

Paths to Understanding Rowmotion on a Product of Two Chains

Tom Roby (UConn)

Describing joint research with Gregg Musiker¹ (UMN)

Combinatorics Seminar
Brandeis University, Waltham MA

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Outline & Acknowledgements

- 1 Classical Rowmotion
- 2 Piecewise-linear (PL) and Birational Rowmotion
- 3 Formula in terms of Lattice Paths
- 4 Sketch of Proof
- 5 Applications (Periodicity and Homomesy)

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<http://www.math.uconn.edu/~troby/research.html>

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Please feel free to interrupt with questions or comments.

ご質問コメント等があれば、ご遠慮なくおっしゃって下さい。

- The combinatorial rowmotion map has liftings (via a decomposition into involutions called *toggles*) to the piecewise-linear (order polytope) and then birational settings. Proving results at the birational level implies them at the other levels.
- For rectangular posets $P = [0, r] \times [0, s]$, we give a formula in terms of NILPs that allows us to compute ρ_B^k , the k th iteration of birational rowmotion.
- The key lemma is a Plücker-like relation satisfied by certain polynomials we define, proven by a colorful combinatorial bijection on pairs of NILPs (along the lines of Fulmek-Kleber).
- Using our formula, we obtain more direct proofs of the periodicity and “antipodal-reciprocity” of this system, as well as the first proof of “homomesy along files”.

Classical rowmotion is the rowmotion studied by Striker-Williams (2012), who coined the term. It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or “nonnesting partitions”, with relations to Lie theory).
- Propp-Roby (2015), as one of several actions that displays the homomesy phenomenon on the product of two chains.

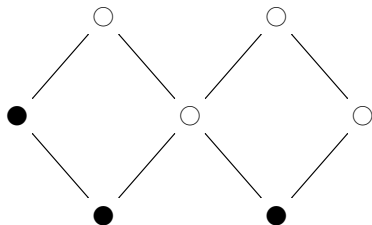
Classical rowmotion

Let P be a finite poset. **Classical rowmotion** is the map $r : J(P) \rightarrow J(P)$

sending every **order ideal** S to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let S be the following order ideal

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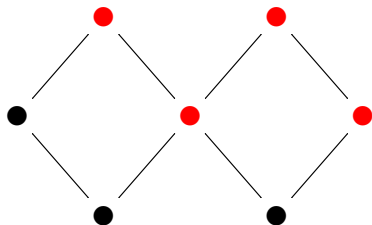
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Mark the complement in **red**.

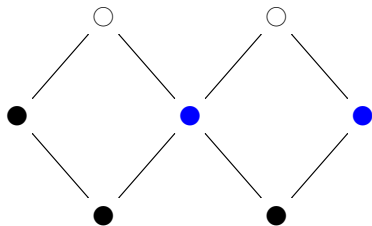


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Mark M (the minimal elements of the complement) in **blue**.



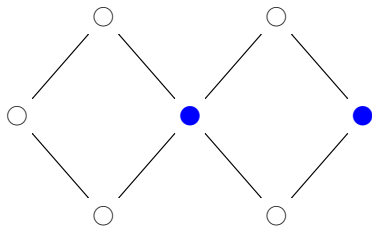
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Remove the old order ideal:

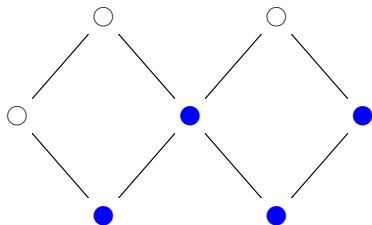


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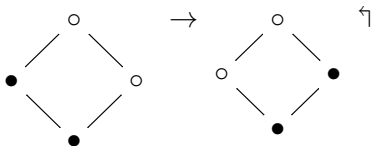
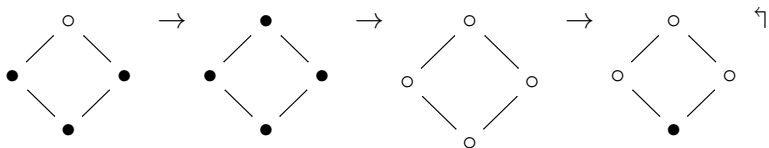
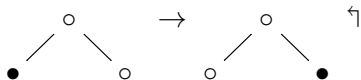
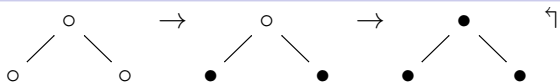
sending every **order ideal** S to a new order ideal $r(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let S be the following order ideal

$r(S)$ is the order ideal generated by M ("everything below M):



Examples of Orbits of this Dynamic on Order Ideals.



Definition ([PR15])

Given an (invertible) action τ on a finite set of objects S , call a statistic $f : S \rightarrow \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ -orbit \mathcal{O} is the same constant c for all \mathcal{O} , i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of \mathcal{O} .

(Call f c -mesic for short.) **Greek for "same-middle"**

Theorem ([PR15])

For the action of rowmotion on order ideals $J(P)$ of rectangular posets $P = [p] \times [q]$, the cardinality statistic is homomesic (with average $pq/2$).

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- Birational rowmotion can be related to Y -systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PR15, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.
- Galashin & Pylyavskyy have a very interesting extension of birational rowmotion to directed graphs and “ R -systems.” [GaPy17].

Classical rowmotion: Periodicity

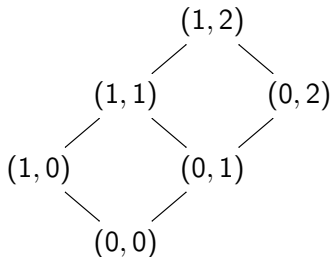
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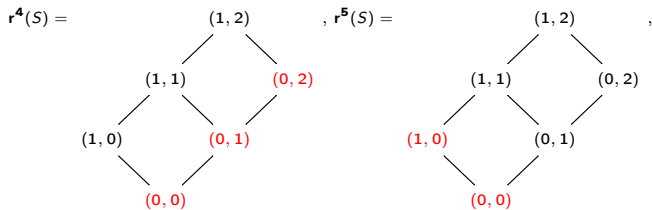
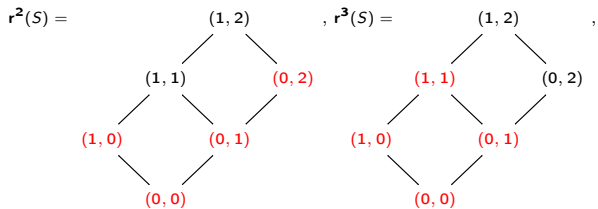
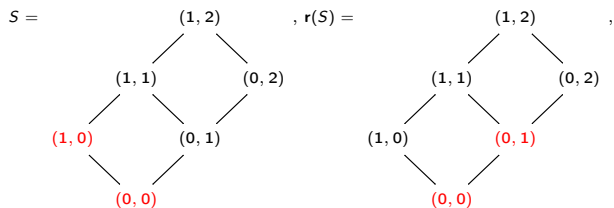
However, **for some types of P** , the order can be explicitly computed or bounded from above. See Striker-Williams [StWi11] (and the **very recent** Thomas-Williams [TW17]) for an exposition of known results.

- If P is a $p \times q$ -rectangle:



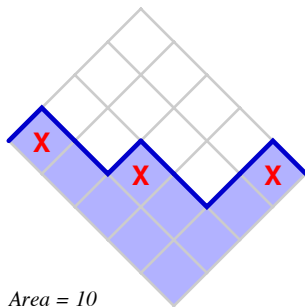
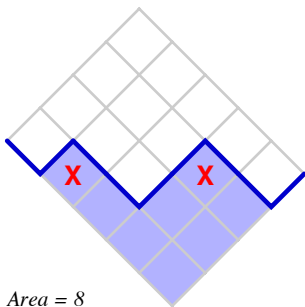
(shown here for $p = 2$ and $q = 3$), then $\text{ord}(\mathbf{r}) = p + q$.

Classical rowmotion: Periodicity (Example)



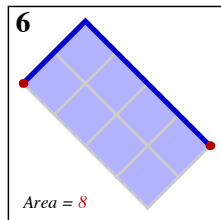
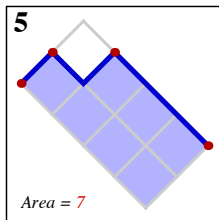
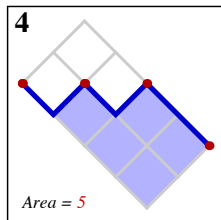
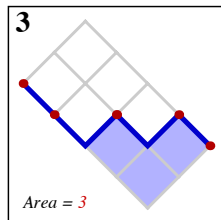
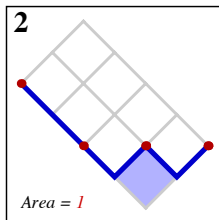
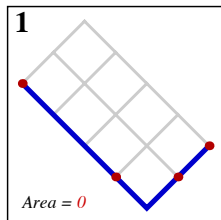
Example in lattice cell form

Viewing the elements of the poset as **squares** below, we would map:



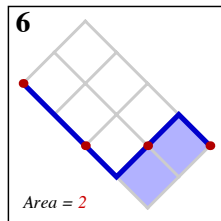
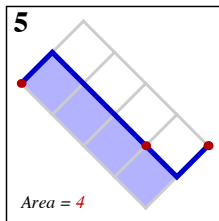
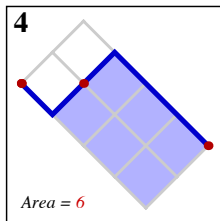
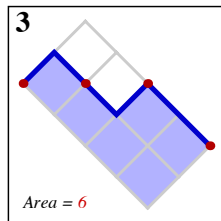
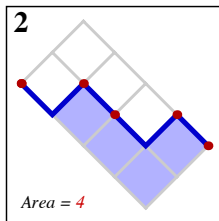
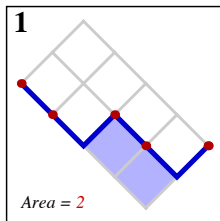
Rowmotion on $[4] \times [2]$ A

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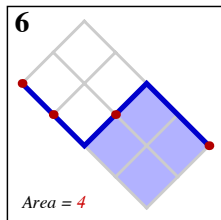
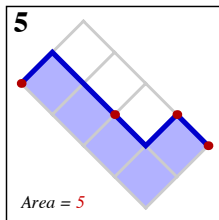
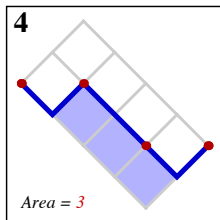
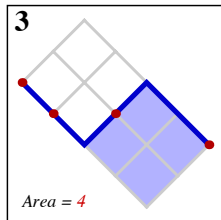
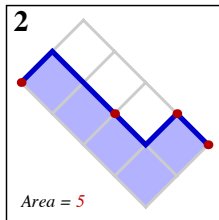
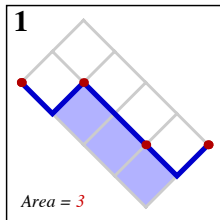


$$(0+1+3+5+7+8) / 6 = 4$$

Rowmotion on $[4] \times [2]$ B

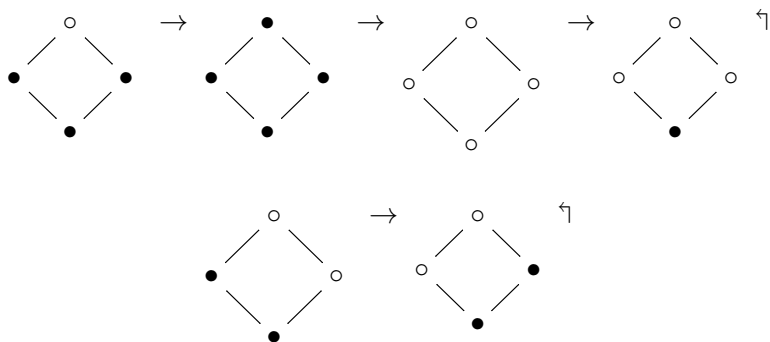


$$(2+4+6+6+4+2) / 6 = 4$$



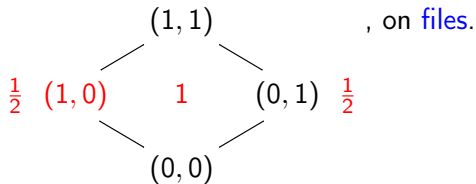
$$(3+5+4+3+5+4) / 6 = 4$$

Classical rowmotion: Antipodal and File Homomieses



The **average value** along **antipodal (N-S, E-W) pairs** is 1 for both **orbits**,

and is also **constant**, as



There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
 - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
 - S otherwise.

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- (“Try to add or remove v from S , as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave S fixed.”)
- More formally, if P is a poset and $v \in P$, then the v -toggle is the map $\mathbf{t}_v : J(P) \rightarrow J(P)$ which takes every order ideal S to:
 - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
 - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S ;
 - S otherwise.
- Note that $\mathbf{t}_v^2 = \text{id}$.

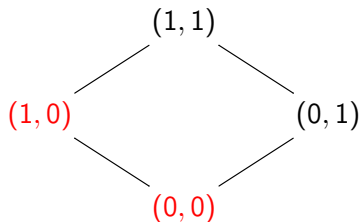
Classical rowmotion via toggling (“rowmotion in slowmotion”)

- Let (v_1, v_2, \dots, v_n) be a **linear extension** of P ; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass [CaFl95] showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \dots \circ \mathbf{t}_{v_n}.$$

Example:

Start with this order ideal S :



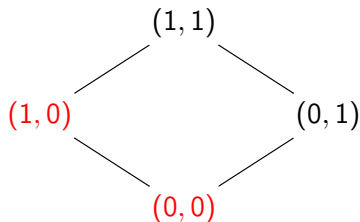
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Example:

First apply $\mathbf{t}_{(1,1)}$, which changes nothing:



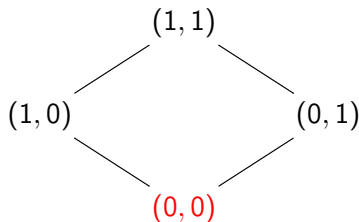
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Example:

Then apply $\mathbf{t}_{(1,0)}$, which removes $(1,0)$ from the order ideal:



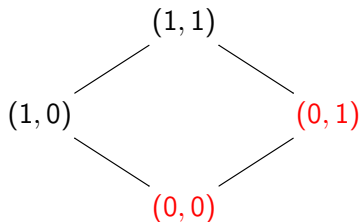
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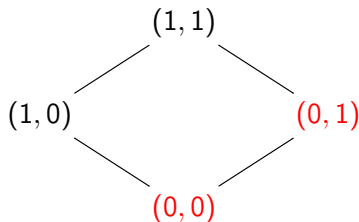
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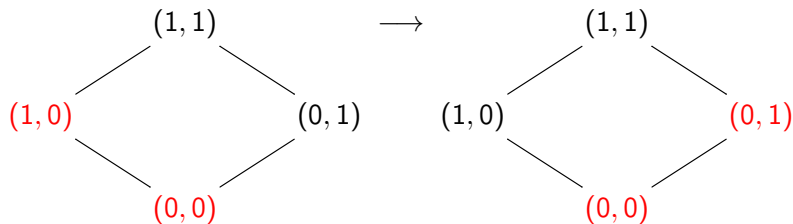
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Example:

So this is $S \rightarrow \mathbf{r}(S)$:



Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley [Stan86]) is the set of functions $f : P \rightarrow [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$. (Compare with $J(P) = \{f : P \rightarrow \{0, 1\} : f \text{ is monotone}\}$)

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \rightarrow \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w .

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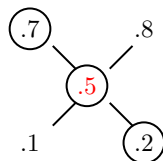
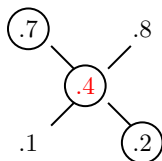
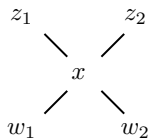
Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

Example of flipping at a node

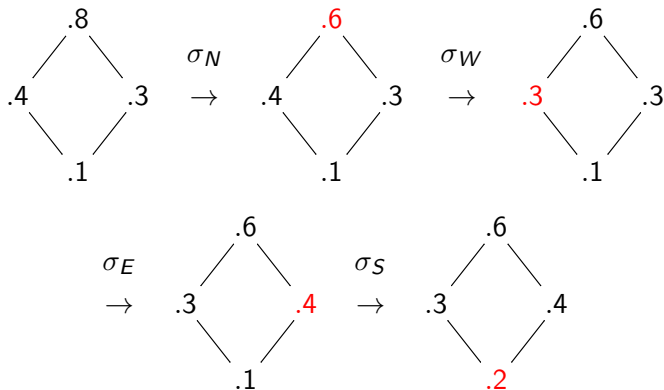


$$\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1, 1)$, $W = (1, 0)$, $E = (0, 1)$, and $S = (0, 0)$ in order to get $\rho_{PL}(f)$.)

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \rightarrow [0, 1]$ at a point $x \in P$ with f' , where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can “detropicalize” this flip map and apply it to an assignment $f : P \rightarrow \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

- Let P be a finite poset. We define \widehat{P} to be the poset obtained by adjoining two new elements $\widehat{0}$ and $\widehat{1}$ to P and forcing
 - $\widehat{0}$ to be less than every other element, and
 - $\widehat{1}$ to be greater than every other element.
- Let \mathbb{K} be a field.
- A \mathbb{K} -labelling of P will mean a function $\widehat{P} \rightarrow \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of \widehat{P} .
- For any $v \in P$, define the **birational v -toggle** as the rational map

$$T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}} \text{ defined by } (T_v f)(w) = \frac{\sum_{\widehat{P} \ni u < \cdot v} f(u)}{f(v) \sum_{\widehat{P} \ni u > \cdot v} \frac{1}{f(u)}} \text{ for } w = v.$$

(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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- We define **birational rowmotion** as the rational map

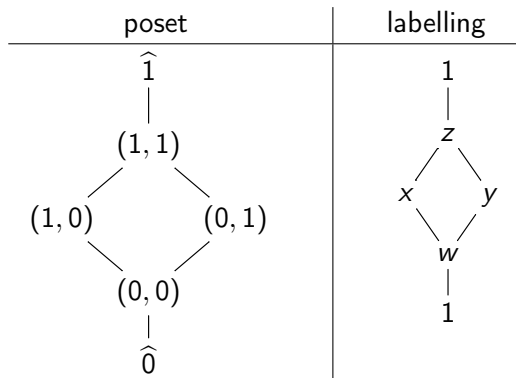
$$\rho_B := T_{v_1} \circ T_{v_2} \circ \dots \circ T_{v_n} : \mathbb{K}^{\hat{P}} \dashrightarrow \mathbb{K}^{\hat{P}},$$

where (v_1, v_2, \dots, v_n) is a linear extension of P .

- This is indeed independent of the linear extension, because
 - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
 - we can get from any linear extension to any other by switching incomparable adjacent elements.
- This is originally due to Einstein and Propp [EiPr13, EiPr14], following the lead of Kirillov-Berenstein [KiBe95].

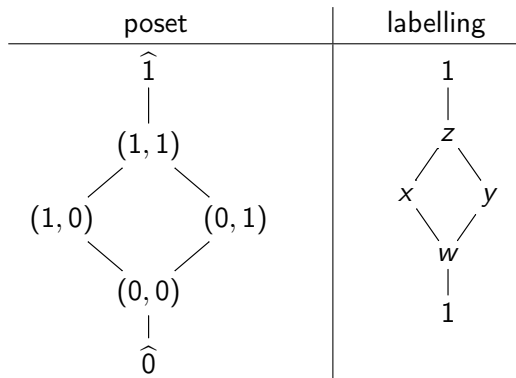
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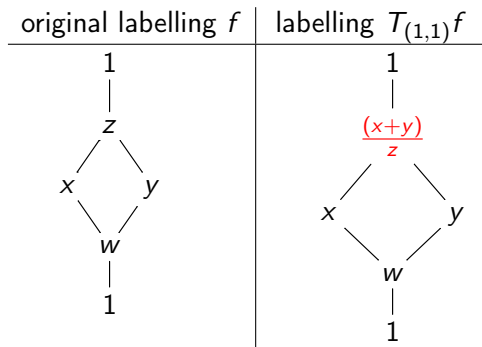


We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$
 using the linear extension
 $((1, 1), (1, 0), (0, 1), (0, 0))$.

That is, toggle in the order “top, left, right, bottom”.

Example:

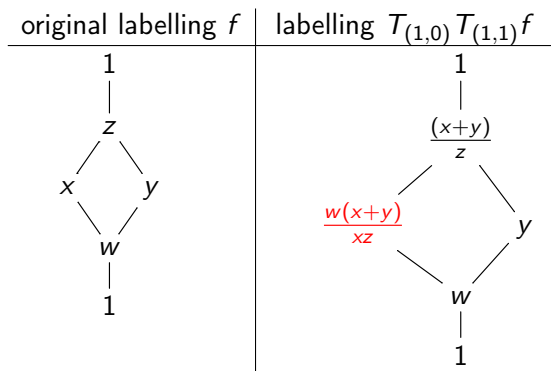
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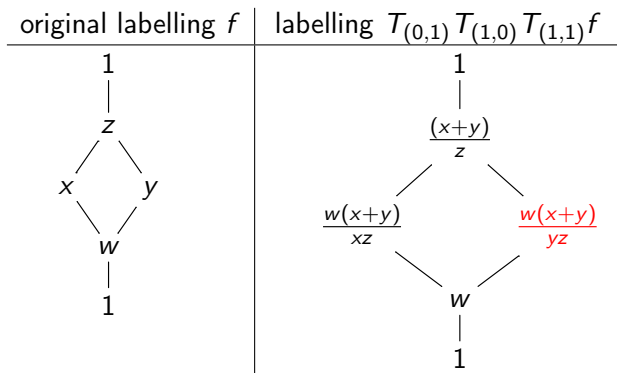
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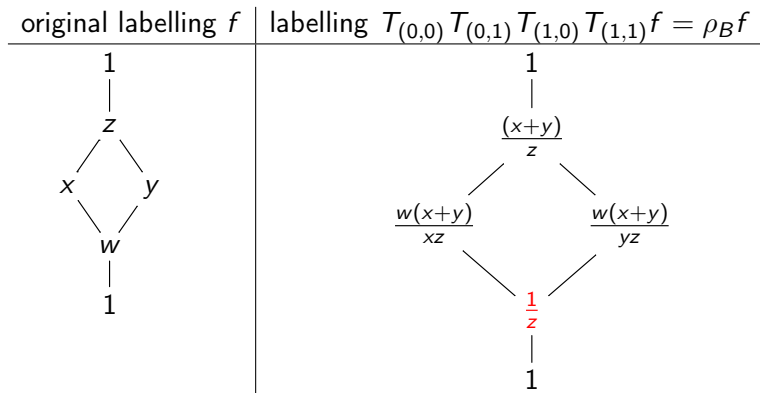
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Example: Iterating this procedure we get

$$\rho_B f = \begin{array}{ccc} & \frac{(x+y)}{z} & \\ & / \quad \backslash & \\ \frac{(x+y)w}{xz} & & \frac{(x+y)w}{yz} \\ & \backslash \quad / & \\ & \frac{1}{z} & \end{array},$$

$$\rho_B^3 f = \begin{array}{ccc} & \frac{1}{w} & \\ & / \quad \backslash & \\ \frac{yz}{(x+y)w} & & \frac{xz}{(x+y)w} \\ & \backslash \quad / & \\ & \frac{xy}{(x+y)w} & \end{array},$$

$$\rho_B^2 f = \begin{array}{ccc} & \frac{(x+y)w}{xy} & \\ & / \quad \backslash & \\ \frac{1}{y} & & \frac{1}{x} \\ & \backslash \quad / & \\ & \frac{z}{x+y} & \end{array},$$

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Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also “antipodal reciprocity”.

The poset $[0, 1] \times [0, 1]$ has **three files**, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all **iterates of birational rowmotion** in a given **file**:

$$\prod_{k=1}^4 \rho_B^k(f)(1, 0) = \frac{(x+y)w}{xz} \frac{1}{y} \frac{yz}{(x+y)w} (x) = 1,$$

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Birational homomorphism on files

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Each of these **products equalling one** is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the **birational level**.

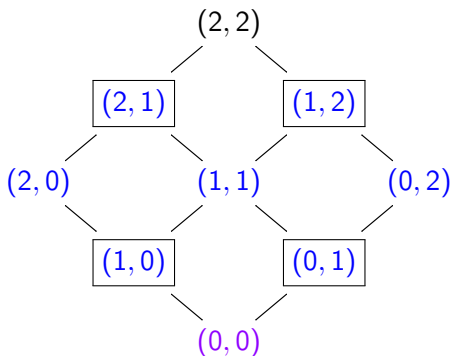
Birational Rowmotion on the Rectangular Poset

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i, j)$ for $(i, j) \in [0, r] \times [0, s]$ and $k \in [0, r + s + 1]$.

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1) Let $\mathcal{V}_{(m,n)} := \{(u, v) : (u, v) \geq (m, n)\}$ be the *principal order filter at (m, n)* , $\square_{(m,n)}^k$ be the *rank-selected subposet*, of elements in $\mathcal{V}_{(m,n)}$ whose rank (within $\mathcal{V}_{(m,n)}$) is at least $k - 1$ and whose corank is at most $k - 1$.



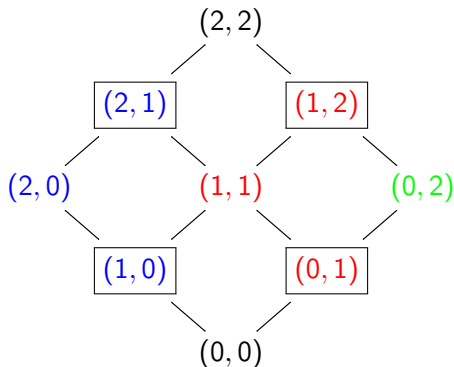
2) Let s_1, s_2, \dots, s_k be the k minimal elements and let t_1, t_2, \dots, t_k be the k maximal elements of $\square_{(m,n)}^k$. (For $k \leq \min\{r - m, s - n\} + 1$.)

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Let $A_{ij} := \frac{\sum_{z \ll (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i,j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in \widehat{P}).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_{\mathbf{k}}(\mathbf{m}, \mathbf{n})$ in terms of the A_{ij} 's as follows.

We define a **lattice path of length k** within $P = [0, r] \times [0, s]$ to be a sequence v_1, v_2, \dots, v_k of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1, 0)$ or $(0, 1)$ for each $i \in [k]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex.



Birational Rowmotion on the Rectangular Poset

3) Let $S_k(m, n)$ be the set of all NILPs (non-intersecting lattice paths) in $\square_{(m,n)}^k$, from $\{s_1, s_2, \dots, s_k\}$ to $\{t_1, t_2, \dots, t_k\}$. Let $\mathcal{L} = \{L_1, L_2, \dots, L_k\} \in S_k^k(m, n)$ denote a **lsingle** such k -collection of NILPs.

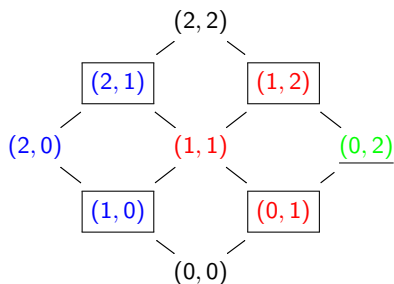
4) Define $\varphi_k(m, n) := \sum_{\mathcal{L} \in S_k^k(m, n)} \prod_{\substack{(i,j) \in \square_{(m,n)}^k \\ (i,j) \notin L_1 \cup L_2 \cup \dots \cup L_k}} A_{ij}$.

Theorem* (approx.):

$$\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

EG: $\rho_B^2(1, 1) = \frac{\varphi_1(0, 0)}{\varphi_2(0, 0)}$.

$$= \frac{\text{sum of 6 quartic terms in } A_{ij}}{A_{20} + A_{11} + A_{02}}$$



(*) Caveats explained and general statement given in the next few slides.

Main Theorem (M-Roby 2018)

Fix $k \in [0, r + s + 1]$, and let $\rho_B^{k+1}(i, j)$ denote the rational function associated to the poset element (i, j) after $(k + 1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0, r] \times [0, s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k - i]_+ + [k - j]_+$.

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(a1) When $M = 0$, i.e., $(i - k, j - k)$ still lies in the poset $[0, r] \times [0, s]$:

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(a2) When $0 < M \leq k$:

$$\rho_B^{k+1}(i, j) = \mu^{([k-j]_+, [k-i]_+)} \left(\frac{\varphi_{k-M}(i - k + M, j - k + M)}{\varphi_{k-M+1}(i - k + M, j - k + M)} \right)$$

where $\mu^{(a,b)}$ is the operator that takes a rational function in $\{A_{(u,v)}\}$ and simply shifts each index in each factor of each term:

$$A_{(u,v)} \mapsto A_{(u-a, v-b)}.$$

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Fix $k \in [0, r + s + 1]$ and set $M = [k - i]_+ + [k - j]_+$. After $(k + 1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0, r] \times [0, s]$ we get:

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(b) When $M \geq k$: $\rho_B^{k+1}(i, j) = 1/\rho_B^{k-i-j}(r - i, s - j)$, which is well-defined by part (a).

Remark: We prove that our formulae in (a) and (b) agree when $M = k$, allowing us to give a new proof of periodicity:

$\rho_B^{r+s+2+d} = \rho_B^d$; thus we get a formula for **all** iterations of the birational rowmotion map.

Corollary

For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$.

Corollary ([GrRo15, Thm. 30])

The birational rowmotion map ρ_B on the product of two chains $P = [0, r] \times [0, s]$ is periodic, with period $r + s + 2$.

Corollary ([GrRo15, Thm. 32])

The birational rowmotion map ρ_B on the product of two chains $P = [0, r] \times [0, s]$ satisfies the following reciprocity:

$$\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i, s-j}}.$$

Theorem

Given a file F in $[0, r] \times [0, s]$,
$$\prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i,j) = 1.$$

The poset $[0, 1] \times [0, 1]$ has **three files**, $\{(1, 0)\}$, $\{(0, 0), (1, 1)\}$, and $\{(0, 1)\}$.

Multiplying over all **iterates of birational rowmotion** in a given **file**:

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$$\prod_{k=1}^4 \rho_B^k(f)(0,1) = \frac{(x+y)w}{yz} \frac{1}{x} \frac{xz}{(x+y)w} (y) = 1.$$

Example

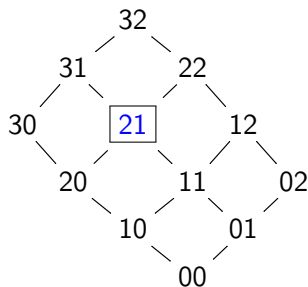
We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

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Recall that in the case where shifting $(i, j) \mapsto (i - k, j - k)$ (straight down by $2k$ ranks) still gives a point in P , we get a simpler formula.

Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$.



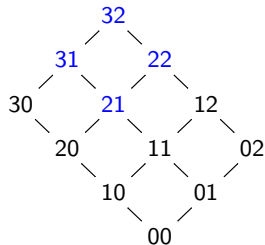
Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

Recall that in the case where shifting $(i, j) \mapsto (i - k, j - k)$ (straight down by $2k$ ranks) still gives a point in P , we get a simpler formula.

Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$.

When $k = 0$, $M = 0$ and we get



$$\rho_B^1(2, 1) = \frac{\varphi_0(2, 1)}{\varphi_1(2, 1)} = \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22} + A_{31}}.$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

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Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$.

When $k = 1$, we still have $M = 0$, and $\rho_B^2(2, 1) = \frac{\varphi_1(1, 0)}{\varphi_2(1, 0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}$$

For the numerator, $s_1 = (1, 0)$, $t_1 = (3, 2)$, and there are six lattice paths from s_1 to t_1 , each of which covers 5 elements and leaves 4 uncovered.

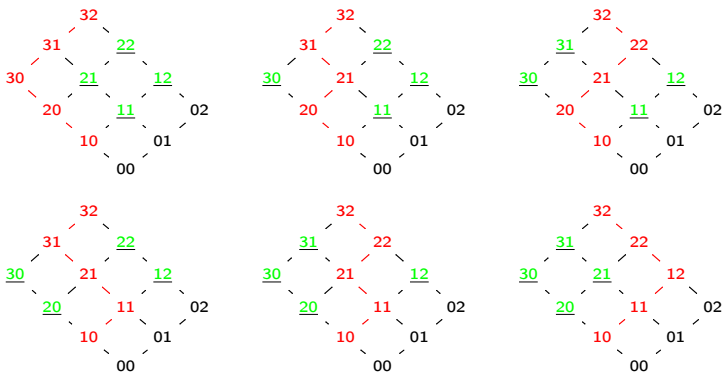
For the denominator, $s_1 = (2, 0)$, $s_2 = (1, 1)$, $t_1 = (3, 1)$, and $t_2 = (2, 2)$, and each pair of lattice paths leaves exactly one element uncovered.

Example

When $k = 1$, we still have $M = 0$, and $\rho_B^2(2, 1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}$$

For the numerator, $s_1 = (1, 0)$, $t_1 = (3, 2)$, and there are six lattice paths from s_1 to t_1 , each of which covers 5 elements and leaves 4 uncovered.

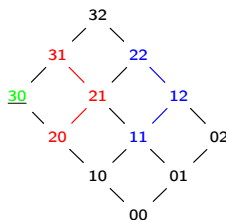
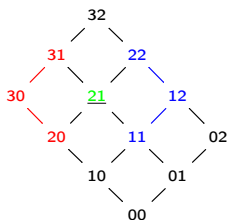
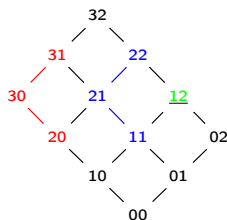


Example

When $k = 1$, we still have $M = 0$, and $\rho_B^2(2, 1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

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For the denominator, $s_1 = (2, 0)$, $s_2 = (1, 1)$, $t_1 = (3, 1)$, and $t_2 = (2, 2)$, and each pair of lattice paths leaves exactly one element uncovered.



Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where shifting $(i, j) \mapsto (i - k, j - k)$ (straight down by $2k$ ranks) gives a point outside of P , we must also apply a μ -translation.

When $k = 2$, we get $M = [2 - 2]_+ + [2 - 1]_+ = 1 \leq 2 = k$. So by part (a) of the main theorem we have

$$\rho_B^3(2, 1) = \mu^{(1,0)} \left[\frac{\varphi_1(1, 0)}{\varphi_2(1, 0)} \right] = (\text{just shifting indices in the } k = 1 \text{ formula})$$

$$\frac{A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}}{A_{02} + A_{11} + A_{20}}.$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where shifting $(i, j) \mapsto (i - k, j - k)$ (straight down by $2k$ ranks) gives a point outside of P , we must also apply a μ -translation.

When $k = 3$, we get $M = [3 - 2]_+ + [3 - 1]_+ = 3 = k$. Therefore,

$$\rho_B^4(2, 1) = \mu^{(2,1)} \left[\frac{\varphi_0(2, 1)}{\varphi_1(2, 1)} \right] = \mu^{(2,1)} \left[\frac{A_{21}A_{22}A_{31}A_{32}}{A_{22} + A_{31}} \right] = \frac{A_{00}A_{01}A_{10}A_{11}}{A_{01} + A_{10}}.$$

In this situation, we can also use part (b) of the main theorem to get

$$\rho_B^4(2, 1) = 1/\rho_B^{3-2-1}(3-2, 2-1) = 1/\rho_B^0(1, 1) = \frac{1}{x_{11}}.$$

The equality between these two expressions is easily checked.

Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where μ -translation would lead to negative subscripts for the φ 's, i.e., $M > k$, part (a) of the Theorem does not apply.

When $k = 4$, we get $M = [4 - 2]_+ + [4 - 1]_+ = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^5(2, 1) = 1/\rho_B^{4-2-1}(3-2, 2-1) = 1/\rho_B^1(1, 1) = \frac{\varphi_1(1, 1)}{\varphi_0(1, 1)} = \frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}}.$$

Each term in the numerator is associated with one of the three lattice paths from $(1, 1)$ to $(3, 2)$ in P , while the denominator is just the product of all A -variables in the principal order filter $\vee(1, 1)$.

Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where μ -translation would lead to negative subscripts for the φ 's, i.e., $M > k$, part (a) of the Theorem does not apply.

When $k = 5$, we get $M = [5 - 2]_+ + [5 - 1]_+ = 7 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^6(2, 1) = 1/\rho_B^{5-2-1}(3-2, 2-1) = 1/\rho_B^2(1, 1) = \frac{\varphi_2(0, 0)}{\varphi_1(0, 0)} =$$

$$(A_{02}A_{12} + A_{02}A_{21} + A_{11}A_{21} + A_{30}A_{02} + A_{30}A_{11} + A_{30}A_{20}) /$$

$$(A_{01}A_{11}A_{02}A_{21}A_{12}A_{22} + A_{01}A_{11}A_{02}A_{30}A_{12}A_{22} + A_{01}A_{11}A_{02}A_{30}A_{12}A_{31} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{01}A_{20}A_{02}A_{30}A_{12}A_{31} + A_{01}A_{20}A_{02}A_{30}A_{21}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{12}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{21}A_{31} + A_{10}A_{20}A_{02}A_{30}A_{12}A_{22} + A_{10}A_{20}A_{11}A_{30}A_{21}A_{31})$$

Example

We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

In the case where μ -translation would lead to negative subscripts for the φ 's, i.e., $M > k$, part (a) of the Theorem does not apply.

When $k = 6$, we get $M = [6 - 2]_+ + [6 - 1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\begin{aligned}\rho_B^7(2, 1) &= 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1, 1) = \mu^{(1,1)} \left[\frac{\varphi_1(1, 1)}{\varphi_0(1, 1)} \right] \\ &= \mu^{(1,1)} \left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}} \right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = \dots\end{aligned}$$

Example

When $k = 6$, we get $M = [6 - 2]_+ + [6 - 1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\begin{aligned}\rho_B^7(2, 1) &= 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1, 1) = \mu^{(1,1)} \left[\frac{\varphi_1(1, 1)}{\varphi_0(1, 1)} \right] \\ &= \mu^{(1,1)} \left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}} \right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = \dots\end{aligned}$$

The lattice paths involved here are the same as for the $k = 4$ computation.

We can deduce this by $A_{00} = 1/x_{00}$, $A_{10} = x_{00}/x_{10}$, $A_{01} = x_{00}/x_{01}$, $A_{11} = (x_{10} + x_{01})/x_{11}$, $A_{20} = x_{10}/x_{20}$, and $A_{21} = (x_{20} + x_{11})/x_{21}$.

Periodicity also kicks in: $\rho_B^7(2, 1) = \rho_B^0(2, 1) = x_{21}$ using $(r + s + 2) = 7$.

Sketch of Proof

By definition of birational rowmotion,

$$\rho_B^{k+1}(i, j) = \frac{\left(\rho_B^k(i, j-1) + \rho_B^k(i-1, j) \right) \cdot \left(\rho_B^{k+1}(i+1, j) \parallel \rho_B^{k+1}(i, j+1) \right)}{\rho_B^k(i, j)}$$

where

$$A \parallel B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

Sketch of Proof

By definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \parallel \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

$$A \parallel B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

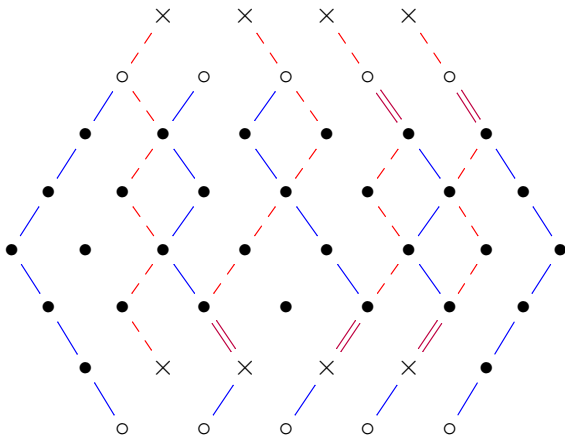
By induction on k , and the fact that we apply birational rowmotion from top to bottom, we can apply algebraic manipulations to reduce our result to proving the following **Plücker-like identity**:

$$\begin{aligned} \varphi_k(i-k, j-k) \varphi_{k-1}(i-k+1, j-k+1) = \\ \varphi_k(i-k, j-k+1) \varphi_{k-1}(i-k+1, j-k) \\ + \varphi_k(i-k+1, j-k) \varphi_{k-1}(i-k, j-k+1). \end{aligned}$$

It is sufficient to verify the following Plücker-like identity

$$\begin{aligned} \varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1) = \\ \varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k) \\ + \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1). \end{aligned}$$

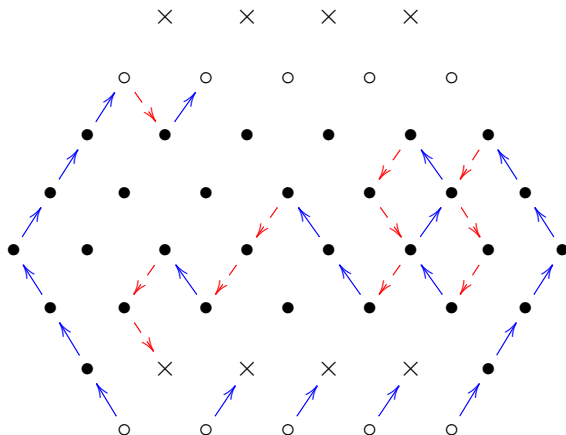
Example (k=5):



Sketch of Proof

We build **bounce paths** and **twigs** (paths of length one from \circ to \times) starting from the bottom row of \circ 's.

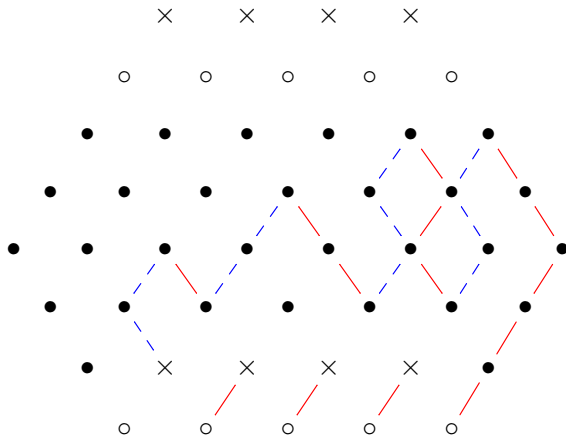
Example (k=5):



Sketch of Proof

We then reverse the colors along the $(k - 2)$ twigs and the one bounce path from \circ to \times (rather than \circ to \circ).

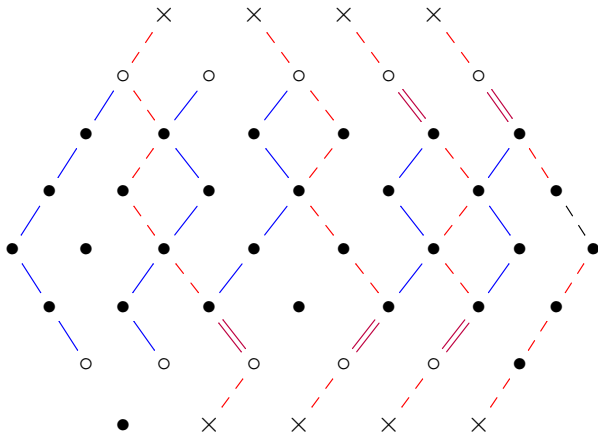
Example ($k=5$):



Sketch of Proof

Swap in the new colors and shift the \circ 's and \times 's in the bottom two rows.

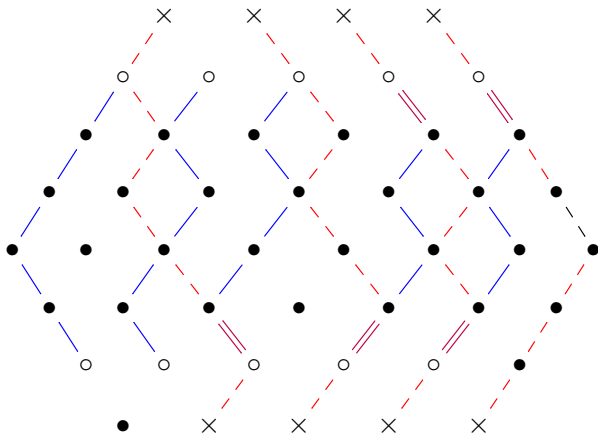
Example ($k=5$):



Sketch of Proof

$$\begin{aligned} \varphi_k(i-k, j-k) \varphi_{k-1}(i-k+1, j-k+1) = \\ \varphi_k(i-k, j-k+1) \varphi_{k-1}(i-k+1, j-k) \\ + \varphi_k(i-k+1, j-k) \varphi_{k-1}(i-k, j-k+1). \end{aligned}$$

Example (k=5):



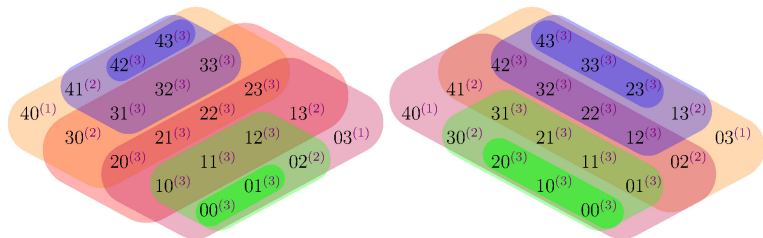
Theorem

Given a file F in $[0, r] \times [0, s]$,
$$\prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i,j) = 1.$$

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Sketch of Proof: Double-counting argument, followed by color-coded cancellations and several entries immediately equal to 1, as in ensuing table.



Further Application: Birational File Homomesy

Let $(r, s) = (4, 3)$, $d = 2$, and consider the file $F = \{(4, 2), (3, 1), (2, 0)\}$. The following table displays the values of $\rho_B^k(i, j)$ for $0 \leq k \leq 8$, $(i, j) \in F$.

	(4, 2)	(3, 1)	(2, 0)
$k = 0$	$\frac{\varphi_0(4, 2)}{\varphi_1(4, 2) = 1}$	$\frac{\varphi_0(3, 1)}{\varphi_1(3, 1)}$	$\frac{\varphi_0(2, 0)}{\varphi_1(2, 0)}$
$k = 1$	$\frac{\varphi_1(3, 1)}{\varphi_2(3, 1) = 1}$	$\frac{\varphi_1(2, 0)}{\varphi_2(2, 0)}$	$\mu^{(1,0)} \left[\frac{\varphi_0(2, 0)}{\varphi_1(2, 0)} \right]$
$k = 2$	$\frac{\varphi_2(2, 0)}{\varphi_3(2, 0) = 1}$	$\mu^{(1,0)} \left[\frac{\varphi_1(2, 0)}{\varphi_2(2, 0)} \right]$	$\mu^{(2,0)} \left[\frac{\varphi_0(2, 0)}{\varphi_1(2, 0)} \right] = \frac{1}{x_{23}}$
$k = 3$	$\mu^{(1,0)} \left[\frac{\varphi_2(2, 0)}{\varphi_3(2, 0) = 1} \right]$	$\mu^{(2,0)} \left[\frac{\varphi_1(2, 0)}{\varphi_2(2, 0)} \right]$	$\frac{\varphi_1(2, 3) = 1}{\varphi_0(2, 3)}$
$k = 4$	$\mu^{(2,0)} \left[\frac{\varphi_2(2, 0)}{\varphi_3(2, 0) = 1} \right]$	$\mu^{(3,1)} \left[\frac{\varphi_0(3, 1)}{\varphi_1(3, 1)} \right] = \frac{1}{x_{12}}$	$\frac{\varphi_2(1, 2) = 1}{\varphi_1(1, 2)}$
$k = 5$	$\mu^{(3,1)} \left[\frac{\varphi_1(3, 1)}{\varphi_2(3, 1) = 1} \right]$	$\frac{\varphi_1(1, 2)}{\varphi_0(1, 2)}$	$\frac{\varphi_3(0, 1) = 1}{\varphi_2(0, 1)}$
$k = 6$	$\mu^{(4,2)} \left[\frac{\varphi_0(4, 2)}{\varphi_1(4, 2) = 1} \right] = \frac{1}{x_{01}}$	$\frac{\varphi_2(0, 1)}{\varphi_1(0, 1)}$	$\mu^{(0,1)} \left[\frac{\varphi_3(0, 1) = 1}{\varphi_2(0, 1)} \right]$
$k = 7$	$\frac{\varphi_1(0, 1)}{\varphi_0(0, 1)}$	$\mu^{(0,1)} \left[\frac{\varphi_2(0, 1)}{\varphi_1(0, 1)} \right]$	$\mu^{(1,2)} \left[\frac{\varphi_2(1, 2) = 1}{\varphi_1(1, 2)} \right]$
$k = 8$	$\mu^{(0,1)} \left[\frac{\varphi_1(0, 1)}{\varphi_0(0, 1)} \right] = x_{42}$	$\mu^{(1,2)} \left[\frac{\varphi_1(1, 2)}{\varphi_0(1, 2)} \right] = x_{31}$	$\mu^{(2,3)} \left[\frac{\varphi_1(2, 3) = 1}{\varphi_0(2, 3)} \right] = x_{20}$

The final slide of this talk (before the references)

- Combinatorial rowmotion is an well-studied action, that can be written as of involutions called “toggles”.
- Generalizing toggling to the piecewise-linear setting, then lifting to the birational setting, gives birational rowmotion. Results (about periodicity or homomesy) at this level imply results at the PL and combinatorial level.
- We give a formula in terms of NILPs for birational rowmotion, and use it to prove periodicity and birational homomesy.
- Open questions include: Can we generalize this to other “nice” posets?

We're happy to talk about this further with anyone who's interested.

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Thanks very much for coming to this talk!

どうも有り難う御座いました。

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