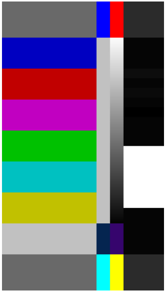
Birational Rowmotion: order, homomesy, and cluster connections

Tom Roby (University of Connecticut)

Describing joint research with Darij Grinberg

Combinatorics Seminar
University of Minnesota
Minneapolis, MN USA



22 May 2015

Slides for this talk are available online (or will be soon) at

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Abstract

Within dynamical algebraic combinatorics, an action of particular interest on the set of order ideals of a finite poset P is *rowmotion* (aka the “Fon-der-Flaass map” aka “Panyushev complementation”). Various surprising properties of rowmotion have been exhibited in work of Brouwer/Schrijver, Cameron/Fon der Flaass, Panyushev, Armstrong/Stump/Thomas, Striker/Williams, and Propp/R. For example, its order is $p + q$ when P is the product $[p] \times [q]$ of two chains, and several natural statistics have the same average over every rowmotion orbit (i.e., are "homomesic").

Recent work of Einstein/Propp generalizes rowmotion twice: first to the piecewise-linear setting of a poset's "order polytope", defined by Stanley in 1986, and then via detropicalization to the birational setting.

In these latter settings, generalized rowmotion no longer has finite order in the general case. Results of Grinberg and the speaker, however, show that it still has order $p + q$ on the product $[p] \times [q]$ of two chains, and still has finite order for a wide class of forest-like ("skeletal") graded posets and for some triangle-shaped posets. Our methods of proof are partly based on those used by Volkov to resolve the type AA (rectangular) Zamolodchikov Periodicity Conjecture, and recently a workgroup at an AIM workshop found a more direct connection between Y-systems and birational rowmotion on $[p] \times [q]$.

Acknowledgments

This seminar talk discusses work with Darij Grinberg, including ideas and results from Arkady Berenstein, David Einstein, Max Glick, Gregg Musiker, Jim Propp, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures. Darij Grinberg & Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, Miriam Farber, Sam Hopkins, Svante Linusson, Gregg Musiker, Alexander Postnikov, Pavlo Pylyavskyy Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.

Please feel free to interrupt with questions, comments, or general heckling.

Outline

- Review of classical and piecewise-linear rowmotion;
- Detropicalizing to birational toggles and rowmotion on a finite poset P;
- Periodicity and order of birational rowmotion on P, particularly products of chains and graded forests;
- A sketch of some proof ideas;
- Homomesy in the birational context;

Classical rowmotion

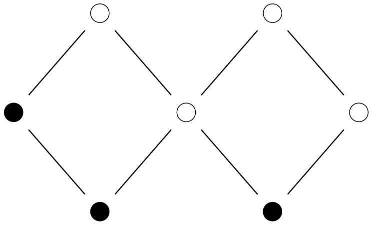
Classical rowmotion is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

- Let P be a finite poset.
  Classical rowmotion is the map $\mathbf{r} : J(P) \to J(P)$ which sends every order ideal S to the order ideal obtained as follows:
  Let M be the set of minimal elements of the complement $P \setminus S$.
  Then, $\mathbf{r}(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Example:

Let S be the following order ideal ($\bullet$ = inside order ideal):



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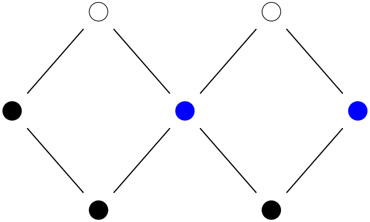
Example:
Mark M (= minimal elements of complement) blue.



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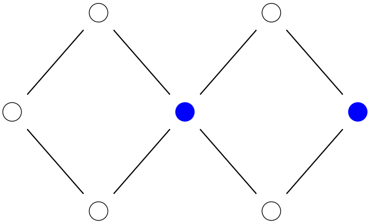
Example:
Forget about the old order ideal:



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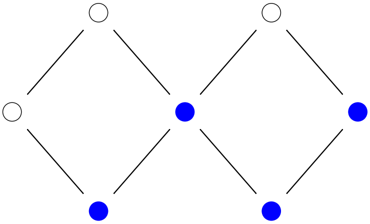
Example:
$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):

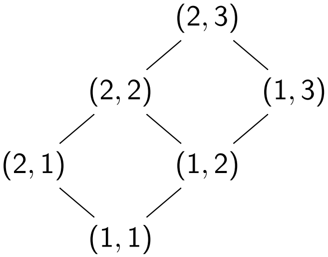


Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.
However, **for some types of** P, the order can be explicitly computed or bounded from above.
See Striker-Williams for an exposition of known results.

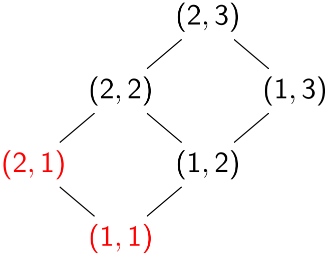
- If P is a $p \times q$-rectangle:



(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

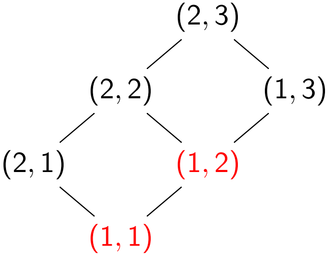
Classical rowmotion: properties

Example:
Let S be the order ideal of the 2×3-rectangle given by:

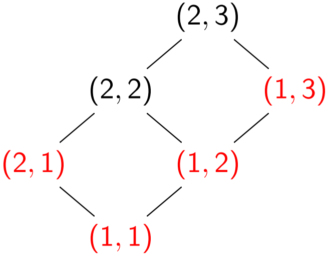


Classical rowmotion: properties

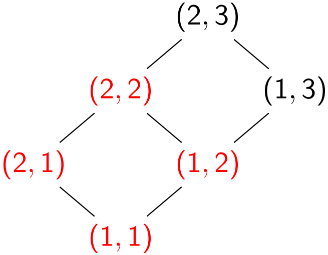
Example:
$\mathbf{r}(S)$ is



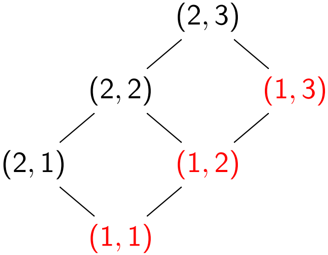
Example:
$\mathbf{r}^2(S)$ is



Example:
$\mathbf{r}^3(S)$ is

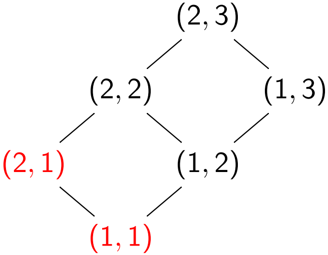


Example:
$\mathbf{r}^4(S)$ is



Classical rowmotion: properties

Example:
$\mathbf{r}^5(S)$ is



which is precisely the S we started with.

$\mathrm{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

There is an alternative definition of classical rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

Classical rowmotion: the toggling definition

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  - S otherwise.

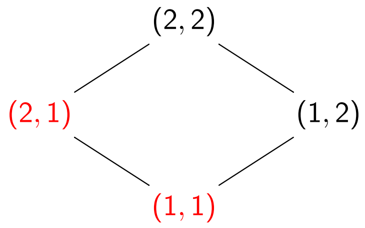
  ("Try to add or remove v from S, as long as the result remains within $J(P)$; otherwise, leave S fixed.")
- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.

Classical rowmotion: the toggling definition

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Example:
Start with this order ideal S:



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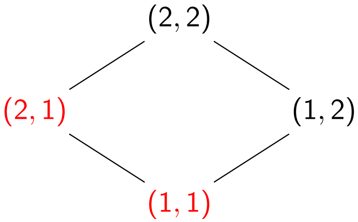
Example:
First apply $\mathbf{t}_{(2,2)}$, which changes nothing:



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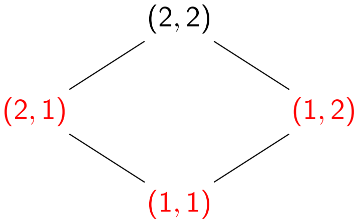
Example:

Then apply $\mathbf{t}_{(1,2)}$, which adds $(1, 2)$ to the order ideal:



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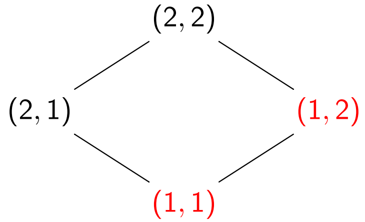
Example:

Then apply $\mathbf{t}_{(2,1)}$, which removes $(2,1)$ from the order ideal:



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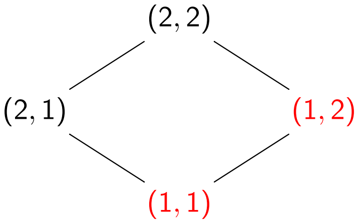

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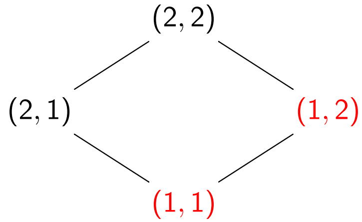
Example:
So this is $\mathbf{r}(S)$:



Generalizing to PL setting: the order polytope of a poset

We can generalize this idea of composition of toggles to define a **piecewise-linear (PL)** version of rowmotion on an infinite set of functions on a poset.

Generalizing to PL setting: the order polytope of a poset

We can generalize this idea of composition of toggles to define a **piecewise-linear (PL)** version of rowmotion on an infinite set of functions on a poset.

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.
The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

Flipping-maps in the order polytope

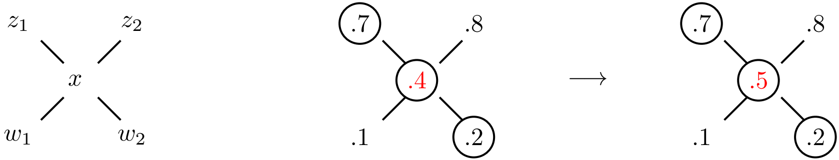
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w <\cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w <\cdot x$ means x covers w.

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w <\cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w <\cdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example of flipping at a node

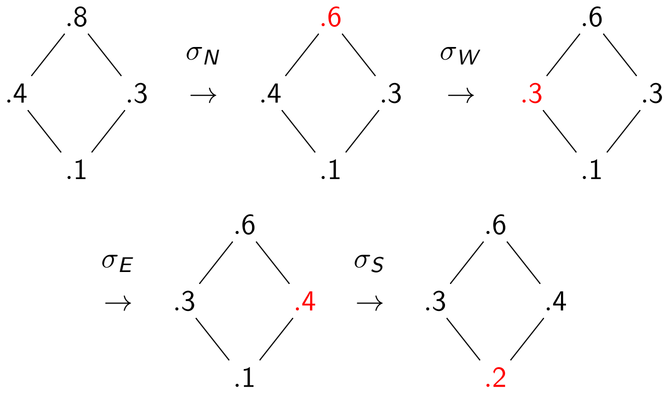


$$\min_{z \cdot > x} f(z) + \max_{w <\cdot\, x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:



(Here we successively flip values at the North, West, East, and South.)

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0,1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

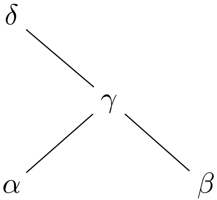
We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset (using that $\min(z_i) = -\max(-z_i)$) to get

$$f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
  - 0 to be less than every other element, and
  - 1 to be greater than every other element.

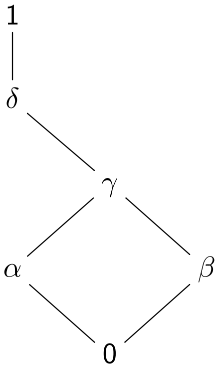
Example:

$P =$



$\Longrightarrow$

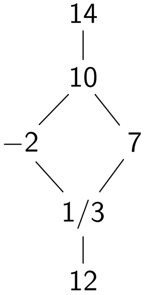
$\widehat{P} =$



Birational rowmotion: definition

- Let $\mathbb{K}$ be a field.
- A $\mathbb{K}$-**labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

Example: This is a $\mathbb{Q}$-labelling of the 2×2-rectangle:



Birational rowmotion: definition

- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$\left(T_v f\right)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

for all $w \in \widehat{P}$.

- That is,
  - **invert** the label at v,
  - **multiply** by the **sum** of the labels at vertices **covered by** v,
  - **multiply** by the **parallel sum** of the labels at vertices **covering** v.

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  for all $w \in \widehat{P}$.

- Notice that this is a **local change** to the label at v; all other labels stay the same.
- We have $T_v^2 = \mathrm{id}$ (on the range of T_v), and T_v is a birational map.

Birational rowmotion: definition

- We define **birational rowmotion** as the rational map

$$R := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

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- This is indeed independent on the linear extension, because:

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- This is indeed independent on the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (even when they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

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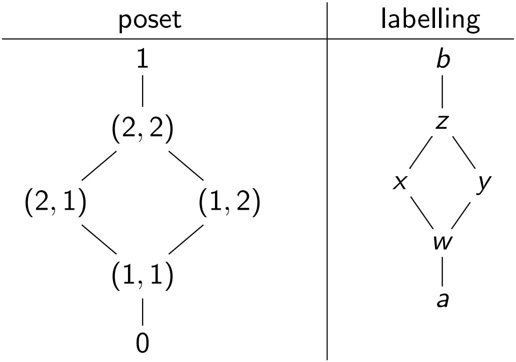
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- For more information about the lifting of rowmotion from classical to PL to birational, see, Einstein-Propp [EiPr13], where R is denoted ρ_B.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



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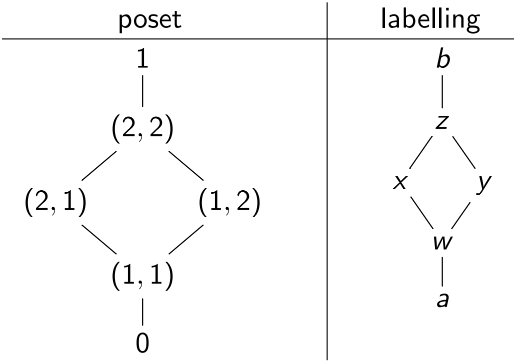

We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1),(1,2),(2,1),(2,2))$).

That is, toggle in the order "top, left, right, bottom".

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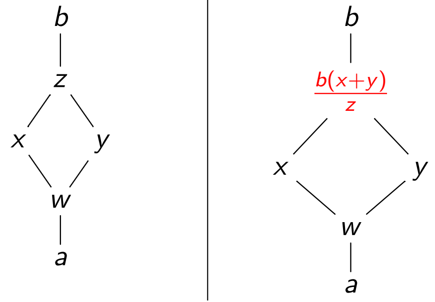
original labelling f	labelling $T_{(2,2)} f$



We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

Birational rowmotion: example

Example:
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

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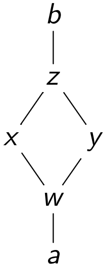	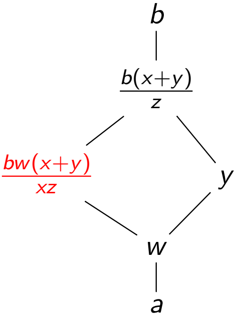

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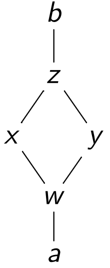	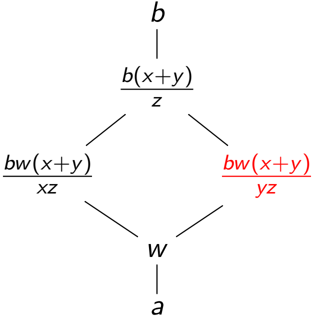

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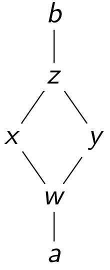
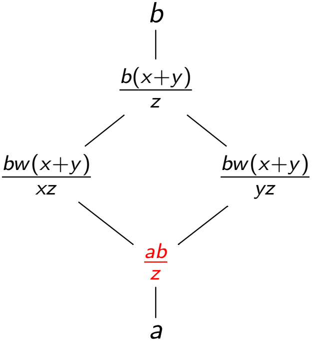
original labelling f	labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$





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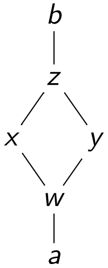
Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

$R^0 f =$



Birational rowmotion orbit on a product of chains

Example:

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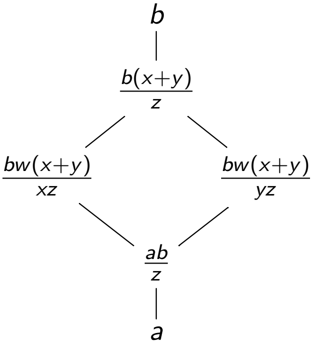
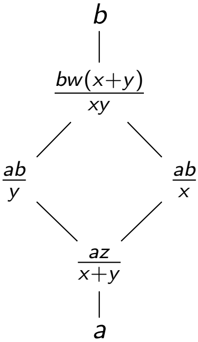


Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

$R^2 f =$



Birational rowmotion orbit on a product of chains

Example:

Iteratively apply R to a labelling of the 2×2-rectangle.

$R^3 f =$

$$
\begin{tikzcd}[column sep=small]
 & b \arrow[d, no head] & \\
 & \frac{ab}{w} \arrow[dl, no head] \arrow[dr, no head] & \\
\frac{ayz}{w(x+y)} \arrow[dr, no head] & & \frac{axz}{w(x+y)} \arrow[dl, no head] \\
 & \frac{xy}{aw(x+y)} \arrow[d, no head] & \\
 & a &
\end{tikzcd}
$$

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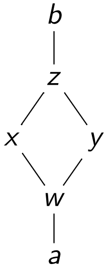

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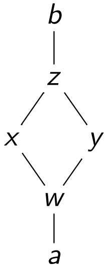

So we are back where we started.

$$\mathrm{ord}(R) = 4.$$

Birational rowmotion: order

- Let $\operatorname{ord} \phi$ denote the order of a map or rational map ϕ. This is the smallest positive integer k such that $\phi^k = \mathrm{id}$ (on the range of ϕ^k), or ∞ if no such k exists.
- A straightforward argument shows that $\operatorname{ord}(\mathbf{r}) \mid \operatorname{ord}(R)$ for every finite poset P.
- Do we have equality?

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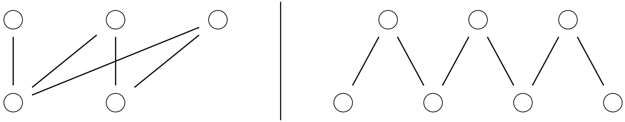

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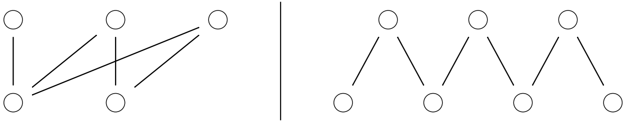
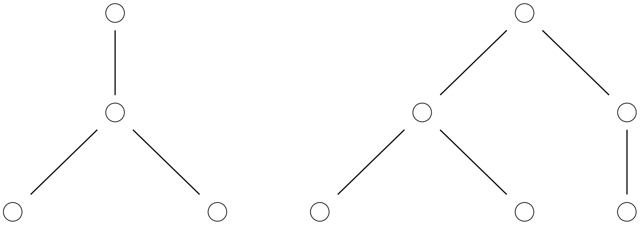


- **Nevertheless**, equality holds for many special types of P.

- **Theorem.** Assume that $n \in \mathbb{N}$, and P is a poset which is a forest (made into a poset using the "descendant" relation) having all leaves on the same level n (i.e., each maximal chain of P has n vertices). Then,

$$\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, \ldots, n+1) .$$

Example: For P as shown, $\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, 3, 4) = 12$.

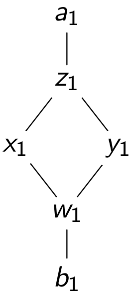
$P =$ 

- Even the $\operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, ..., n + 1)$ part of this result seems to be new.
- The proof that $\operatorname{ord}(R) \mid \operatorname{lcm}(1, 2, ..., n + 1)$ is essentially inductive, but with a few complications. We consider the interplay between the map $\overline{R}$, defined on *homogenous equivalence classes of labelings* and R itself.
- In fact, our proof handles the wider class of posets we call "skeletal posets". (These can be regarded as a generalization of forests where we are allowed to graft existing forests on roots on the top and on the bottom, and to use antichains instead of roots. An example is the 2×2-rectangle.)

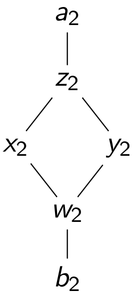
- Two $\mathbb{K}$-labellings f and g of P are said to be **homogeneously equivalent** if there is a $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in (\mathbb{K} \setminus 0)^n$ such that

$$g(v) = \lambda_i f(v) \qquad \text{for all } i \text{ and all } v \in P_i.$$

Example: These two labellings:



and



are homogeneously equivalent if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.

- Let $\overline{\mathbb{K}^{\widehat{P}}}$ denote the set of all $\mathbb{K}$-labellings of P (with no zero labels) modulo homogeneous equivalence.
  Let $\pi : \mathbb{K}^{\widehat{P}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ be the canonical projection.
- There exists a rational map $\overline{R} : \overline{\mathbb{K}^{\widehat{P}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ such that the diagram

$$\begin{tikzcd}
\mathbb{K}^{\widehat{P}} \arrow[r, dashed, "R"] \arrow[d, dashed, "\pi"'] & \mathbb{K}^{\widehat{P}} \arrow[d, dashed, "\pi"] \\
\overline{\mathbb{K}^{\widehat{P}}} \arrow[r, dashed, "\overline{R}"'] & \overline{\mathbb{K}^{\widehat{P}}}
\end{tikzcd}$$

  commutes.
- Hence $\operatorname{ord}\left(\overline{R}\right) \mid \operatorname{ord}(R)$.

- **Theorem (periodicity):** If P is the $p \times q$-rectangle (i.e., the poset $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$ with coordinatewise order), then
$$\operatorname{ord}(R) = p + q.$$

Example: For the 2×2-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

Birational rowmotion: the rectangle case

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Example: For the 2×2-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

- **Theorem (reciprocity):** If P is the $p \times q$-rectangle, and $(i, k) \in P$ and $f \in \mathbb{K}^{\widehat{P}}$, then

$$f\left(\underbrace{(p+1-i, q+1-k)}_{\substack{=\text{antipode of } (i,k) \\ \text{in the rectangle}}}\right) = \frac{f(0) f(1)}{\left(R^{i+k-1} f\right)((i, k))}.$$

- These were conjectured (independently) by James Propp and R.

Birational rowmotion: the rectangle case, example

Example: Here is the generic R-orbit on the 2×2-rectangle again:

$$\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& a &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{bw(x+y)}{xz} \arrow[dr, dash] & & \frac{bw(x+y)}{yz} \arrow[dl, dash] \\
& \frac{ab}{z} \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& \frac{bw(x+y)}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ab}{y} \arrow[dr, dash] & & \frac{ab}{x} \arrow[dl, dash] \\
& \frac{az}{x+y} \arrow[d, dash] & \\
& a &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& b \arrow[d, dash] & \\
& \frac{ab}{w} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ayz}{w(x+y)} \arrow[dr, dash] & & \frac{axz}{w(x+y)} \arrow[dl, dash] \\
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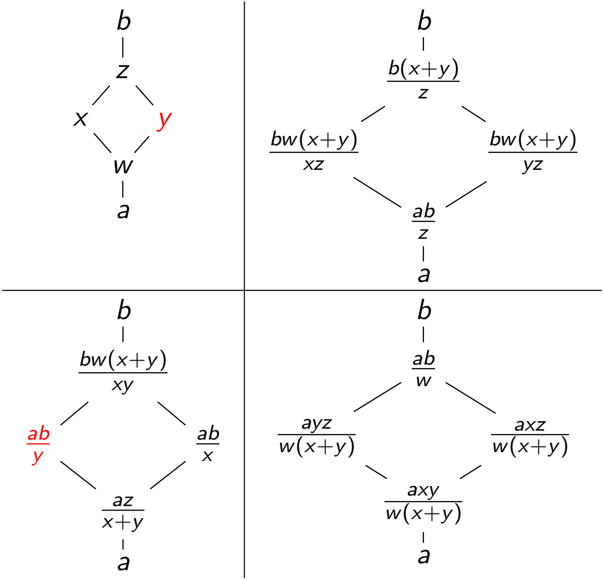

Birational rowmotion: the rectangle case, example

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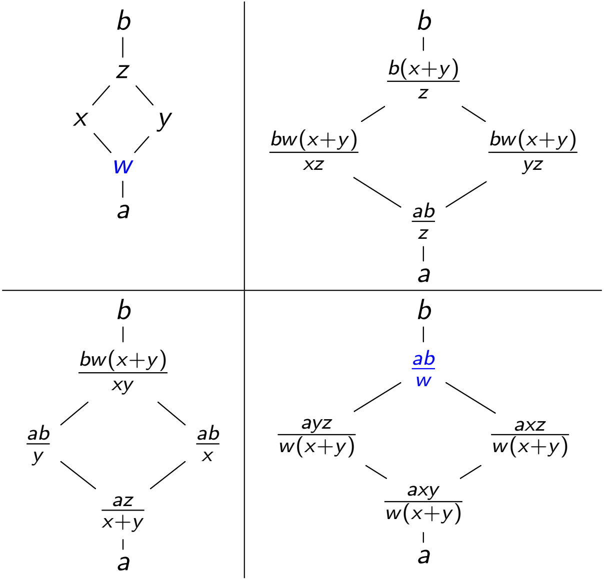

Birational rowmotion: the rectangle case, proof idea

- Inspiration: Alexandre Yu. Volkov, *On Zamolodchikov's Periodicity Conjecture*, `arXiv:hep-th/0606094`.
- We reparametrize our assignments $f : \widehat{P} \to \mathbb{K}$ through $p \times (p+q)$-matrices in such a way that birational rowmotion corresponds to "cycling" the columns of the matrix.
- This uses a 3-term Plücker relation.
- Lots of technicalities to be managed, particularly around birational maps not necessarily being defined everywhere.

- Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix with p rows and $p+q$ columns.
- Let A_i be the i-th column of A. Extend to all $i \in \mathbb{Z}$ by setting

$$A_{p+q+i} = (-1)^{p-1} A_i \qquad \text{for all } i.$$

- Let $A[a:b \mid c:d]$ be the matrix whose columns are A_a, A_{a+1}, ..., A_{b-1}, A_c, A_{c+1}, ..., A_{d-1} from left to right.

Birational rowmotion: the rectangle case, proof

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- Let $A[a : b \mid c : d]$ be the matrix whose columns are $A_a, A_{a+1}, \ldots, A_{b-1}, A_c, A_{c+1}, \ldots, A_{d-1}$ from left to right.
- For every $j \in \mathbb{Z}$, we define a $\mathbb{K}$-labelling $\operatorname{Grasp}_j A \in \mathbb{K}^{\widehat{P}}$ by

$$\left(\operatorname{Grasp}_j A\right)((i,k)) = \frac{\det\left(A[j+1 : j+i \mid j+i+k-1 : j+p+k]\right)}{\det\left(A[j : j+i \mid j+i+k : j+p+k]\right)}$$

for every $(i,k) \in P$ (this is well-defined for a Zariski-generic A) and $\left(\operatorname{Grasp}_j A\right)(0) = \left(\operatorname{Grasp}_j A\right)(1) = 1$.

Birational rowmotion: the rectangle case, proof

- The proof of $\operatorname{ord}(R) = p + q$ now rests on four claims:
  - **Claim 1:** $\operatorname{Grasp}_j A = \operatorname{Grasp}_{p+q+j} A$ for all j and A.
  - **Claim 2:** $R\left(\operatorname{Grasp}_j A\right) = \operatorname{Grasp}_{j-1} A$ for all j and A.
  - **Claim 3:** For almost every $f \in \mathbb{K}^{\widehat{P}}$ satisfying $f(0) = f(1) = 1$, there exists a matrix $A \in \mathbb{K}^{p \times (p+q)}$ such that $\operatorname{Grasp}_0 A = f$.
  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 1 is immediate from the definitions.

Birational rowmotion: the rectangle case, proof

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- Claim 2 is a computation with determinants, which boils down to the three-term Plücker identities:

$$\det\left(A\left[a-1:b \mid c:d+1\right]\right) \cdot \det\left(A\left[a:b+1 \mid c-1:d\right]\right)$$
$$+ \det\left(A\left[a:b \mid c-1:d+1\right]\right) \cdot \det\left(A\left[a-1:b+1 \mid c:d\right]\right)$$
$$= \det\left(A\left[a-1:b \mid c-1:d\right]\right) \cdot \det\left(A\left[a:b+1 \mid c:d+1\right]\right).$$

for $A \in \mathbb{K}^{u \times v}$ and $a \leq b$ and $c \leq d$ and $b - a + d - c = u - 2$.

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  - **Claim 4:** In proving $\operatorname{ord}(R) = p + q$ we can WLOG assume that $f(0) = f(1) = 1$.

- Claim 3 is an annoying (nonlinear) triangularity argument: With the ansatz $A = (I_p \mid B)$ for $B \in \mathbb{K}^{p \times q}$, the equation $\operatorname{Grasp}_0 A = f$ translates into a system of equations in the entries of B which can be solved by elimination.

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- Claim 4 follows by noting that for an n-graded poset we have $\operatorname{ord}(R) = \operatorname{lcm}\left(n + 1, \operatorname{ord}\left(\overline{R}\right)\right)$.

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- The reciprocity statement can be proven in a similar vein.

Connection with Y-systems

Although a number of people suspected a possible connection between Y-systems and birational rowmotion, it was only recently (in March 2015) that an explicit connection was uncovered by Max Glick, Darij Grinberg, and Gregg Musiker (and possibly others who were in the room that afternoon). This gives another path to proving the periodicity of birational rowmotion, and connects it with the theory of cluster algebras.

Informally, a Y-system is a dynamical system of rational functions defined on a graph coming from root systems. The setup is as follows:

- Let Δ, Δ' be *Dynkin diagrams* on vertex sets I, I' and let C, C' be the corresponding *Cartan matrices*. Set the graph $\Gamma := I \times I'$.
- Define $A = (a_{i,j}) := 2\mathrm{Id}_{\#I} - C$ and $A' = (a'_{i',j'}) := 2\mathrm{Id}_{\#I'} - C'$.
- The matrices A and A' controls how this dynamical system updates.

The only example we consider here is Type A, e.g.,

$$\Delta = A_4 \Rightarrow C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \implies A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Connection with Y-systems

- Let h, h' denote the Coxeter numbers of Δ, Δ' (i.e., order of the product of all the simple reflections in any order).
- The $\Delta \times \Delta'$ Y-system is then the collection $\{Y_{i,i',t} \colon (i,i') \in I \times I', t \in \mathbb{Z}\}$ satisfying the relations

$$Y_{i,i',t+1} Y_{i,i',t-1} = \frac{\prod_{j \in I} (1 + Y_{j,i',t})^{a_{i,j}}}{\prod_{j' \in I'} (1 + Y_{i,j',t}^{-1})^{a_{i',j'}}}.$$

- Periodicity Theorem (Keller): $Y_{i,i',t+2(h+h')} = Y_{i,i',t}$.
- In type A_n, $h = n+1$, so the $A_{p-1} \times A_{q-1}$ Y-system has order $2(p+q)$.

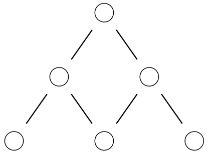
when $f(i,i')$ are rational functions on the vertices of the Hasse diagram of P, and $R \colon \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ is birational rowmotion on P, then we have

$$Y_{i,i',i+i'-2k} = \frac{R^k f(i, i'+1)}{R^k f(i+1, i')}$$

where the $Y_{i,i',t}$ belong to the $A_{p-1} \times A_{q-1}$ Y-system.

- **Theorem (periodicity):** If P is the triangle $\Delta(p) = \{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i+k > p+1\}$ with $p > 2$, then
$$\operatorname{ord}(R) = 2p.$$

Example: The triangle $\Delta(4)$:



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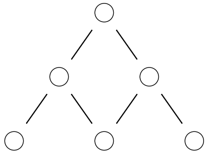
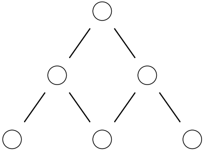


- **Theorem (reciprocity):** R^p reflects any $\mathbb{K}$-labelling across the vertical axis.
- This is precisely the same result as for classical rowmotion.

Birational rowmotion: the Δ-triangle case

- **Theorem (periodicity):** If P is the triangle $\Delta(p) = \{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i+k > p+1\}$ with $p > 2$, then
$$\operatorname{ord}(R) = 2p.$$

Example: The triangle $\Delta(4)$:



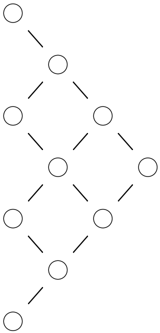
- **Theorem (reciprocity):** R^p reflects any $\mathbb{K}$-labelling across the vertical axis.
- This is precisely the same result as for classical rowmotion.
- The proofs use a "folding"-style argument to reduce this to the rectangle case.

Birational rowmotion: the $\triangleright$-triangle case

- **Theorem (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k\}$, then

$$\operatorname{ord}(R) = 2p.$$

Example: For $p = 4$, this P has the form:



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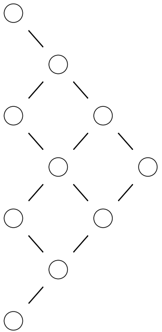

.

- Again this is reduced to the rectangle case.

- **Conjecture (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k; \; i+k > p+1\}$, then
$$\operatorname{ord}(R) = p.$$

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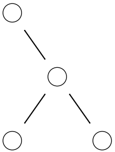

Birational rowmotion: the right-angled triangle case

- **Conjecture (periodicity):** If P is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1\}$, then
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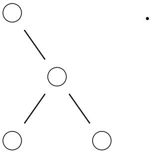
Example: For $p = 4$, this P has the form:



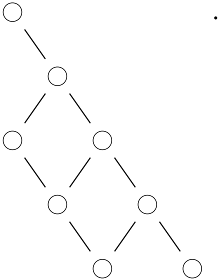
- We proved this for p odd.
- Note that for p even, this is a type-B positive root poset. Armstrong-Stump-Thomas did this for classical rowmotion.

Birational rowmotion: the trapezoid case (Nathan Williams)

- **Conjecture (periodicity):** If P is the trapezoid $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1;\ k \geq s\}$ for some $0 \leq s \leq p$, then

$$\operatorname{ord}(R) = p.$$

Example: For $p = 6$ and $s = 5$, this P has the form:



- This was observed by Nathan Williams and verified for $p \leq 7$.
- Motivation comes from Williams's "Cataland" philosophy.

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- **Not true:** for all those P that have nice and small $\text{ord}(\mathbf{r})$'s.
- **However** it seems that $\text{ord}(R) < \infty$ holds if P is **the positive root poset of a coincidental-type root system** (A_n, B_n, H_3), or a **minuscule heap** (see Rush-Shi, section 6).
- But the positive root system of D_4 has $\text{ord}(R) = \infty$.

Application: Promotion on SSYTs

- The following is an application of our result on rectangle-shaped posets.
- It is well known (see Striker-Williams) that **classical** rowmotion ($=$ birational rowmotion over the boolean semiring $\{0,1\}$) is related to promotion on **two-rowed** semistandard Young tableaux.
- Similarly, **birational** rowmotion over the tropical semiring $\operatorname{Trop}\mathbb{Z}$ relates to **arbitrary** semistandard Young tableaux.
- As an application of the periodicity theorem, we obtain the classical result that promotion done n times on a rectangular semistandard Young tableau with "ceiling" n does nothing.

QUESTIONS: What is this related to?

This line of work appears (at least superficially) to be related to several other areas of research:

- Y-systems and Zomolodchikov Periodicity?
- Cluster mutations?
- bounded octohedron recurrence?
- Kirillov-Berenstein RSK?
- Other Coxeter or Catalan combinatorics?

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Q: What orders of toggling lead to finite-order birational maps?

Birational rowmotion: noncommutative generalization?

- This is new and unproven, and inspired by Iyudu/Shkarin, `arXiv:1305.1965v3` (Kontsevich's periodicity conjecture).
- Work in a **skew field**. Write $\overline{m}$ for m^{-1}.
- Define the v-toggle by

$$\left(T_v f\right)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

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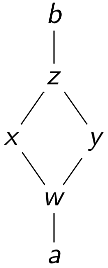
Iteratively apply R to a labelling of the 2×2-rectangle.
$R^0 f =$



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$R^1 f =$

$$\begin{tikzcd}
 & b \arrow[d, dash] & \\
 & (x+y)\overline{z}b \arrow[dl, dash] \arrow[dr, dash] & \\
w\overline{x}(x+y)\overline{z}b \arrow[dr, dash] & & w\overline{y}(x+y)\overline{z}b \arrow[dl, dash] \\
 & a\overline{z}b \arrow[d, dash] & \\
 & a &
\end{tikzcd}$$

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$R^2 f =$

$$\begin{tikzcd}
& b \arrow[d, dash] & \\
& w\left(\overline{x}+\overline{y}\right)b \arrow[dl, dash] \arrow[dr, dash] & \\
a\cdot\overline{x+y}\cdot x\left(\overline{x}+\overline{y}\right)b \arrow[dr, dash] & & a\cdot\overline{x+y}\cdot y\left(\overline{x}+\overline{y}\right)b \arrow[dl, dash] \\
& a\overline{b}z\cdot\overline{x+y}\cdot b \arrow[d, dash] & \\
& a &
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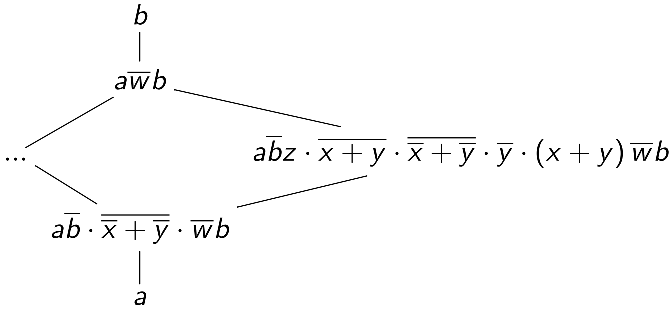

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$R^4 f =$

$$b$$

$$a\overline{b}z\overline{a}b$$

$$\ldots \qquad a\overline{b} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{x+y} \cdot y\,(\overline{x}+\overline{y})\,(x+y)\,\overline{a}b$$

$$a\overline{b}w\overline{a}b$$

$$a$$

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\end{tikzcd}$$

(after nontrivial simplifications).

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Iteratively apply R to a labelling of the 2×2-rectangle.
$R^4 f =$

$$\begin{tikzcd}[arrows=dash]
 & b & \\
 & a\overline{b}z\overline{a}b \arrow[u] & \\
a\overline{b}x\overline{a}b \arrow[ur] & & a\overline{b}y\overline{a}b \arrow[ul] \\
 & a\overline{b}w\overline{a}b \arrow[ul] \arrow[ur] & \\
 & a \arrow[u] &
\end{tikzcd}$$

That is, all of our labels got conjugated by $a\overline{b}$. **Is R^{p+q} always conjugation by $f(0) \cdot (f(1))^{-1}$ on a $p \times q$-rectangle?** This is similar to Kontsevich's periodicity. (Noncommutative determinants?)

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DEF: Given an (invertible) action τ on a finite set of objects S, call a statistic $\varphi : S \to \mathbb{C}$ **homomesic** [Gk., "same middle"] with respect to (S, τ) iff the average of φ over each τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$, i.e., $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{s \in \mathcal{O}} \varphi(s)$ does not depend on the choice of $\mathcal{O}$.

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For example, the statistic $\#I$ (cardinality of the ideal) is *homomesic* with respect to rowmotion, $\mathbf{r}$, acting on $J([4] \times [2])$.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\mathbf{r}$*-orbit in* $J([p] \times [q])$*. Then*

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{pq}{2},$$

i.e., the cardinality statistic is homomesic with respect to the action of rowmotion on order ideals.

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It turns out that to show a similar statement for rowmotion acting on the *antichains* of P, the right tool is an equivariant bijection from Stanley's "Promotion and Evacuation" paper, as rephrased by Hugh Thomas.

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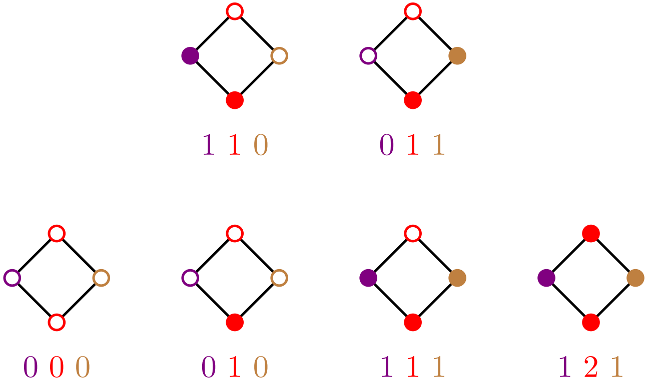
Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has
$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

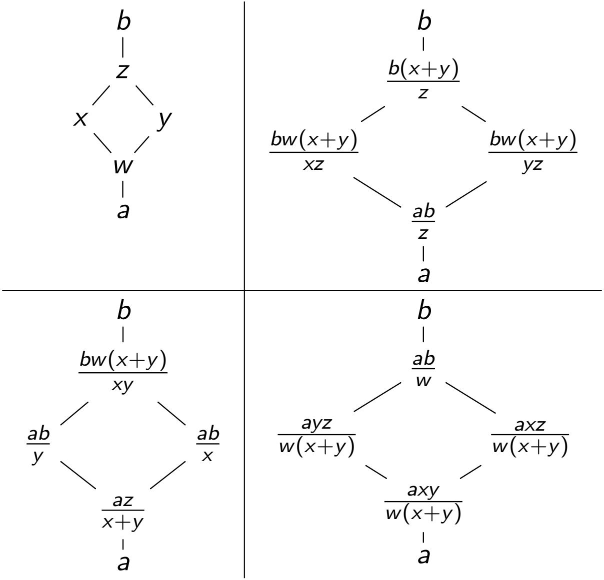
We have more refined homomesies for combinatorial rowmotion on $J([p] \times [q])$.
For $1 - b \leq k \leq a - 1$, let $f_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k f_k(I)$.

Theorem (Propp, R.): If $\mathcal{O}$ is any ∂-orbit in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} f_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Homomesy for Birational rowmotion on $J([2] \times [2])$:

Example: Consider the **geometric** means of products in each file:



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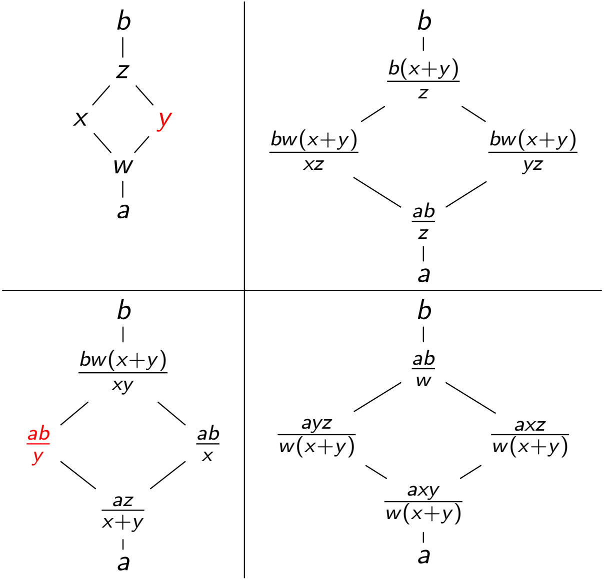

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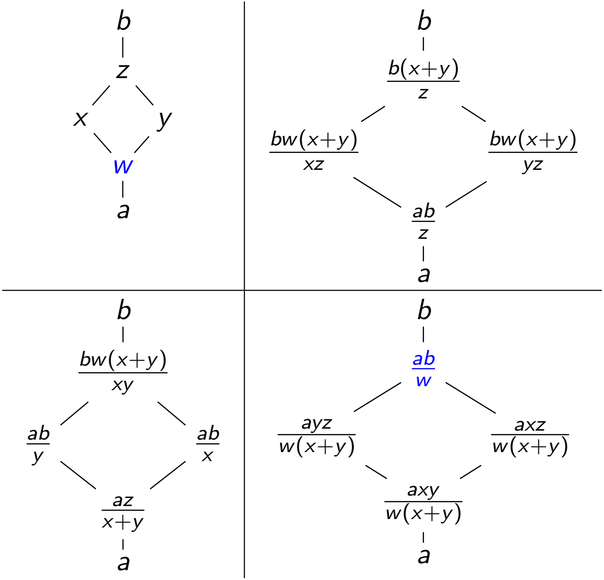

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Thanks very much for coming to this talk!