] \times [2]$ A

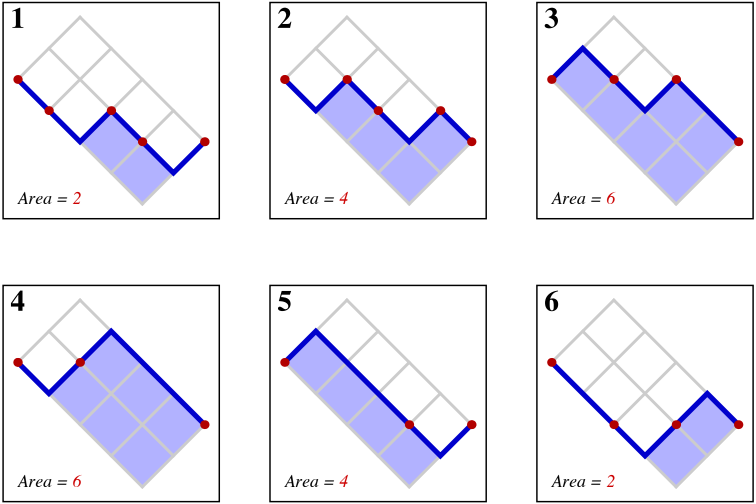
Rowmotion on $[4] \times [2]$ A



$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ B

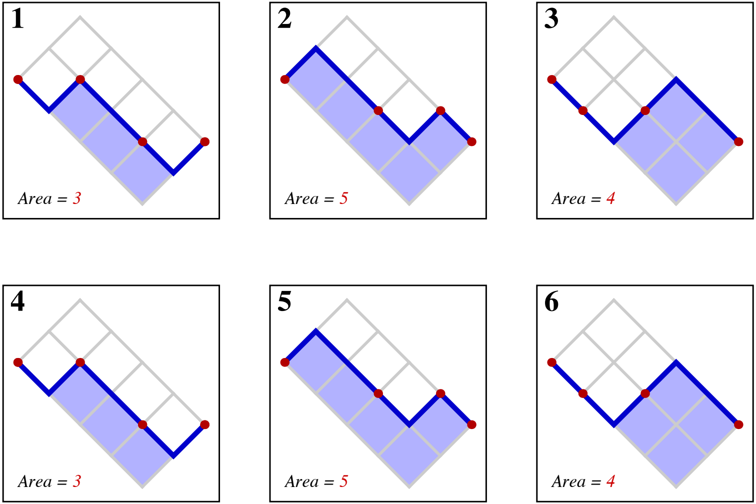
Rowmotion on $[4] \times [2]$ B



$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$ C

Rowmotion on $[4] \times [2]$ C



$$(3+5+4+3+5+4)\ /\ 6 = 4$$

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The last slide of this talk

We've found lots of examples of conjectural homomesies in all branches of combinatorics, starting at the level of the twelve-fold way and progressing through spanning trees, parking functions, abelian sandpiles (aka chip-firing), rotor-routing, etc.

For more information, see:

http://jamespropp.org/ucbcomb12.pdf
http://jamespropp.org/mathfest12a.pdf
http://www.math.uconn.edu/∼troby/combErg2012kizugawa.pdf
http://jamespropp.org/mitcomb13a.pdf
http://jamespropp.org/propp-roby.pdf