Video Lecture M4: Linear Transformations in Practice

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Outline & Objectives

- Memorize the fundamental properties of linear transformations: *one-to-one* and *onto*, interpret them geometrically, and recognize them in concrete examples.
- Connect these concepts with properties of the columns of a matrix representing a transformation.
- Analyze whether a given matrix transformation has these properties by transforming it to RREF.
One-to-one & Onto Functions

Definition (one-to-one, onto for functions)

A function $f : X \rightarrow Z$ is called one-to-one if $f(x) = f(y)$ in $Z \implies x = y$ in $X$. $f$ is called onto if for every $z \in Z$, there exists $x \in X$ such that $f(x) = z$.

Less formally:
A mapping $T$ is one-to-one if each $z \in Z$ is the image of at most one $x \in X$ (uniqueness).
A mapping $T$ is onto if each $z \in Z$ is the image of at least one $x \in X$ (existence).
**Definition (one-to-one, onto for linear transformations)**

A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called **one-to-one** if $T(\vec{x}) = T(\vec{y})$ in $\mathbb{R}^m \implies \vec{x} = \vec{y}$ in $\mathbb{R}^n$. $T$ is called **onto** if for every $\vec{b} \in \mathbb{R}^m$, there exists $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

1. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 0 & \Box & * & * & * & * & * \\ 0 & 0 & 0 & \Box & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \Box & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Theorem (1-1 at $\vec{0}$ means 1-1 everywhere)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one $\iff$ the equation $T(\vec{x}) = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.

Theorem (1-1, onto for matrix transformations)

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(\vec{x}) = A\vec{x}$. Then

(a) $T$ is one-to-one $\iff$ columns of $A$ are lin. independent.
(b) $T$ is onto $\iff$ columns of $A$ span $\mathbb{R}^m$.

1-1 $\iff$ no free variables; onto $\iff$ pivot in every row.