Video Lecture G6: Creating Orthogonal Bases with the Gram–Schmidt Algorithm

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Outline & Objectives

- Leverage our understanding of orthogonal projection to develop a process (called *Gram–Schmidt orthogonalization*) for creating an orthogonal basis from a given basis for a subspace W of $\mathbb{R}^n$.

Creating orthogonal bases

$W = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} \right\}$.

Given $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ lin indep in $\mathbb{R}^4$, construct orthog basis for $W = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Set $\vec{v}_1 = \vec{x}_1$.
Let $\vec{v}_2 = \vec{x}_2 - \mathrm{Proj}_{\vec{v}_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$.
Let $\vec{v}_3 = \vec{x}_3 - \mathrm{Proj}_{\mathrm{Span}\{\vec{v}_1, \vec{v}_2\}} \vec{x}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$.

Why is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a orthogonal basis for W?

How would we make it an orthonormal basis for W?

$\vec{x}_1 = (1, 1, 1, 1)$, $\vec{x}_2 = (3, 2, 3, 4)$, $\vec{x}_3 = (2, 0, 4, 2)$,

"Gram-Schmidt" Orthogonalization

Theorem (Gram-Schmidt orthogonalization)

Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p\}$ *be a basis for subspace* W *of* $\mathbb{R}^n$. *Define*
$\vec{v}_1 = \vec{x}_1$.
$\vec{v}_2 = \vec{x}_2 - \mathrm{Proj}_{\vec{v}_1} \vec{x}_2 = \vec{v}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$.
$\vec{v}_3 = \vec{x}_3 - \mathrm{Proj}_{\mathrm{Span}\{\vec{v}_1, \vec{v}_2\}} \vec{x}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$.
$\vdots \qquad \vdots$

$$\vec{v}_p = \vec{x}_p - \mathrm{Proj}_{\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}\}} \vec{x}_p$$

$$= \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}.$$

Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ *is an orthogonal basis for* W. *Also,*
$\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ *for all* $k \in [p]$.

NOTES: (1) Can rescale $\vec{v}_i$ at any point (e.g., to avoid fractions);
(2) Usually best to normalize vectors at the very end;
(3) The output ON basis is far from unique, depends on initial $\{\vec{x}_i\}$, and the ordering.