Video Lecture M8: Matrix Inverses: Properties, Elementary Matrices, & Algorithms

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Outline & Objectives

- Analyze algebraic properties of matrix inversion.
- Reconceptualize row operations as multiplication by elementary matrices to create an algorithm for computing matrix inverses.
Proposition

Suppose that \( A \) and \( B \) are invertible \( mxs \) in \( \mathbb{R}^{n \times n} \). Then

1. \( A^{-1} \) is invertible and \((A^{-1})^{-1} = A\).
2. \( A^T \) is invertible and \((A^T)^{-1} = (A^{-1})^T\).
3. \( AB \) is invertible, and \((AB)^{-1} = B^{-1}A^{-1}\)

Inverse of any product of matrices in \( \mathbb{R}^{n \times n} \) is the product of the inverses in the reverse order: \((A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}\).
Elementary Matrices

**Definition**

An *elementary matrix* $E \in \mathbb{R}^{n \times n}$ is one obtained from $I_n$ by a single row operation.

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

Multiplying $E$ by $A$ just applies corresponding row-op to $A$!

Every elementary matrix is invertible. (why?)

**Theorem**

$A \in \mathbb{R}^{n \times n}$ is invertible $\iff$ $A$ is row-equivalent to $I_n$. Then any sequence of row ops that takes $A$ to $I_n$ also takes $I_n$ to $A^{-1}$.
Row-reduction algorithm for computing matrix inverses

Form “super-augmented” matrix \([A \mid I_n]\) and row-reduce until \(A\) is in RREF. If get \([I_n \mid B]\), then \(B = A^{-1}\). If get \([\text{NOT} \ I_n \mid B]\), then \(A\) is not invertible.

\[
\begin{bmatrix}
1 & -3 & 0 & 1 & 0 & 0 \\
-1 & 2 & -1 & 0 & 1 & 0 \\
0 & -2 & 2 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -1/2 & -3/2 & -3/4 \\
0 & 1 & 0 & -1/2 & -1/2 & -1/4 \\
0 & 0 & 1 & -1/2 & -1/2 & 1/4
\end{bmatrix}
\]

CHECK!