Combinatorial Ergodicity

Tom Roby (University of Connecticut)

Describing joint research with Jim Propp

23 July 2012

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

Acknowledgments

This talk describes ongoing work with Jim Propp.

Special thanks to Mike LaCroix for making animations and pictures.

Thanks also to Drew Armstrong, Karen Edwards, Bob Edwards, Svante Linusson, Richard Stanley, and Ben Young for useful conversations.

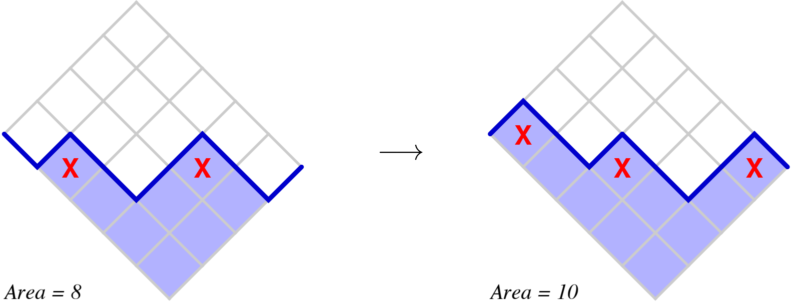
Overview

- The rowmotion operator on a poset;
- Panyushev's Conjecture for rowmotion on root posets;
- Definition of combinatorial ergodicity;
- Rowmotion and Promotion on products of chains; and
- Further directions.

The rowmotion operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the order ideal (downward-saturation) of A.
For example, viewing elements of the poset as squares below, we would map:



Rowmotion on posets 2

τ is invertible since it is a composition of three invertible operations:

antichains $\leftrightarrow$ order ideals (down-sets) $\leftrightarrow$ up-sets $\leftrightarrow$ antichains

This also shows that the same map, call it $\overline{\tau}$, can be thought of as operating on the set of order ideals in $J(P)$ as well as $\mathcal{A}(P)$.

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, and Striker and Williams (who coined the term "rowmotion").

Panyushev's conjecture

Most of the work on rowmotion has focussed on its orbit structure, with the notable exception of some conjectures of Panyushev, e.g.,

Conjecture (Panyushev, Conj. 2.1(iii))

Let Δ be a reduced irreducible root system in $\mathbf{R}^n$.
Choose a system of positive roots and make it a poset by decreeing that y covers x iff $y - x$ is a simple root.
Let $\mathcal{O}$ be an arbitrary τ-orbit. Then

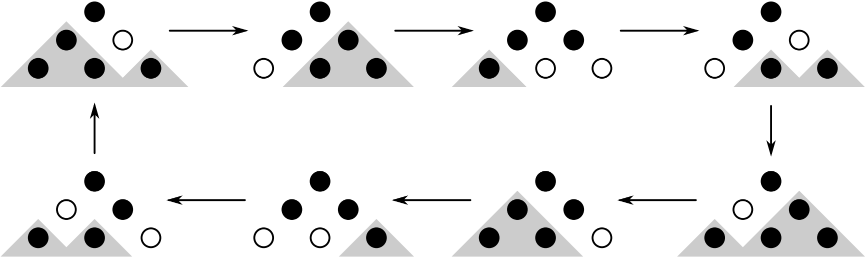
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{n}{2}.$$

In other words, *the average size of an antichain over any rowmotion orbit is independent of the orbit*.

This was proved by Armstrong, Stump, and Thomas in their 2011 article "A uniform bijection between nonnesting and noncrossing partitions".

An example of AST's result

Here's an example orbit taken from [AST] for the A_3 root poset:



For A_3 this action has three orbits (sized 2, 4, and 8), and the average cardinality of an antichain is

$$\frac{1}{8}(2+1+1+2+2+1+1+2) = \frac{3}{2}$$

Definition of Combinatorial Ergodicity

- Let $\xi : S \to S$ be a map (action) on a finite set of combinatorial objects S.
- Under this action, S naturally decomposes as a (disjoint) union of finitely many distinct ξ-orbits: $S = \bigcup \mathcal{O}_k$.
- Let $\phi : S \to \mathbf{F}$ (usually $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$) be some natural statistic on S.

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We say that the triple (S, ξ, ϕ) exhibits **combinatorial ergodicity** if the average of ϕ over each ξ-orbit $\mathcal{O}$ in S is the same as the average of ϕ over the whole set S:

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A simple example

Set $S = \binom{[n]}{k}$, thought of as length n binary strings with k 1's. $\tau := C_R : S \to S$ by $b = b_1 b_2 \cdots b_n \mapsto b_n b_1 b_2 \cdots b_{n-1}$ (cyclic shift), and $\phi(b) = \#\text{inversions}(b) = \#\{i < j : b_i > b_j\}$.
Then over any orbit $\mathcal{O}$ we have:

$$\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \phi(s) = \frac{k(n-k)}{2} = \frac{1}{\#S} \sum_{s \in S} \phi(s).$$

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EG: $n = 4, k = 2$ gives us two orbits:

$$\begin{array}{cc} 0011 & 0101 \\ \hline \end{array}$$

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$0110 \mapsto 2$	AVG $= \frac{4}{2} = 2$
$0011 \mapsto 0$	
AVG$= \frac{8}{4} = 2$	

A larger example

EG: $n = 6, k = 2$ gives us three orbits:

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$001100 \mapsto 4$	$010100 \mapsto 5$	
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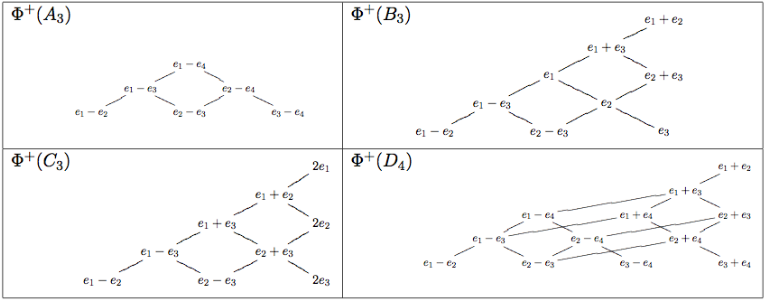
$001100 \mapsto 4$	$010100 \mapsto 5$	
$000110 \mapsto 2$	$001010 \mapsto 3$	
$000011 \mapsto 0$	$000101 \mapsto 1$	
AVG$= \frac{24}{6} = 4$	AVG$= \frac{24}{6} = 4$	AVG$= \frac{12}{3} = 4$

Summary so far

We've defined (S, ξ, ϕ) to be **combinatorial ergodic** if the average of ϕ over every ξ-orbit $\mathcal{O}$ in S is the same: $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \phi(s) = \frac{1}{\#S} \sum_{s \in S} \phi(s)$. The two examples we've seen:

- (binary n-strings with k 1's, $C_R =$ cyclic shift, $\#$inversions)
- $(\mathcal{A}(P), \tau =$ rowmotion, $\#A)$ where P is a $(+)$-root poset.



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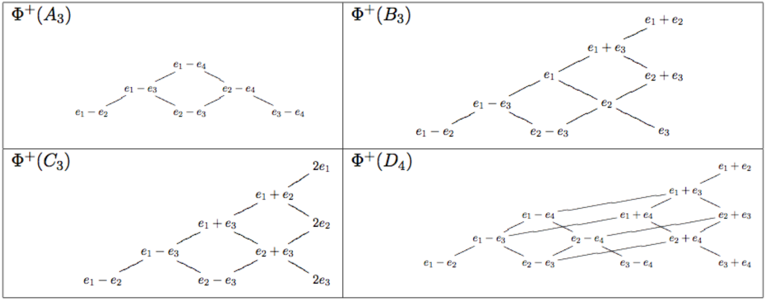


Q: In what other situations does this phenomenon arise?

Products of two chains

Consider the poset $[a] \times [b]$ (where $[n]$ denotes the linear ordering of $\{1, 2, \ldots, n\}$).

Proposition

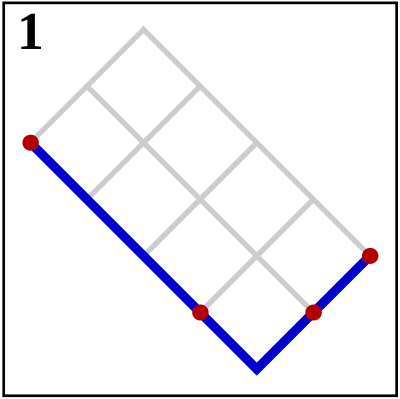
Let $\mathcal{O}$ be an arbitrary τ-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{ab}{a+b}.$$

In other words, $(\mathcal{A}([a] \times [b]), \tau, \#A)$ is combinatorially ergodic.

But even more is true.

Rowmotion on $[4] \times [2]$ A



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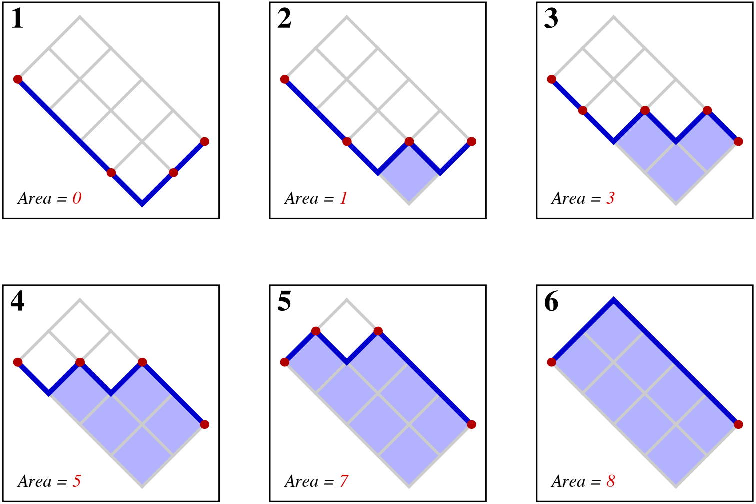
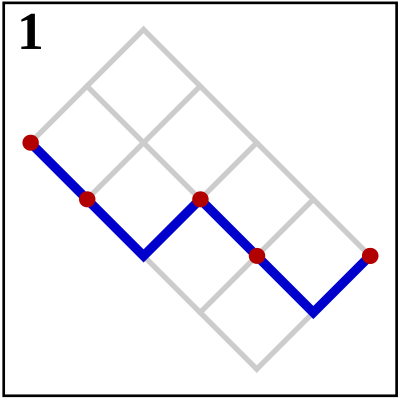


$$(0+1+3+5+7+8)\ /\ 6 = 4$$



Rowmotion

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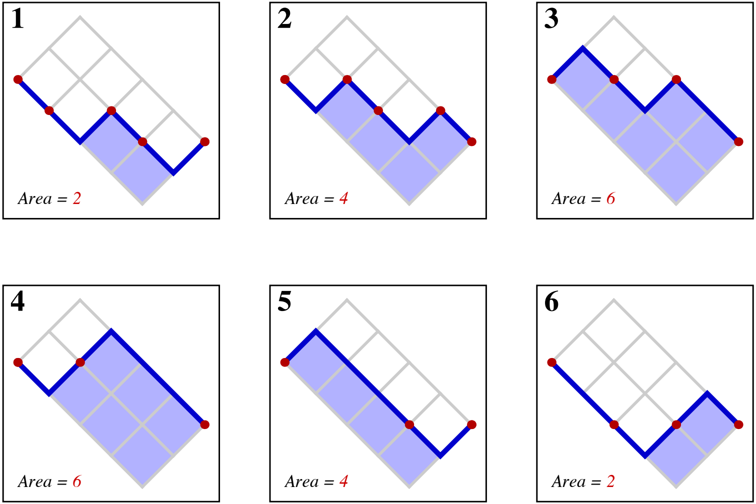

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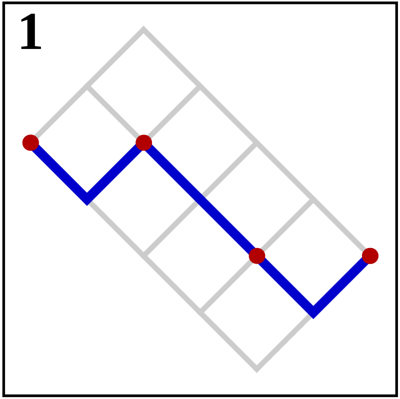
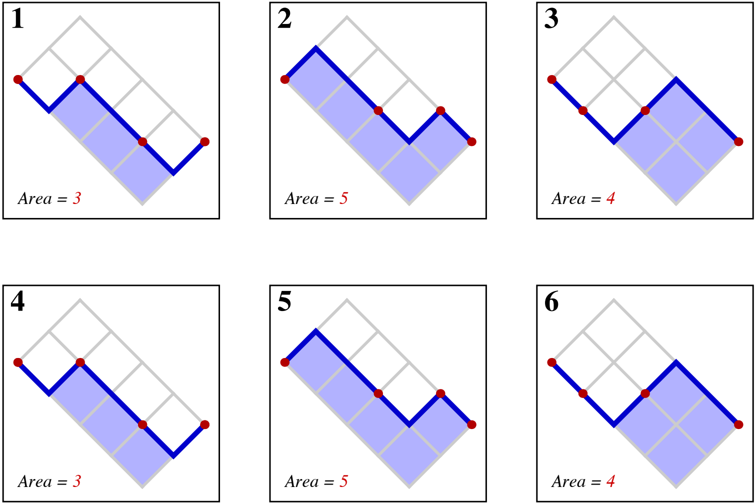


Rowmotion

Rowmotion on $[4] \times [2]$ C



$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Combinatorial Ergodicity of order ideal sizes under rowmotion

Theorem (Propp-R.)

Let $\mathcal{O}$ be an arbitrary $\overline{\tau}$-orbit in $J([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

I.e., $(J([a] \times [b]), \overline{\tau}, \#I)$ is combinatorially ergodic.

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What else can we vary here? How about the operation!

Toggling

In their 1995 article "Orbits of antichains revisited", Cameron and Fon-der-Flaass give an alternative description of τ in terms of toggle-operations applied to order ideals.

Given $I \in J(P)$ and $x \in P$, let $\tau_x(I) = I \triangle \{x\}$ provided that $I \triangle \{x\}$ is an order ideal of P; otherwise, let $\tau_x(I) = I$.

We call the involution τ_x "toggling at x".

The involutions τ_x and τ_y commute *unless* x covers y or y covers x.

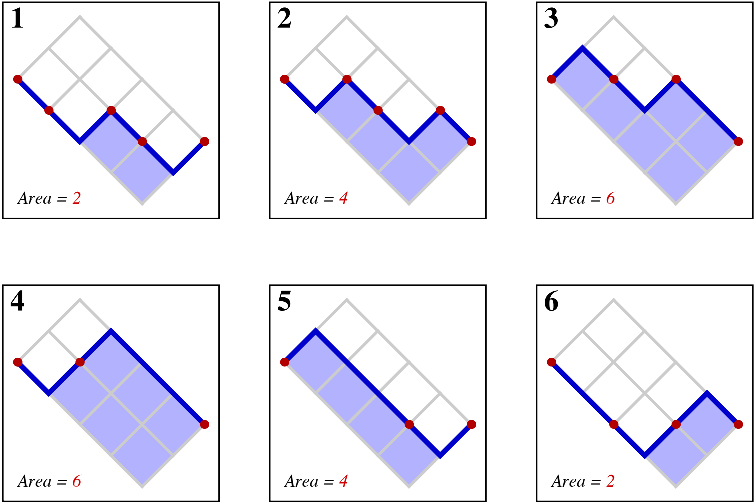
Toggling from top to bottom

Theorem (Cameron and Fon-der-Flaass): Let $x_1, x_2, \ldots, x_n$ be any order-preserving enumeration (linear extension) of the elements of the poset P. Then the action on $J(P)$ given by the composition $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ coincides with the action of $\overline{\tau}$.

In the particular case $P = [a] \times [b]$, we can enumerate P rank-by-rank; that is, we can list the (i,j)'s in order by $i + j$.

Note that all the involutions coming from a given rank of P commute with one another, since no two of them are in a covering relation. We compute τ from top to bottom, τ^{-1} from bottom-to-top.

Rowmotion on $[4] \times [2]$ B



$$(2+4+6+6+4+2)\ /\ 6 = 4$$

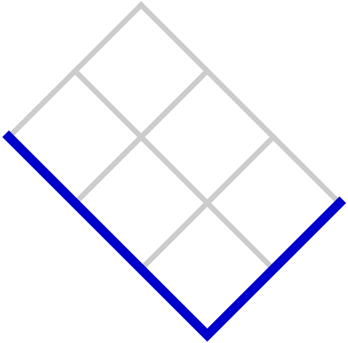
Toggling from side to side

Define a **file** of $P = [a] \times [b]$ as the set of all $(i, j) \in P$ with $i - j$ fixed. Note that all toggles in a given file commute with one another, since no two of them are in a covering relation.

Theorem (Striker-Williams)

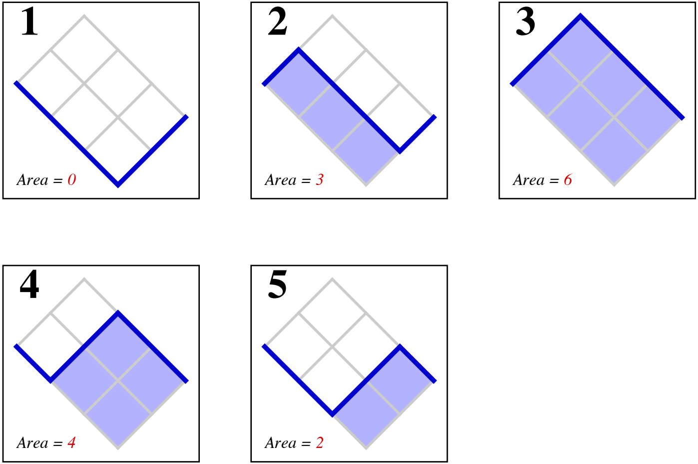
Let $x_1, x_2, \ldots, x_n$ be any enumeration of the elements of the poset $[a] \times [b]$ arranged in order of decreasing $i - j$. Then the action on $J(P)$ given by $\partial := \tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ viewed as acting on the paths (or binary strings representing them) is just a leftward cyclic shift.

Striker and Williams call this well-defined composition **promotion** since it is closely related to Schützenberger's notion of promotion on linear extensions of posets. This definition and their results apply more generally to the class of *rc-posets*, whose elements fit neatly into rows & columns.



Promotion

Promotion on $[3] \times [2]$ A



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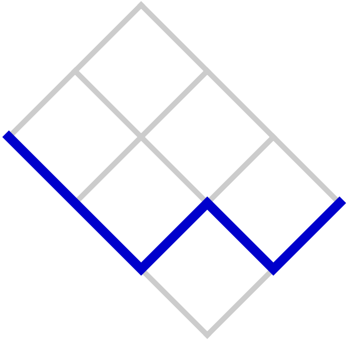

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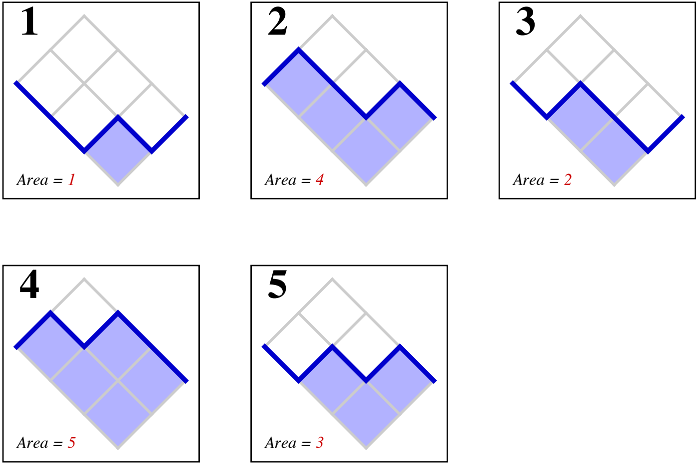

$$(1+4+2+5+3)\ /\ 5 = 3$$

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Combinatorial Ergodicity for promotion

Theorem (Propp-R.)

Let $\mathcal{O}$ be an arbitrary orbit in $J([a] \times [b])$ under the action of promotion ∂. Then

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

(I.e., $(J([a] \times [b]), \partial, \#I)$ satisfies combinatorial ergodicity.)

Proof that $(J([a] \times [b]), \partial, \#I)$ is comb. ergodic

Proof: (Sketch) Each order ideal I is defined by a lattice path, which can be represented as a binary string w of a 0's and b 1's (where 0=Down (SE), 1=Up (NE)). Then $\#I = \mathrm{inv}(w)$, so we want to show that

$$\frac{1}{\#\mathcal{O}} \sum_{w \in \mathcal{O}} \mathrm{inv}(w) = \frac{ab}{2}.$$

There are several short proofs of this, e.g., one can write the number of inversions in w as $\sum_{i<j} w_i(1 - w_j)$ and then perform algebraic manipulations.

The story thus far

We've looked at 2 different actions (rowmotion and promotion) and 2 different notions of cardinality (statistics) for the objects they act on (antichains and order ideals).

In 3 of the 2×2 cases, we've seen that actions and statistics satisfy **combinatorial ergodicity**, i.e., *the average cardinality along an orbit doesn't depend on the orbit.* How about the 4th?

Comb. Erg.?	#I	#A
Rowmotion	Y	Y
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Our framework allows for a good deal of flexibility: one can vary

- the statistic (ϕ),
- the action or map (τ),
- the poset P,
- or look beyond the setting of posets.

Future Directions and Questions

We've just begun to explore the territory here, so there's lots left to do, including:

1. Look for other interesting cases of combinatorially ergodicity;
2. Try to construct frameworks that make it easier to find and prove examples (e.g., building up more complicated instances from simpler ones);
3. Clarify the relationship with the Cyclic Sieving Phenomenon of Reiner, Stanton, & White. Comb. ergodicity often arise in situations where there is also a CSP.

We expect to put a paper on the arXiv later this summer that will lay out the basic framework, including the examples from this talk and a number of others.

Why "Ergodicity"?

This may seem like a misnomer: A measurable action is ergodic iff the only invariant sets have measure zero or full measure, so in the combinatorial setting, an action is ergodic iff it is transitive.

However, the coinage makes more sense if you think back to Boltzmann's original notion of the equality between space-averages and long-term time-averages.

Note that if x is a periodic point for the invertible map τ (and there is no other kind of point if τ is a permutation!) we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(\tau^k(x)) = \frac{1}{\#(\mathcal{O})} \sum_{y\in\mathcal{O}} \phi(y)$$

where $\mathcal{O}$ is the orbit of x.

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The final slide of this talk

I'm happy to talk about this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html