Video Lecture M3: Linear Transformations in Theory

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Outline & Objectives

- Characterize linear transformations independently of matrices, as functions $\mathbb{R}^n \to \mathbb{R}^m$ satisfying two basic axioms.
- Prove that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented (uniquely!) by a matrix.

Linear Transformations algebraically

For matrices: $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$ and $A(c\vec{v}) = cA\vec{v}$.

Definition

Call a function $T : \mathbb{R}^n \to \mathbb{R}^m$ linear if it satisfies:
(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and
(b) $T(c\vec{v}) = cT(\vec{v})$.

1. $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x_1, x_2, x_3) = (x_2 - x_3, x_1 + 2x_3)$.
2. $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x_1, x_2, x_3) = (x_2 - x_3, x_1^2 + 2x_3)$.
3. $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x_1, x_2, x_3) = (x_2 - x_3 + 2, x_1 + 2x_3)$.

Proposition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then $T(\vec{0}) = \vec{0}$.

Linear Transformations Geometrically

By definition, a LT preserves the parallelogram law.

So if we know where T sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then we know where it sends $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Any Linear Transformation is a matrix transformation

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}\ \forall x \in \mathbb{R}^n$.
In fact, $A = [T(\vec{e}_1)\ \ \ldots\ \ T(\vec{e}_n)]$, where $\vec{e}_j$ is jth column of I_n.

Proof: Write $\vec{x} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$. By linearity

$$T(\vec{x}) = T\left(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n\right) = T(x_1\vec{e}_1) + \cdots + T(x_n\vec{e}_n)$$

$$= x_1T(\vec{e}_1) + \cdots + x_nT(\vec{e}_n) = [T(\vec{e}_1)\ \ \ldots\ \ T(\vec{e}_n)]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that

(1) reflects through the line $x_1 = x_2$, then scales horizontally by 2.

(2) scales by 2, then rotates 45° CCW.