Video Lecture G4: Orthogonal Projection in $\mathbb{R}^n$

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Outline & Objectives

- Define the *orthogonal projection* of one vector onto another vector or onto a subspace and analyze its properties.
- Prove a formula for $\mathrm{Proj}_W \vec{y}$ in terms of inner products.

Orthogonal Projection

How to write given $\vec{y}$ as $\vec{y} = c\vec{u} + \vec{z}$, with $\vec{z} \perp \vec{u}$? Define the *orthogonal projection* of $\vec{y}$ onto $\vec{u}$ as $\widehat{y} = \mathrm{Proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

Find the projection of $y = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ onto (line spanned by) $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

This is the same formula as for coefficients when write $\vec{y}$ in terms of an orthogonal basis; we are projecting onto each basis vector.

Theorem (Orthogonal Decomposition Theorem)

Given a subspace W of $\mathbb{R}^n$, any $\vec{y} \in \mathbb{R}^n$ has a unique representation as $\vec{y} = \widehat{y} + \vec{z}$, *with* $\widehat{y} \in W$, $\vec{z} \in W^{\perp}$.
If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ *is an orthogonal basis for* W, *then*

$$\widehat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

$\widehat{y}$ *is called the orthogonal projection of* $\vec{y}$ *onto* W.

Pf: Assume basis, write $\widehat{y}$ as above. Compute $\vec{z}\cdot\vec{u}_j = (\vec{y} - \widehat{y})\cdot\vec{u}_j \ldots$

If $\vec{y} = \widehat{y} + \vec{z} = \widehat{y}_1 + \vec{z}_1$, then $\widehat{y} - \widehat{y}_1 = \vec{z}_1 - \vec{z} \in W \cap W^{\perp}$.

An Example

Definition (Orthogonal Projection)

Given a subspace W *of* $\mathbb{R}^n$*, any* $\vec{y} \in \mathbb{R}^n$ *has a* *unique* *representation as* $\vec{y} = \widehat{y} + \vec{z}$*, with* $\widehat{y} \in W$ *and* $\vec{z} \in W^{\perp}$*.* *We call* $\widehat{y}$ *the* *orthogonal projection of* $\vec{y}$ *onto* W*. If* $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ *is an orthogonal basis for* W*, then*

$$\widehat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

$W = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. Find $\mathrm{Proj}_W \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix}$. Find $\mathrm{Proj}_W \begin{bmatrix} 7 \\ 2 \\ -2 \end{bmatrix}$.

NB: $\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & -1 & 4 \\ -1 & 1 & 0 \end{array}\right] \overset{\mathcal{R}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. Why did we know this already?

Q: What other vectors have same projection?

In general, to compute all vectors that give same projection, need a basis for $W^{\perp}$.