Video Lecture G3: Orthonormality

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Outline & Objectives

- Define the concepts of an *orthonormal set* and an *orthonormal basis* of vectors in $\mathbb{R}^n$ and analyze their properties and advantages over sets which are just *orthogonal*.
- Analyze properties of $m \times n$ matrices whose columns are orthonormal.
- Define an *orthogonal matrix* $U \in \mathbb{R}^{n\times n}$ as one that satisfies the property $U^{-1} = U^T$ and prove that its rows and columns are orthonormal bases for $\mathbb{R}^n$.

Orthonormal sets

Definition (Orthonormal sets and bases)

Call an orthogonal set of vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ an orthonormal set if $\|\vec{u}_i\| = 1\ \forall\, i \in [k]$. If additionally S is a basis for a subspace W, call S an orthonormal basis for W.

The standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is orthonormal. If $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is orthogonal (and all $\neq \vec{0}$), then $\left\{\frac{\vec{u}_1}{\|\vec{u}_1\|}, \ldots, \frac{\vec{u}_1}{\|\vec{u}_k\|}\right\}$ is orthonormal. $\left(\frac{-2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{3}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \right)$, $\left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)$ in $\mathbb{R}^4$.

Theorem (Orthonormal bases rock ON!)

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ be an orthonormal basis for W and suppose $\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$. Then $c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}\vec{y}\cdot\vec{u}_j\ \forall\, j \in [p]$.

Matrices with orthonormal columns

Theorem

$U \in \mathbb{R}^{m\times n}$ has orthonormal columns $\iff$ $U^T U = I$.

Theorem

Let $U \in \mathbb{R}^{m\times n}$ have ON columns and $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then (1) $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$; hence, (2) $\|U\vec{x}\| = \|\vec{x}\|$ and (3) $(U\vec{x})\cdot(U\vec{y}) = 0 \iff \vec{x}\cdot\vec{y} = 0$;

Pf: (1) $(U\vec{x})\cdot(U\vec{y}) = (U\vec{x})^T(U\vec{y}) = \vec{x}^T U^T U\vec{y} = \vec{x}^T\vec{y} = \vec{x}\cdot\vec{y}$. ■

Proposition

A square mx $U \in \mathbb{R}^{n\times n}$ has orthonormal columns $\iff$ $U^T = U^{-1}$ $\Leftrightarrow$ U has ON rows. Such a matrix is called orthogonal.

$$P = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$