Video Lecture M4: Linear Transformations in Practice

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Outline & Objectives

- Memorize the fundamental properties of linear transformations: *one-to-one* and *onto*, intepret them geometrically, and recognize them in concrete examples.
- Connect these concepts with properties of the columns of a matrix representing a transformation.
- Analyze whether a given matrix transformation has these properties by transforming it to RREF.

One-to-one & Onto Functions

Definition (one-to-one, onto for functions)

A function $f : X \to Z$ is called one-to-one if $f(x) = f(y)$ in $Z \implies x = y$ in X. f is called onto if for every $z \in Z$, there exists $x \in X$ such that $f(x) = z$.

Less formally:
A mapping T is *one-to-one* if each $z \in Z$ is the image of *at most* one $x \in X$ (*uniqueness*).
A mapping T is *onto* if each $z \in Z$ is the image of *at least* one $x \in X$ (*existence*).

One-to-one & Onto Linear Transformations

Definition (one-to-one, onto for linear transformations)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ *is called* one-to-one *if* $T(\vec{x}) = T(\vec{y})$ *in* $\mathbb{R}^m \implies \vec{x} = \vec{y}$ *in* $\mathbb{R}^n$. T *is called* onto *if for every* $\vec{b} \in \mathbb{R}^m$*, there exists* $\vec{x} \in \mathbb{R}^n$ *such that* $T(\vec{x}) = \vec{b}$.

1. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Theorem (1-1 at $\vec{0}$ means 1-1 everywhere)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ *is one-to-one* $\iff$ *the equation* $T(\vec{x}) = \vec{0}$ *has only the solution* $\vec{x} = \vec{0}$.

Theorem (1-1, onto for matrix transformations)

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ *is given by* $T(\vec{x}) = A\vec{x}$. *Then*

(a) T *is one-to-one* $\iff$ *columns of* A *are lin. independent.*

(b) T *is onto* $\iff$ *columns of* A *span* $\mathbb{R}^m$.

1-1 $\iff$ no free variables; *onto* $\iff$ pivot in every row.