Outline & Objectives

- Analyze the reasoning that connects different statements that are all equivalent to a matrix $A \in \mathbb{R}^{n \times n}$ being invertible, thereby reviewing a number of crucial course concepts.
- Apply these in examples.
The Invertible Matrix Theorem

**Theorem**

For any *square* matrix $A \in \mathbb{R}^{n \times n}$ TFAE (The Following Are Equiv.):

1. $A$ is invertible.
2. $A$ is row-equivalent to $I_n$.
3. $A$ has $n$ pivot positions.
4. $A\vec{x} = \vec{0}$ has only the trivial soln. $\vec{x} = \vec{0}$.
5. Columns of $A$ are linearly independent.
6. $A \mapsto A\vec{x}$ is one-to-one.
7. $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has $\leq 1$ soln.
8. $\forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has $\geq 1$ soln.
9. $A \mapsto A\vec{x}$ is onto.
10. Columns of $A$ span $\mathbb{R}^n$.
11. $\exists C \in \mathbb{R}^{n \times n}$ such that $CA = I_n$.
12. $\exists D \in \mathbb{R}^{n \times n}$ such that $AD = I_n$.
13. $A^T$ is invertible.
14. $\ldots$ More to come!
Applications of the Invertible Matrix Theorem

Suppose $M$ is a $4 \times 4$ matrix and \[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}
\] is not a linear combination of the columns of $M$. Could $M$ be one-to-one? Can you say anything about the set of columns of $M$? What if $M$ were $4 \times 3$?

**Definition**

Call a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ invertible if there exists a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T(\vec{x})) = \vec{x} = T(S(\vec{x}))$.

**Theorem**

Let $A$ be the standard matrix for the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$. Then $T$ is invertible (with inverse $S$) if and only if $A$ is invertible (with $S\vec{x} = A^{-1}\vec{x}$).