Video Lecture M6: Matrix Operations 2

Tom Roby
Outline & Objectives

- Memorize the definition of the transpose $A^T$ of a matrix $A$ and compute examples.
- Generalize some algebraic properties for numbers to matrix multiplication, and demonstrate that others fail.
- Apply properties of these operations in examples and as part of reasoning in proofs.
Algebraic properties of matrix multiplication

Recall: \[ AB = [A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_p] \text{ where } (m \times n)(n \times p) \mapsto (m \times p) \]

Proposition

Let \( A, B, C \) be matrices of appropriate sizes, \( r \in \mathbb{R} \):

1. \( A(BC) = (AB)C \)
2. \( A(B + C) = AB + BC \)
3. \( (A + B)C = AC + BC \)
4. \( r(AB) = (rA)B = A(rB) \)
5. \( I_mA = A = AI_n \)
6. \( BUT \ AB \neq BA \text{ OFTEN} \)

Think about dimensions for \( AB \) versus \( BA \).

Can we make sense of \( A^4 \)?
If \( AB = 0 \), then must \( A = 0 \) or \( B = 0 \)?
If \( AC = BC \) and \( C \neq 0 \) then must \( A = B \)?

**Ex:** Find examples of \( A, B, C \in \mathbb{R}^{2 \times 2} \) where (1) \( AB \neq BA \), (2) \( AB = 0 \), but neither \( A \) nor \( B \) is zero, and (3) \( AC = BC \), but \( A \neq B \).
Matrix Transposes

Definition

For $m \times n$ matrix $A$, define $A^T$ to be the $n \times m$ matrix whose rows are the columns of $A$, i.e., $A = (a_{ij}) \iff A^T = (a_{ji})$.

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{bmatrix}^T = 
\begin{bmatrix}
1 & 5 & 9 \\
2 & 6 & 10 \\
3 & 7 & 11 \\
4 & 8 & 12
\end{bmatrix},
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & j
\end{bmatrix}^T = 
\begin{bmatrix}
a & d & g \\
b & e & h \\
c & f & j
\end{bmatrix}.
$$

Proposition

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T \neq A^T B^T$

Ex: Find an example of two $2 \times 2$ matrices $A$ and $B$ such that $(AB)^T \neq A^T B^T$. 